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Étude de l'existence de solutions faibles d'énergie infinie pour les équations de Navier-Stokes incompressibles

Study of the existence of infinite energy weak solutions to the incompressible Navier-Stokes equations

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Chapter 1

Introduction

Pour l'étude des fluides visqueux incompressibles, les équations de Navier–Stokes sont un modèle fondamental. Dans cette thèse, nous nous proposons d'étudier les équations de Navier–Stokes sur l'espace tout entier, ainsi on oublie les conditions de bord, et on dispose d'un arsenal d'outils provenant de l'analyse harmonique pour son étude. Nous présentons les équations :

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0, \end{cases}$$

où, la vitesse du fluide dénotée par $\mathbf{u} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ et la pression du fluide $p : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ sont les inconnues, tandis que les données initiales sont la vitesse initiale du fluide dans le temps $t = 0$: $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$; et le tenseur de force $\mathbb{F} = (F_{i,j})_{1\leq i,j\leq 3}$ (avec $F_{i,j} : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$) dont la divergence $\nabla \cdot \mathbb{F}$ définie par $(\sum_{i=1}^3 \partial_i F_{i,1}, \sum_{i=1}^3 \partial_i F_{i,2}, \sum_{i=1}^3 \partial_i F_{i,3})$, représente la force appliquée au fluide. Dans la littérature, nous trouvons souvent le terme $(\mathbf{u} \cdot \nabla)\mathbf{u}$ réécrit comme $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ où l'on définit $\mathbf{u} \otimes \mathbf{u} = (u_i u_j)_{1\leq i,j\leq 3}$.

Même si les équations (NS) ci-dessus sont un système simplifié après avoir considéré l'espace $\mathbb{R}^3$ tout entier au lieu d'un domaine et d'avoir pris la constante de viscosité du fluide comme étant $\nu = 1$, la problématique de fond continue à être la même, et la compréhension des notions comme la turbulence et l'explosion sont encore le vif du sujet.

Une manière importante de diviser l'étude de ces équations se fait en considérant la notion d'énergie.

Une solution $\mathbf{u}$ des équations (NS) sans force extérieure, c'est à dire avec $\mathbb{F} = 0$, issue d'une donné initiale $\mathbf{u}_0$, est dite une solution d'énergie finie pour une famille de fonctionnelles d'énergie $E(t, \cdot)$, indexée par t, si

$$E(0, \mathbf{u}_0) < +\infty \qquad \text{et} \qquad t \to E(t, \mathbf{u}) \text{ est décroissante,}$$

ces fonctionnelles étant définies positives sur l'espace de fonctions de $[0, t] \times \mathbb{R}^3$ dans $\mathbb{R}^3$, pour tout $t \geq 0$.

L'estimation de fonctionnelles d'énergie pour obtenir un contrôle des approximations de solutions et pouvoir ainsi passer à la limite par un argument de compacité de type Aubin-Lions est une idée qui remonte aux travaux fondateurs de Leray en 1934 (Leray, 1934). Dans l'étude de solutions du système de Navier-Stokes incompressible, un rôle clé est joué par l'inégalité d'énergie de Leray

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2\int_0^t \|\nabla \mathbf{u}(s)\|_{L^2}^2 ds \leq \|\mathbf{u}(0)\|_{L^2}^2 - 2\sum_{1\leq i,j\leq 3}\int_0^t \int F_{i,j}\partial_i u_j \; dx\, ds,$$

Leray a construit, pour une donné initiale L^2, des solutions faibles globales qui satisfont cette inégalité d'énergie, où le terme énergie fait référence à la fonctionnelle $E(t,\mathbf{u}) = \|\mathbf{u}(t)\|_2$.

L'étude de l'unicité et régularité pour les solutions d'énergie finie de Leray est encore un problème principal.

Récemment, dans (Feichtinger et al., 2020), H. Feichtinger, K. Gröchenig, Kuijie Li and Baoxiang Wang ont donné un sens à l'unicité du problème pour certains données initiales dans L^2, en se restreignant aux fonctions dont les fréquences vivent dans le premier octant de l'espace $\mathbb{R}^3$, et aussi en sortant du cadre de distributions car ils ont utilisé un espace différent de fonctions test.

Pour le cas périodique, c'est à dire, lorsque l'on cherche une solution $\mathbf{u} : [0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ aux équations de Navier–Stokes, il y a eu de nombreuses avancées par rapport a la non-unicité des solutions faibles d'énergie finie.

Quelques travaux qui ont fait partie de ces récentes avancées sont (Lellis and Székelyhidi-Jr, 2019), (Buckmaster and Vicol, 2019b) et (Buckmaster and Vicol, 2019a). Ils ont développé une technique, qu'on appelle intégration convexe, pour construire des solutions non uniques.

Cependant, ces résultats ne sont pas exactement liés aux solutions de Leray dans l'espace tout entier. L'appartenance de ces nouvelles solutions non uniques à l'espace $L^2((0,T),H^1)$ n'est pas connue, et donc le problème de la non-unicité des solutions de Leray dans l'espace tout entier reste ouvert.

Il existe une version locale de l'inégalité d'énergie de Leray liée à l'existence d'une mesure μ, positive et localement finie, pour laquelle la solution $\mathbf{u}$ satisfait

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{u}\right) - \nabla\cdot(p\mathbf{u}) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

Cette balance d'énergie locale a été exploitée en 1995 par Farwig et Sohr (Farwig and Sohr, 1995) pour montrer l'existence de solutions faibles dans le cas du domaine extérieur.

Les solutions faibles d'énergie infinie ont été introduites ensuite par Lemarié-Rieusset en 1999 (Lemarié-Rieusset, 1999). Cela a permis de montrer l'existence locale de solutions faibles pour une donnée $\mathbf{u}_0$ uniformément localement de carré intégrable, c'est à dire, telle que

$$\|\mathbf{u}_0\|_{L^2_{\text{uloc}}} = \sup_{x\in\mathbb{R}^3}\int_{B(x,1)} |\mathbf{u}_0|^2\, dy < +\infty,$$

et un tenseur de force $\mathbb{F} \in (L^2_t L^2_x)_{\text{uloc}}((0,1)\times\mathbb{R}^3)$, c'est à dire qui satisfait

$$\sup_{x_0\in\mathbb{R}^3}\int_0^1\int_{B(x_0,1)} |\mathbb{F}|^2\, dxds < +\infty.$$

Nous introduisons une des définitions pionniers.

Définition. *(Solution locale de Leray) Un champ de vecteurs* $\boldsymbol{u} = (u_1, u_2, u_3) \in L^2_{\mathrm{loc}}([0, T_0) \times \mathbb{R}^3)$ *est appelée une solution locale de Leray du problème de Navier–Stokes (NS) associé a une vitesse initiale à divergence nulle* $\boldsymbol{u}_0 \in L^2_{\mathrm{uloc}}$ *et à un tenseur de force*

$$\mathbb{F} = (F_{i,j}) \in (L^2_t L^2_x)_{\mathrm{uloc}}((0, T_0) \times \mathbb{R}^3)$$

si :

- *Pour tout* $t < T < T_0$,

$$\sup_{0 \le t < T} \sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0,1)} |\boldsymbol{u}|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^T \int_{B(x_0,1)} |\nabla \boldsymbol{u}|^2 \, dx ds < +\infty.$$

- *Il existe une distribution* p *sur* $(0, T_0) \times \mathbb{R}^3$ *tel que* $(\boldsymbol{u}, p)$ *est solution faible du système*

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases}$$

c'est à dire, pour toute $\varphi \in \mathcal{D}((0, T_0) \times \mathbb{R}^3)$ *et pour toute* $j \in \{1, 2, 3\}$, *on a*

$$\begin{aligned} - \iint_{(0,T) \times \mathbb{R}^3} u_j \partial_t \varphi = \iint_{(0,T) \times \mathbb{R}^3} u_j \Delta \varphi + \sum_{i=1}^3 \iint_{(0,T) \times \mathbb{R}^3} u_i u_j \partial_i \phi \\ + \langle p, \partial_j \varphi \rangle - \sum_{i=1}^3 \iint_{(0,T) \times \mathbb{R}^3} F_{i,j} \partial_i \varphi \end{aligned}$$

et

$$\sum_{i=1}^3 \iint_{(0,T) \times \mathbb{R}^3} u_i \partial_i \varphi = 0$$

- *Pour tout compact* $K \subset \mathbb{R}^3$, *on a*

$$\lim_{t \to 0} \|\boldsymbol{u}(t, .) - \boldsymbol{u}_0\|_{L^2(K)} = 0$$

- $\boldsymbol{u}$ *est adaptée au sens de Caffarelli-Kohn-Nirenberg :* $p \in (L^{\frac{3}{2}} L^{\frac{3}{2}})_{\mathrm{loc}}$ *et il existe une mesure positive localement finie pour laquelle*

$$\partial_t (\frac{|\boldsymbol{u}|^2}{2}) = \Delta (\frac{|\boldsymbol{u}|^2}{2}) - |\nabla \boldsymbol{u}|^2 - \nabla \cdot \left(\frac{|\boldsymbol{u}|^2}{2} \boldsymbol{u} \right) - \nabla \cdot (p \boldsymbol{u}) + \boldsymbol{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

1.1 Motivation

À ce point, nous soulignons le fait qu'un traitement astucieux pour le terme de la pression est nécessaire dans le contexte L^2_{uloc}, par exemple, pour obtenir les contrôles à priori des solutions du problème approché. En (Lemarié-Rieusset, 2016), on fait varier la définition de la pression, selon la partie de l'espace où l'on veut obtenir un contrôle uniforme pour les solutions approchées.

Dans les dernières années, cette notion de solution locale de Leray a été légèrement modifiée plusieurs fois, soit pour rajouter des conditions sur la pression ou soit pour considérer des données initiales dans des espaces qui ne sont pas contenus dans L^2_{uloc} et où on ne connaissait pas à priori si la pression était caractérisée. On trouve, dans la littérature l'utilisation des expansions locales de la pression pour analyser l'existence, unicité et régularité des solutions du système de Navier–Stokes, comme l'ont fait (Bradshaw and Tsai, 2020a), (Bradshaw and Tsai, 2020b) et (Kang, Miura, and Tsai, 2020).

En réfléchissant à ces aspects, on pourrait se demander sous quelles hypothèses la pression est déterminée, modulo les constantes, par la vitesse et la force. Si on détermine, dans un cadre assez général, une formule globale qui donne la pression en fonction de la vitesse et la force, les différences entre ces récentes définitions des solutions adaptées seront raccourcies.

D'autres constructions de ces solutions d'énergie infinie à donnée initiale uniformément localement L^2 ont été données en 2006 par Basson (Basson, 2006b) et en 2007 par Kikuchi et Seregin (Kikuchi and Seregin, 2007).

Basson a étudié le cas d'une donnée initiale appartenant à $L^2_{\mathrm{uloc}}(\mathbb{R}^2)$ dans sa thèse (Basson, 2006b), où il a démontré existence globale, unicité, régularité et dépendance continue de la donnée initiale. Basson a aussi étudié un espace de données légèrement plus grand, appelé $B_2(\mathbb{R}^2)$, et définie par la norme

$$\|\mathbf{u}_0\|^2_{B_2} = \sup_{R\geq 1} \frac{1}{R^2} \int_{B(0,R)} |\mathbf{u}_0|^2 dx.$$

Dans ce cas, pour une solution $\mathbf{u}_\epsilon \in L^2_{loc}$ d'un système de Navier-Stokes approché et pour φ une fonction test, le terme $\int (\nabla p_\epsilon \cdot \mathbf{u}_\epsilon)\varphi dx$, qui fait intervenir la pression est divisé en deux morceaux, un où p_ϵ se contrôle comme $|\mathbf{u}_\epsilon|^2$ et l'autre où ∇p_ϵ est bien comporté, afin d'obtenir contrôles uniformes pour les solutions approchées $\mathbf{u}_\epsilon$. Avec cette astuce, il arrive à démontrer existence d'une solution locale, mais une formule pour la pression n'est pas explicitée.

Dans les dernières années l'étude du terme de la pression a été un sujet actif, voir par exemple (Kukavica, 2003), (Kukavica and Vicol, 2008) et (Wolf, 2017). On dédiera une bonne partie de cette thèse à l'analyse du terme de la pression.

Jia et Šverák ont surpris en 2014 (Jia and Šverák, 2014) en donnant une construction de solutions auto-similaires à grandes données régulières. L'ingrédient principal et l'idée originale a été de faire intervenir la théorie du degré de Leray–Schauder. Le résultat a été étendu en 2016 par Lemarié-Rieusset (Lemarié-Rieusset, 2016) pour des solutions à données localement L^2. L'observation cruciale à ce stade est qu'une donnée auto-similaire (homogène de degré -1) et localement L^2 est automatiquement uniformément localement L^2.

En 2017 et 2018, Bradshaw et Tsai (Bradshaw and Tsai, 2017) et Chae et Wolf (Chae and Wolf, 2018) ont exploré le cas des solutions auto-similaires pour un sous-groupe discret de dilatations et ils ont donné une réponse positive. Une donnée initiale localement L^2 n'est plus nécessairement uniformément localement L^2, donc les idées précédentes de Lemarié–Rieusset ne pouvaient pas être appliquées. Dans le travail de Chae et Wolf, où ils considèrent des hypothèses très faibles pour la donnée initiale, juste être discrètement auto-similaire et localement L^2, une inégalité d'énergie modifiée a été introduite dans les définitions, un détail corrigé en (Bradshaw and Tsai, 2019a). Un beau résumé de la série de résultats en rapport avec la notion d'auto-similarité et d'auto-similarité discrète est donné en (Bradshaw and Tsai, 2019b).

Nous observons à ce point le fait suivant : si on considère un nombre $\lambda > 0$ fixé et $\mathbf{u}$ un champ de vecteurs tel que $\lambda\mathbf{u}(\lambda x) = \mathbf{u}(x)$, alors $\mathbf{u}$ appartient à $L^2((1+|x|)^{-\gamma}dx)$ pour tout $\gamma > 1$. Cette remarque et les travaux récents de recherche autour des solutions auto-similaires nous ont amené à développer dans (Fernández-Dalgo and Lemarié-Rieusset, 2020b) une extension de la procédure de Leray (pour obtenir des solutions faibles) de L^2 à $L^2((1+|x|)^{-\gamma}dx)$ avec $0 \leq \gamma \leq 2$; on a aussi révisité les résultats en rapport avec les solutions discrètement auto-similaires.

Des améliorations et des raffinements ultérieurs de ces résultats ont été obtenus plus récemment, voir par exemple (Bradshaw, Kukavica, and Tsai, 2019). Nous présentons dans ce mémoire les résultats ultérieurs.

En rapport avec les solutions axisymétriques, nous savons que Ladyzhenskaya (Ladyzhenskaya, 1968) à utilisé un contrôle d'énergie sur la vorticité pour démontrer l'existence d'une solution axisymétrique globale sans tourbillon pour une donnée initiale appartenant à H^1. Bien que ce contrôle soit sous-jacent à la structure des champs de vecteurs axisymétriques sans tourbillon, il est raisonnable de chercher à adapter des résultats analogues pour des espaces à poids radial. C'est ce qu'on a fait en (Fernández-Dalgo and Lemarié-Rieusset, 2021).

1.2 Organisation du document

Nous dédions la première partie de cette thèse, le Chapitre 2, à exposer les résultats obtenus en (Fernández-Dalgo and Lemarié-Rieusset, 2020a) qui donnent une caractérisation du terme de la pression dans un contexte assez général pour être appliquée aux solutions d'énergie infinie étudiées jusqu'à présent.

Dans le Chapitre 3, en suivant (Fernández-Dalgo and Lemarié-Rieusset, 2021), nous développons une procédure générale, pour obtenir des solutions faibles dans des espaces L^2 à poids. Cette procédure générale se base en certaines propriétés du poids qui permettent de contrôler le terme de la pression. L'importance de ces résultats se met en évidence dans le chapitre suivant.

Nous présentons des résultats récents d'existence de solutions régulières axisymétriques sans tourbillon dans le Chapitre 4, où nous considérons de données initiales appartenant à des espaces L^2 à poids et dont la vorticité elle aussi appartient à un espace L^2 à poids. Les contrôles obtenus dans le chapitre précédent, le Chapitre 3, sont cruciales parce que les hypothèses sur les poids permettent de considérer des poids avec une plus grande décroissance à l'infini, par rapport à des contrôles obtenus précédemment.

Pour l'étude de l'existence des solutions discrètement auto-similaires nous dédions le Chapitre 5. Nous remarquons que dans le théorème d'existence intervient les espaces à poids $L^2(w_\gamma dx)$, avec $w_\gamma = (1+|x|)^{-\gamma}$ et $\gamma > 1$ et la condition $\gamma > 1$ est optimale dans le sens qu'une donné initiale discrètement auto-similaire non nulle n'appartient pas à $L^2((1+|x|)^{-1}dx)$, Ces résultats ont été présentés dans

(Fernández-Dalgo and Jarrín, 2021a) dans le contexte des équations de la magnétohydrodynamique.

Dans le chapitre 6, nous revenons sur plusieurs résultats, obtenues pour les équations de Navier–Stokes dans les chapitres précédentes, mais cette fois dans le contexte des équations de la magnétohydrodynamique. Nous aussi ajoutons un nouveau résultat d'unicité fort-faible qui considère des espaces L^2 à poids.

Dans la suite, nous allons décrire de manière plus précise quelques-uns des résultats les plus importants.

Nos premiers efforts visent à donner une formule pour le terme de la pression dans des conditions très générales. Les conditions qu'on a trouvées sont décrites dans le théorème suivant :

Théorème. *Considérons la dimension $d \in \{2,3\}$ et $0 < T < +\infty$. Soit $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leq i,j\leq d}$ un tenseur qui appartient à $L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\,dx))$.*

Soit $\boldsymbol{u}$ une solution du problem

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) - \mathbf{S} + \nabla\cdot\mathbb{F} \\ \nabla\cdot\boldsymbol{u} = 0, \quad \nabla\wedge\mathbf{S} = 0, \quad \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \end{cases}$$

qui satisfait : $\boldsymbol{u}$ appartient à $L^2((0,T), L^2_{w_{d+1}}(\mathbb{R}^d))$, $\lim_{t\to 0}\boldsymbol{u}(t,.) = \boldsymbol{u}_0 \in L^2_{w_{d+1}}$ dans $\mathcal{D}'$ et $\mathbf{S}$ appartient à $\mathcal{D}'((0,T)\times\mathbb{R}^d)$.

On considère $\varphi \in \mathcal{D}(\mathbb{R}^d)$ telle que $\varphi(x) = 1$ dans un voisinage de 0 et on note

$$A_{i,j,\varphi} = (1-\varphi)\partial_i\partial_j G_d.$$

Alors, il existe $g(t) \in L^1((0,T))$ telle que

$$\mathbf{S} = \nabla p_\varphi + \partial_t g$$

avec

$$\begin{aligned} p_\varphi =& \sum_{i,j}(\varphi\partial_i\partial_j G_d) * (u_i u_j - F_{i,j}) \\ &+ \sum_{i,j}\int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y))\,dy. \end{aligned}$$

En plus,

- *∇p_φ ne dépend pas du choix de φ : Si on change φ par ψ, on obtient*

$$p_\varphi(t,x) - p_\psi(t,x) = \sum_{i,j}\int (A_{i,j,\psi}(-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y))\,dy.$$

- *∇p_φ est l'unique solution du problème de Poisson*

$$\Delta \boldsymbol{w} = -\nabla(\nabla\cdot(\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u} - \mathbb{F}))$$

qui satisfait

$$\lim_{\tau\to+\infty} e^{\tau\Delta}\boldsymbol{w} = 0.$$

- *Si on suppose que* $\mathbb{F}$ *appartient à* $L^1((0,T), L^1_{w_d}(\mathbb{R}^d))$ *et* $\boldsymbol{u} \in L^2((0,T), L^2_{w_d}(\mathbb{R}^d))$, *alors on a que* g *est constante et on peut prendre* $g = 0$, *et* $\nabla p_\varphi = \nabla p_0$ *où*

$$p_0 = \sum_{i,j}(\varphi \partial_i \partial_j G_d) * (u_i u_j - F_{i,j}) + \sum_{i,j}((1-\varphi)\partial_i \partial_j G_d) * (u_i u_j - F_{i,j}),$$

p_0 ne dépend pas du choix de φ et on pouvait définir $p_0 = \sum_{i,j}(\partial_i \partial_j G_d) * (u_i u_j - F_{i,j})$.

Dans le chemin pour arriver à cette caractérisation du terme de la pression, on passe par l'étude de deux problèmes de Poisson, et cela nous amène à un corollaire intéressant par rapport au projecteur de Leray. En effet, on obtient que la définition suivante nous donne bien un projecteur sur les champs de vecteurs à divergence nulle.

Définition. *Soit* $\mathbb{H} \in L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\, dx))$ *un tenseur et soit* $\boldsymbol{w} = \nabla \cdot \mathbb{H}$. *Le projecteur de Leray* $\mathbb{P}(\boldsymbol{w})$ *de* $\boldsymbol{w}$ *défini par*

$$\mathbb{P}\boldsymbol{w} = \boldsymbol{w} - \nabla p_\varphi$$

où ∇p_φ *est l'unique solution de*

$$-\Delta \nabla p = -\nabla(\nabla \cdot \boldsymbol{w})$$

telle que

$$\lim_{\tau \to +\infty} e^{\tau \Delta} \nabla p = 0,$$

est un champ des vecteurs à divergence nulle.

Une fois qu'on a éclairé la relation entre le terme de la pression, la vitesse et la force extérieure, nous procédons à établir des contrôles uniformes dans des espaces à poids pour les solutions du système régularisé de Navier–Stokes.

On considère dans un premier temps, un bilan d'énergie pour la vitesse, ce qui nous permet d'obtenir des solutions globales pour des données initiales appartenant à des espaces L^2 à poids. Afin de rendre ces contrôles sur la vitesse réutilisables, pour obtenir des contrôles uniformes pour la vorticité dans des espaces à poids, et ceci sans perte de généralité dans l'ordre de décroissance du poids, nous considérons les propriétés suivantes pour les poids :

Définition. *Soit* Φ *une fonction sur* $\mathbb{R}^d$ $(2 \le d \le 4)$. *On dit que* Φ *est un poids adapté si* Φ *est une fonction Lipschitz continue et satisfait :*

- $(\boldsymbol{H}1)$ $0 < \Phi \le 1$.
- $(\boldsymbol{H}2)$ *Il existe* $C_1 > 0$ *telle que* $|\nabla \Phi| \le C_1 \Phi^{\frac{3}{2}}$
- $(\boldsymbol{H}3)$ *Il existe* $r \in (1,2]$ *telle que* $\Phi^r \in \mathcal{A}_r$ *(où* $\mathcal{A}_r$ *est la classe des poids de Muckenhoupt). Pour* $d = 4$, *on a besoin de la condition* $r < 2$.
- $(\boldsymbol{H}4)$ *Il existe* $C_2 > 0$ *telle que* $\Phi(x) \le \Phi(\frac{x}{\lambda}) \le C_2 \lambda^2 \Phi(x)$, *pour tout* $\lambda \ge 1$. *En particulier,* Φ *est décroissante.*

Nos exemples de base, des poids adaptés, sont :

- $d = 2$, $\Phi(x) = \frac{1}{(1+|x|)^\gamma}$ où $0 \le \gamma < 2$
- $d = 3$ ou $d = 4$, $\Phi(x) = \frac{1}{(1+|x|)^\gamma}$ où $0 \le \gamma \le 2$

- $d=3$, $\Phi(x)=\frac{1}{(1+r)^\gamma}$ où $r=\sqrt{x_1^2+x_2^2}$ et $0\leq\gamma<2$.

Notre résultat principal d'existence globale dans le cadre de régularité locale L^2 pour la donné initiale, s'écrit de la manière suivante :

Théorème. *Considérons la dimension $d\in\{2,3,4\}$ et un poids adapté Φ. Soit $\boldsymbol{u}_0$ un champ de vecteurs à divergence nulle appartenant à $L^2(\Phi\,dx,\mathbb{R}^d)$. Soit $\mathbb{F}=(F_{i,j}(t,x))_{i,j}$ un tenseur appartenant à $L^2((0,+\infty),L^2(\Phi\,dx,\mathbb{R}^d))$.*

Alors, il existe une solution globale $\boldsymbol{u}$ du problème

$$(NS)\begin{cases}\partial_t\boldsymbol{u}=\Delta\boldsymbol{u}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\nabla p+\nabla\cdot\mathbb{F}\\ \nabla\cdot\boldsymbol{u}=0, \qquad\qquad \boldsymbol{u}(0,.)=\boldsymbol{u}_0\end{cases}$$

qui satisfait :

- *$\boldsymbol{u}$ appartient à $L^\infty((0,T),L^2(\Phi dx))$ et $\nabla\boldsymbol{u}$ appartient à $L^2((0,T),L^2(\Phi dx))$, pour tout $T>0$,*
- *$p=\sum_{1\leq i,j\leq d}\mathcal{R}_i\mathcal{R}_j(u_iu_j-F_{i,j})$,*
- *la transformation $t\in[0,+\infty)\mapsto\boldsymbol{u}(t,.)$ est faiblement continue de $[0,+\infty)$ dans $L^2(\Phi\,dx)$, et fortement continue en $t=0$,*
- *l'inégalité d'énergie locale pour $d\in\{2,3\}$: il existe une mesure positive localement finie μ telle que*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2})=\Delta(\frac{|\boldsymbol{u}|^2}{2})-|\nabla\boldsymbol{u}|^2-\nabla\cdot\left(\frac{|\boldsymbol{u}|^2}{2}\boldsymbol{u}-p\boldsymbol{u}\right)+\boldsymbol{u}\cdot(\nabla\cdot\mathbb{F})-\mu,$$

et dans le cas de la dimension $d=2$ on a $\mu=0$.

En plus, on obtient que

$$\begin{aligned}&\|\boldsymbol{u}(t,.)\|^2_{L^2(\Phi dx)}+2\int_0^t\|\nabla\otimes\boldsymbol{u}(s,.)\|^2_{L^2(\Phi dx)}\,ds\\ &\leq\|\boldsymbol{u}_0\|^2_{L^2(\Phi dx)}-\int_0^t\int\nabla(|\boldsymbol{u}|^2)\cdot\nabla\Phi\,dx\,ds+\int_0^t\int(|\boldsymbol{u}|^2\boldsymbol{u}+2p\boldsymbol{u})\cdot\nabla\Phi\,dx\,ds\\ &\quad-2\sum_i\sum_j\int_0^t\int F_{i,j}u_j\partial_i\Phi+F_{i,j}\partial_iu_j\Phi\,dx\,ds.\end{aligned}$$

Dans un deuxième temps, nous étudions un bilan d'énergie pour la vorticité, afin d'obtenir dans le cadre des solutions axisymétriques, des solutions globales pour une donnée initiale appartenant à un espace L^2 à poids et dont la vorticité elle aussi appartient à un espace L^2 à poids.

Ainsi, nous arrivons à notre résultat principal d'existence globale dans le cadre d'une donnée initiale de régularité locale H^1. Nous observons que les poids $\Phi(x)=\frac{1}{(1+r)^\gamma}$ et $\Psi(x)=\frac{1}{(1+r^2)^{\delta/2}}$ avec $0\leq\delta\leq\gamma<2$ satisfont les hypothèses du théorème :

Théorème. *On considère un poids Φ qui satisfait $(\boldsymbol{H}1)-(\boldsymbol{H}4)$. On assume que Φ ne dépend que de $r=\sqrt{x_1^2+x_2^2}$ et qu'il existe un poids Ψ qui est une fonction continue qui ne dépend que de r, tel que $\Phi\leq\Psi\leq1$, $\Psi\in\mathcal{A}_2$ et il existe $C_1>0$ tel que*

$$|\nabla\Psi|\leq C_1\sqrt{\Phi}\Psi \text{ and } |\Delta(\sqrt{\Psi})|\leq C_1\sqrt{\Phi}.$$

Soit $\boldsymbol{u}_0$ un champ de vecteurs à divergence nulle sans tourbillon, tel que $\boldsymbol{u}_0$ appartient à $L^2(\Phi dx)$ et $\nabla \otimes \boldsymbol{u}_0$ appartient à $L^2(\Psi dx)$. Alors, il existe une solution globale $\boldsymbol{u}$ du problème

$$(NS)\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p \\ \nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

telle que

- *$\boldsymbol{u}$ est axisymétrique sans tourbillon, $\boldsymbol{u}$ appartient à $L^\infty((0,T), L^2(\Phi\, dx))$, $\nabla \otimes \boldsymbol{u}$ appartient à $L^\infty((0,T), L^2(\Psi\, dx))$ et $\Delta \boldsymbol{u}$ appartient à $L^2((0,T), L^2(\Psi\, dx))$, pour tout $T > 0$,*
- *les transformations $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ et $t \in [0,+\infty) \mapsto \nabla \otimes \boldsymbol{u}(t,.)$ sont faiblement continues de $[0,+\infty)$ dans $L^2(\Phi\, dx)$ et dans $L^2(\Psi\, dx)$ respectivement, et sont fortement continues en $t = 0$.*

Après, dans le Chapitre 5 nous analysons les solutions auto-similaires. Il vaut mieux introduire d'abord les notions suivantes :

Définition. *On considère $\lambda > 1$.*

On dit que $\boldsymbol{u}_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ est un champ de vecteurs λ-discrètement auto-similaire (λ-DSS) si

$$\lambda \boldsymbol{u}_0(\lambda x) = \boldsymbol{u}_0(x).$$

Un champ de vecteurs dépendant du temps $\boldsymbol{u} \in L^2_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^3)$, est appelé λ-DSS s'il vérifie

$$\lambda \boldsymbol{u}(\lambda^2 t, \lambda x) = \boldsymbol{u}(t,x).$$

Un tenseur de force $\mathbb{F} \in L^2_{\mathrm{loc}}([0,+\infty)\times\mathbb{R}^3)$, est appelé λ-DSS si

$$\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t,x).$$

Nous remarquons que dans les contrôles à priori étudiés pour obtenir des solutions discrètement auto-similaires, on considère des espaces convenables pour optimiser l'intégrabilité des solutions obtenues.

On va se rendre compte que l'espace $L^{10/3}_t L^{10/3}_x$ apparaît souvent dans cette analyse et c'est cet espace qui nous permet d'obtenir γ proche de 1 dans le théorème ci-dessous. L'utilité de cette espace $L^{10/3}_t L^{10/3}_x$ est un peu naturel comme il est l'espace $L^p L^p$ plus régulier qui contient l'espace d'énergie naturel pour les équations de Navier–Stokes, $L^\infty_t L^2 \cap L^2_t \dot{H}^1$.

Un détail important c'est que, comme dans toutes les démonstrations précédentes de l'existence des solutions auto-similaires pour des grandes données initiales, nous faisons intervenir la théorie du dégré de Leray–Schauder. Le résultat est le suivant :

Théorème. *Soit $\gamma \in (1,2)$ et $\lambda \in (1,+\infty)$. Si $\boldsymbol{u}_0$ est un λ-DSS champ de vecteurs à divergence nulle qui appartient à $L^2_{w_\gamma}(\mathbb{R}^3)$ et si $\mathbb{F}$ est un λ-DSS tenseur de force $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\le i,j\le 3}$ qui appartient à $L^2_{loc}((0,+\infty), L^2_{w_\gamma})$, alors les équations de Navier–Stokes avec vitesse initiale $\boldsymbol{u}_0$,*

$$(NS)\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

ont une solution globale $\boldsymbol{u}$ qui satisfait les propriétés suivantes :

- *$\boldsymbol{u}$ est un λ-DSS champ de vecteurs*
- *pour tout $0 < T < +\infty$, $\boldsymbol{u}$ appartient à $L^\infty((0,T), L^2_{w_\gamma})$ et $\nabla \boldsymbol{u} \in L^2((0,T), L^2_{w_\gamma})$*
- *la fonction $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ est faiblement continue de $[0,+\infty)$ dans $L^2_{w_\gamma}$, et fortement continue en $t = 0$*
- *$\boldsymbol{u}$ est adaptée : il existe une mesure positive localement finie μ sur $(0,+\infty) \times \mathbb{R}^3$ telle que*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2}{2}) - |\nabla \boldsymbol{u}|^2 - \nabla \cdot \left((\frac{|\boldsymbol{u}|^2}{2} + p)\boldsymbol{u} \right) + \boldsymbol{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

Dans le dernier chapitre, le Chapitre 6, nous étudions le système couplé des équations de la magnétohydrodynamique :

$$(\text{MHD}) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{u} - \nabla q, \\ \nabla \cdot \mathbf{u} = 0,\ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

Ce système décrit le comportement d'un fluide conducteur du courant électrique en présence de champs électromagnétiques. La vitesse du fluide est notée **u** et **b** est un champ magnétique.

La structure des équations (NS) et (MHD) est vraiment très proche. C'est clair que si $\mathbf{b} = 0$ on obtient les équations de Navier Stokes. En plus, ces deux équations ont le même scaling, c'est à dire : Si $\lambda > 0$ alors $(\mathbf{u}, \mathbf{b})$ est une solution du problème de Cauchy pour le système (MHD) sur $(0, T)$ avec donnée initiale $(\mathbf{u}_0, \mathbf{b}_0)$ et tenseur de force $\mathbb{F}$, si et seulement si, $(\mathbf{u}_\lambda, \mathbf{b}_\lambda)(t,x) = (\lambda \mathbf{u}(\lambda^2 t, \lambda x), \lambda \mathbf{b}(\lambda^2 t, \lambda x))$ est une solution du problème de Cauchy pour le système (MHD) sur $(0, T/\lambda^2)$ avec donnée initiale $(\mathbf{u}_{0,\lambda}, \mathbf{b}_{0,\lambda})(x) = (\lambda \mathbf{u}_0(\lambda x), \lambda \mathbf{b}_0(\lambda x))$ et tenseur de force $\mathbb{F}_\lambda(t,x) = \lambda^2 \mathbb{F}(\lambda^2 t, \lambda x)$.

C'est surtout sur le bilan d'énergie que nous sommes intéressés à savoir comme ces deux équations, (NS) et (MHD), se ressemblent. En fait, ce qui nous sera utile c'est que la condition d'être une solution adapté pour le système (MHD) fait intervenir des termes en forme de divergence pour pouvoir contrôler les termes qui proviennent de la partie non-linéaire des équations ou qui proviennent du terme de la pression.

Comme nous allons voir, on va considérer des solutions du système (MHD) pour lesquelles $q = 0$ et qui satisfont la propriété suivante :

- La solution $(\mathbf{u}, \mathbf{b}, p)$ est adaptée : il existe une mesure positive localement finie μ sur $(0,+\infty) \times \mathbb{R}^3$ telle que

$$\begin{aligned} \partial_t(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}) =& \Delta(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}) - |\nabla \mathbf{u}|^2 - |\nabla \mathbf{b}|^2 - \nabla \cdot \left((\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2})\mathbf{u} \right) \\ & - \nabla \cdot (p\mathbf{u}) + \nabla \cdot ((\mathbf{u} \cdot \mathbf{b})\mathbf{b}) \\ & + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu. \end{aligned}$$

Ces remarques essentiellement vont nous permettre d'étudier le système (MHD) en suivant les mêmes idées utilisées pour analyser (NS).

Un résultat du dernier Chapitre qui est nouveau même dans le contexte des équations de Navier–Stokes, est un résultat d'unicité fort-faible dans le cadre des espaces à poids.

Théorème. *Soit* $0 \leq \gamma \leq 2$ *et* $w_\gamma = (1+|x|)^{-\gamma}$. *Soit* $0 < T < +\infty$. *Soient* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3) = L^2((1+|x|)^{-\gamma}dx)$ *deux champs de vecteurs à divergence nulle. Aussi, on considère un tenseur de force* $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leq i,j\leq 3} \in L^2((0,T), L^2_{w_\gamma})$.

Soient $(\boldsymbol{u}, \boldsymbol{b}, p)$ *et* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}}, \tilde{p})$ *deux solutions du système (MHD) telles que :*

- $\boldsymbol{u}, \boldsymbol{b}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}}$ *appartient à l'espace* $L^\infty((0,T), L^2_{w_\gamma})$ *et* $\nabla\boldsymbol{u}, \nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{b}, \nabla\tilde{\boldsymbol{b}} \in L^2((0,T), L^2_{w_\gamma})$
- *les transformations* $t \in [0,T) \mapsto (\boldsymbol{u}, \boldsymbol{b})(t,.)$ *et* $t \in [0,T) \mapsto (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}})(t,.)$ *sont faiblement continues de* $[0,T)$ *dans* $L^2_{w_\gamma}(\mathbb{R}^3)$, *et sont fortement continues en* $t = 0$
- $(\boldsymbol{u}, \boldsymbol{b})$ *et* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}})$ *sont adaptées: il existe deux mesures positives localement finies* μ *et* ν *sur* $(0,T) \times \mathbb{R}^3$ *telles que*

$$\partial_t(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - |\nabla\boldsymbol{b}|^2 - \nabla\cdot\left((\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2} + p)\boldsymbol{u}\right) + \nabla\cdot((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{b}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu,$$

et

$$\partial_t(\frac{|\tilde{\boldsymbol{u}}|^2 + |\tilde{\boldsymbol{b}}|^2}{2}) = \Delta(\frac{|\tilde{\boldsymbol{u}}|^2 + |\tilde{\boldsymbol{b}}|^2}{2}) - |\nabla\tilde{\boldsymbol{u}}|^2 - |\nabla\tilde{\boldsymbol{b}}|^2 - \nabla\cdot\left((\frac{|\tilde{\boldsymbol{u}}|^2}{2} + \frac{|\tilde{\boldsymbol{b}}|^2}{2} + \tilde{p})\tilde{\boldsymbol{u}}\right) + \nabla\cdot((\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{b}})\tilde{\boldsymbol{b}}) + \tilde{\boldsymbol{u}}\cdot(\nabla\cdot\mathbb{F}) - \nu.$$

Si $\boldsymbol{u}, \boldsymbol{b} \in L^pL^q$, *avec* $\frac{2}{p} + \frac{3}{q} = 1$ *et* $p \in (2,+\infty)$ *alors on a que* $(\boldsymbol{u}, \boldsymbol{b}, p) = (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}}, \tilde{p})$.

Dans le Chapitre 6, nous introduisons aussi les espaces de Morrey locaux :

Définition. *Pour* $\gamma \geq 0$ *et* $p \in (1,\infty)$. *On note* $B^p_\gamma(\mathbb{R}^d)$ *l'espace de Banach de toutes les fonctions* $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ *telles que :*

$$\|u\|_{B^p_\gamma} = \sup_{R\geq 1}\left(\frac{1}{R^\gamma}\int_{B(0,R)} |u(x)|^p\,dx\right)^{1/p} < +\infty.$$

Nous notons $B^p_\gamma L^p(0,T)$ *l'espace de Banach de toutes les fonctions* $u \subset (L^p_t L^p_x)_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ *telles que*

$$\|u\|_{B^p_\gamma L^p(0,T)} = \sup_{R\geq 1}\left(\frac{1}{R^\gamma}\int_0^T\int_{B(0,R)} |u(t,x)|^p\right)^{\frac{1}{p}} dx\,dt < +\infty.$$

$B^p_{\gamma,0}$ *est défini comme le sous-espace de toutes les fonctions* $u \in B^p_\gamma$ *telles que*

$$\lim_{R\to+\infty} \frac{1}{R^\gamma}\int_{B(0,R)} |u(x)|^p\,dx = 0.$$

On note aussi $B^2_2 = B_2$.

L'intérêt des espaces B^p_γ c'est qu'ils sont très proches des espaces à poids $L^2((1+|x|)^{-\gamma}dx)$, qui nous ont déjà été utiles. En particulier, pour $2<\delta<+\infty$, on obtient les inclusions continues

$$L^2_{w_2}\subset B^2_{2,0}\subset B_2\subset L^2_{w_\delta}.$$

Une fois qu'on connaît le résultat d'existence globale d'une solution pour une donné initial qui appartient à $L^2((1+|x|)^{-2}dx)$ c'est naturel d'essayer à démontrer le résultat pour des données initiales qui appartiennent à B_2. On fait cet étude et cela conduit au résultat ci-dessous.

Théorème. *Considère* $0<T<+\infty$. *Soient* $\boldsymbol{u}_0,\boldsymbol{b}_0\in B_2(\mathbb{R}^3)$ *deux champ de vecteurs à divergence nulle. Soit* $\mathbb{F}$ *un tenseur qui appartient à* $B_2L^2(0,T)$. *Alors, il existe un temps* $T_0\in(0,T)$ *tel que le système (MHD) admette une solution* $(\boldsymbol{u},\boldsymbol{b},p,q)$, *avec* $q=0$, *qui satisfait les propriétés suivantes :*

- $\boldsymbol{u},\boldsymbol{b}$ *appartient à* $L^\infty((0,T_0),B_2)$ *et* $\nabla\boldsymbol{u},\nabla\boldsymbol{b}$ *appartient à* $B_2L^2(0,T_0)$
- *la pression* p *est liée à* $\boldsymbol{u},\boldsymbol{b}$ *et* $\mathbb{F}$ *par la formule :*

$$p=\sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_j(u_iu_j-b_ib_j-F_{i,j})$$

- *la transformation* $t\in[0,T)\mapsto(\boldsymbol{u}(t,\cdot),\boldsymbol{b}(t,\cdot))$ *est* $*$*-faiblement continue de* $[0,T)$ *dans* $B_2(\mathbb{R}^3)$, *et pour tout compact* $K\subset\mathbb{R}^3$,

$$\lim_{t\to 0}\|(\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_0,\boldsymbol{b}(t,\cdot)-\boldsymbol{b}_0)\|_{L^2(K)}=0$$

- *la solution* $(\boldsymbol{u},\boldsymbol{b},p)$ *est adaptée : il existe une mesure non-négative localement finie* μ *sur* $(0,T)\times\mathbb{R}^3$ *telle que*

$$\begin{aligned}\partial_t\Big(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}\Big)=&\Delta\Big(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}\Big)-|\nabla\boldsymbol{u}|^2-|\nabla\boldsymbol{b}|^2-\nabla\cdot\left(\Big(\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2}+p\Big)\boldsymbol{u}\right)\\&+\nabla\cdot[(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{b}]+\boldsymbol{u}\cdot(\nabla\cdot\mathbb{F})-\mu.\end{aligned}$$

On obtient aussi que pour $0\le t\le T_0$,

$$\begin{aligned}&\max\{\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{B_2},\|\nabla(\boldsymbol{u},\boldsymbol{b})\|^2_{B_2L^2(0,T_0)}\}\\&\le C\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|^2_{B_2}+C\|\mathbb{F}\|^2_{B_2L^2(0,t)}+C\int_0^t\|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{B_2}+\|(\boldsymbol{u},\boldsymbol{b})(s)\|^6_{B_2}ds.\end{aligned}$$

En plus, si les données vérifient :

$$\lim_{R\to+\infty}R^{-2}\int_{|x|\le R}|\boldsymbol{u}_0(x)|^2+|\boldsymbol{b}_0(x)|^2\,dx=0$$

$$\lim_{R\to+\infty}R^{-2}\int_0^{+\infty}\int_{|x|\le R}|\mathbb{F}(t,x)|^2\,dx\,ds=0,$$

alors on obtient une solution faible globale $(\boldsymbol{u},\boldsymbol{b},p)$.

Chapter 2

The pressure term

The goal of this chapter is to study the auxiliary unknown ∇p in the Cauchy problem for the Navier–Stokes equations on $\mathbb{R}^d$

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0, \end{cases}$$

more specifically, we seek to propose a formula for the pressure. This term is usually interpreted as a Lagrange multiplier for the constraint of incompressibility.

Remark 2.1. *We use the following notation : $\boldsymbol{u}$ denotes a vector field $(u_1, u_2, ..., u_d)$, $\mathbb{F} = (F_{i,j})_{1\leq i,j\leq d}$ is a tensor. $\nabla \cdot \mathbb{F}$ denotes the vector $(\sum_i \partial_i F_{i,1}, \sum_i \partial_i F_{i,2}, ..., \sum_i \partial_i F_{i,d})$. Thus, for a vector field $\boldsymbol{b}$ such that $\nabla \cdot \boldsymbol{b} = 0$, we have $(\boldsymbol{b} \cdot \nabla)\boldsymbol{u} = \nabla \cdot (\boldsymbol{b} \otimes \boldsymbol{u})$.*

In order to not assume differentiability of $\mathbf{u}$ in our calculations, it is convenient to rewrite the equations as

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \mathbf{u} = 0, \qquad\qquad \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

If we take the Laplacian in the equations (NS), in view of the identity

$$-\Delta \mathbf{w} = \nabla \wedge (\nabla \wedge \mathbf{w}) - \nabla(\nabla \cdot \mathbf{w}),$$

we find

$$0 = -\Delta \nabla p - \nabla(\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F})) = -\Delta \nabla p - \nabla(\sum_{1\leq i,j\leq d} \partial_i \partial_j (u_i u_j - F_{i,j}))$$

and

$$\partial_t \Delta u = \Delta^2 u + \nabla \wedge (\nabla \wedge (\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}))).$$

Then, we can see that the rotational free unknown ∇p obeys a Poisson equation.

We denote G_d the following fundamental solution of the operator $-\Delta$ (which means $-\Delta G_d = \delta$):

$$G_2 = \frac{1}{2\pi} \ln(\frac{1}{|x|}), \quad G_3 = \frac{1}{4\pi|x|},$$

we obtain *formally*

$$\nabla p = G_d * \nabla(\sum_{1\leq i,j\leq d} \partial_i \partial_j (u_i u_j - F_{i,j})) + H \tag{2.1}$$

where H is harmonic in the space variable, in other words $\Delta H = 0$. In the literature, we usually find the hypothesis ∇p vanishes at infinity, which is read as $H = 0$. Another way to write this assumption would be

$$\partial_t \mathbf{u} = G_d * \nabla \wedge (\nabla \wedge \partial_t \mathbf{u}).$$

The operator

$$\mathbb{P} = G_d * \nabla \wedge (\nabla \wedge .)$$

is named as the Leray projection operator and the decomposition (*if it is well-defined*)

$$\mathbf{w} = \mathbb{P}\mathbf{w} - G_d * \nabla(\nabla \cdot \mathbf{w})$$

the Hodge decomposition of the vector field $\mathbf{w}$.

Thereafter, a relevant issue when dealing with the Navier–Stokes equations is to analyse whether in formula (2.1) the first half of the right-hand term is well-defined, and if so what can we say about the second half.

In order to give meaning to the formal convolution $G_d * \nabla\partial_i\partial_j(u_i u_j)$ or to the term $\left(\nabla\partial_i\partial_j G_d\right) * (u_i u_j)$, we should demand u_i to be locally $L^2_t L^2_x$ (to make $u_i u_j$ a distribution) and to have small increase at infinity, since $\nabla\partial_i\partial_j G_d$ has small decay at infinity (far from the origin, it belongs to $L^1 \cap L^\infty$ and is controlled as $O(|x|^{-(d+1)})$). We thus will analyse solutions $\mathbf{u}$ which belong to $L^2((0,T), L^2(\mathbb{R}^d, w_{d+1}\, dx))$ with

$$w_\gamma(x) = (1 + |x|)^{-\gamma}.$$

As we will see $F_{i,j}$ plays a role similar to $u_i u_j$, thus we will take $\mathbb{F}$ belonging to $L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\, dx))$.

The main results in (Fernández-Dalgo and Lemarié-Rieusset, 2020a) are described in this chapter. We first present a Lemma just to clarify the meaning of ∇p in the Navier-Stokes equations, and a definition to be rigorous :

Lemma 2.1. *Let the dimension $d \in \{2,3\}$ and consider a real number $\gamma > 0$. Let $0 < T < +\infty$. Let $\boldsymbol{u}(t,x) = (u_i(t,x))_{1\le i\le d}$ a divergence free vector field which belongs to $L^1((0,T), L^1(\mathbb{R}^d, w_\gamma\, dx))$, and let $\mathbb{H} = (H_{i,j}(t,x))$ be a tensor such that $\mathbb{H}(t,x) \in L^1((0,T), L^1(\mathbb{R}^d, w_\gamma\, dx))$.*

We define the distribution $\mathbf{S}$ as

$$\mathbf{S} = \Delta \boldsymbol{u} - \nabla \cdot \mathbb{H} - \partial_t \boldsymbol{u}.$$

Then, the following statements are equivalents:
(A) S is curl-free : $\nabla \wedge \mathbf{S} = 0$.
(B) There exists a distribution $p \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ such that $\mathbf{S} = \nabla p$.

Remark: to cover the case of the Navier–Stokes equations, we consider $\mathbb{H} = \boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{F}$, with the hypothesis $\boldsymbol{u} \in L^2((0,T), L^2(\mathbb{R}^d, w_\gamma\, dx))$ and $\mathbb{F} \in L^1((0,T), L^1(\mathbb{R}^d, w_\gamma\, dx))$.

In order to write accurately we introduce a definition, which can be found in (Lemarié-Rieusset, 2002).

Definition 2.1. *Consider an increasing sequence of compact intervals $[a_n, b_n]$, with $a_n < b_n$, such that $(0,T) = \cup_{n\in\mathbb{N}}[a_n, b_n]$. Let $\mathcal{T}_n$ the Frechet space of the functions $f \in \mathcal{C}^\infty((0,T) \times \mathbb{R}^d)$ such that $\operatorname{supp} f \subset [a_n, b_n] \times \mathbb{R}^d$ and the the semi-norms*

$$\sup_{a_n<t<b_n} \sup_{x\in\mathbb{R}^d} |x^\alpha \frac{\partial^\beta}{\partial x^\beta} \frac{\partial^p}{\partial t^p} f(t,x)|$$

are finite, where $\alpha, \beta \in \mathbb{N}^d$ and $p \in \mathbb{N}$.

Let us denote $\mathcal{T}((0,T) \times \mathbb{R}^d) = \cup_{n\in\mathbb{N}}\mathcal{T}_n$ the space of test functions on $(0,T) \times \mathbb{R}^d$ which are compactly supported in time and have fast decay in space. We consider the topological structure of $\mathcal{T}$ as inductive limit of the Frechet spaces $\mathcal{T}_n$. We denote T' the dual space of $\mathcal{T}$.

The elements of $\mathcal{T}'$ are then the distributions ω on $(0,T) \times \mathbb{R}^d$, such that for all $[a,b] \subset (0,T)$ there exists $C \geq 0$ and $N \in \mathbb{N}$ such that for all $\phi \in \mathcal{T}$ with $\operatorname{supp}(\varphi) \subset [a,b] \times \mathbb{R}^d$, we have

$$\langle \omega | \varphi \rangle \leq C \sum_{|\alpha|,|\beta|,p\leq N,} \|x^\alpha \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^p}{\partial t^p} f(t,x)\|_\infty.$$

For $f \in \mathcal{T}$, we define the Fourier transform with respect to the spatial variable as

$$\mathscr{F} f(t,\xi) = \int_{\mathbb{N}^d} f(t,x) e^{-ix\cdot\xi} dx.$$

We can define the Fourier transform with respect to the spatial variable for a distribution $w \in \mathcal{T}'$ as usual by duality.

Our principal result with respect to the pressure term in the Navier–Stokes equations is the following one :

Theorem 1. *Consider the dimension $d \in \{2,3\}$. Let $0 < T < +\infty$. Consider a tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leq i,j\leq d}$ belonging to $L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\, dx))$. Let $\boldsymbol{u}$ be a solution of the problem*

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) - \mathbf{S} + \nabla\cdot\mathbb{F} \\ \nabla\cdot\boldsymbol{u} = 0, \quad \nabla\wedge\mathbf{S} = 0, \quad \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \end{cases} \tag{2.2}$$

which satisfies : $\boldsymbol{u}$ belongs to $L^2((0,T), L^2_{w_{d+1}}(\mathbb{R}^d))$, $\lim_{t\to 0} \boldsymbol{u}(t,.) = \boldsymbol{u}_0 \in L^2_{w_{d+1}}$ in $\mathcal{D}'$ and $\mathbf{S}$ belongs to $\mathcal{D}'((0,T)\times\mathbb{R}^d)$.

We consider $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x) = 1$ on a neighborhood of 0 and we denote

$$A_{i,j,\varphi} = (1-\varphi)\partial_i\partial_j G_d.$$

Then, there exist $g(t) \in L^1((0,T))$ such that

$$\mathbf{S} = \nabla p_\varphi + \partial_t g$$

with

$$\begin{aligned} p_\varphi =& \sum_{i,j} (\varphi\partial_i\partial_j G_d) * (u_i u_j - F_{i,j}) \\ &+ \sum_{i,j} \int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y))\, dy. \end{aligned}$$

Moreover,

- *∇p_φ does not depend on the choice of φ : If we change φ by ψ, we find*

$$p_\varphi(t,x) - p_\psi(t,x) = \sum_{i,j} \int (A_{i,j,\psi}(-y) - A_{i,j,\varphi}(-y))(u_i(t,y)u_j(t,y) - F_{i,j}(t,y))\, dy.$$

- ∇p_φ *is the unique solution of the Poisson problem*

$$\Delta \boldsymbol{w} = -\nabla(\nabla \cdot (\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{F}))$$

which satisfy

$$\lim_{\tau \to +\infty} e^{\tau\Delta} \boldsymbol{w} = 0 \text{ in } \mathcal{T}'.$$

- *If we assume that* $\mathbb{F}$ *belongs to* $L^1((0,T), L^1_{w_d}(\mathbb{R}^d))$ *and* $\boldsymbol{u} \in L^2((0,T), L^2_{w_d}(\mathbb{R}^d))$*, then we find that* g *is constant and we can take* $g = 0$*, and* $\nabla p_\varphi = \nabla p_0$ *where*

$$p_0 = \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j - F_{i,j}) + \sum_{i,j} ((1-\varphi)\partial_i \partial_j G_d) * (u_i u_j - F_{i,j}),$$

p_0 *does not depend on* φ *and we could define* $p_0 = \sum_{i,j} (\partial_i \partial_j G_d) * (u_i u_j - F_{i,j})$.

We remark that the same or similar splits for the pressure term have already been studied by many authors, under different hypothesis, for exemple we refer to (Chemin, 1995) and (Bahouri, Chemin, and Danchin, 2011). In the literature, we can also find other formulas for the pressure which are useful to deduce more precise controls, for exemple in the two-dimensional case, Gallay gives in (Gallay, 2017) a formula where vorticity intervenes.

If $\mathbb{F} = 0$, the case $g \neq 0$ can be seen as a referential change of the case $g = 0$, this fact is known as the extended Galilean invariance of the Navier–Stokes equations :

Theorem 2. *Consider the dimension* $d \in \{2,3\}$*. Let* $0 < T < +\infty$*. Let* $\boldsymbol{u}$ *be a solution of*

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) - \mathbf{S} \\ \nabla \cdot \boldsymbol{u} = 0, \quad \nabla \wedge \mathbf{S} = 0, \quad \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \end{cases} \tag{2.3}$$

which satisfies : $\boldsymbol{u}$ *belongs to* $L^2((0,T), L^2_{w_{d+1}}(\mathbb{R}^d))$, $\lim_{t \to 0} \boldsymbol{u}(t,.) = \boldsymbol{u}_0 \in L^2_{w_d}$ *in* $\mathcal{D}'$*, and* $\mathbf{S}$ *belongs to* $\mathcal{D}'((0,T) \times \mathbb{R}^{d+1})$.

We take $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *such that* $\varphi(x) = 1$ *on a neighborhood of* 0 *and we denote*

$$A_{i,j,\varphi} = (1-\varphi)\partial_i \partial_j G_d.$$

We decompose $\mathbf{S}$ *as*

$$\mathbf{S} = \nabla p_\varphi + \partial_t g$$

with

$$\begin{aligned} p_\varphi = & \sum_{i,j} (\varphi \partial_i \partial_j G_d) * (u_i u_j) \\ & + \sum_{i,j} \int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(u_i(t,y) u_j(t,y))\, dy \end{aligned}$$

and

$$g(t) \in L^1((0,T)).$$

We define

$$E(t) = \int_0^t g(\lambda) d\lambda$$

and

$$\boldsymbol{w}(t,x) = \boldsymbol{u}(t, x - E(t)) + g(t).$$

Then, $\boldsymbol{w}$ is a solution of the following Navier–Stokes equations

$$\begin{cases} \partial_t \boldsymbol{w} = \Delta \boldsymbol{w} - \nabla\cdot(\boldsymbol{w}\otimes\boldsymbol{w}) - \nabla q_\varphi \\ \nabla\cdot\boldsymbol{w} = 0, \qquad \boldsymbol{w}(0,x) = \boldsymbol{u}_0(x) \\ q_\varphi = \sum_{i,j}(\varphi\partial_i\partial_j G_d) * (w_i w_j) + \sum_{i,j}\int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(w_i(t,y)w_j(t,y))\,dy \end{cases} \tag{2.4}$$

with initial data $\boldsymbol{u}_0$.

2.1 Curl-free vector fields

Below we prove the Lemma 2.1 which clarify the meaning of the curl-free vector field occurring in the Navier Stokes equations, more precisely we prove that the curl-free vector field occurring in the Navier Stokes is necessarily a gradient :

Proof. Let take a partition of unity on $(0,T)$,

$$\sum_{j\in\mathbb{Z}} \omega_j = 1$$

where ω_j is supported on $(2^{j-2}T, 2^jT)$ for $j<0$, on $(T/4, 3T/4)$ for $j=0$ and on $(T-2^{-j}T, T-2^{-(j+2)}T)$ in the case $j>1$. We let

$$V_j = -\omega_j\mathbf{u} + \int_0^t \omega_j\Delta\mathbf{u} - \omega_j\nabla\cdot\mathbb{H} + (\partial_t\omega_j)\mathbf{u}\,ds.$$

Therefore V_j is a sum of the form $A + \int_0^t \Delta B + \nabla\cdot C + D\,ds$ where A, B, C and D belong to $L^1((0,T), L^1(\mathbb{R}^d, w_\gamma\,dx))$ (the fact of belonging to L^1 in time is stable by integration); so that, by the Fubini theorem, we can see this distribution as a time-dependent tempered distribution.

We have $\partial_t V_j = \omega_j\mathbf{S}$, V_j is equal to 0 for t on a neighborhood of the origin, and $\nabla\wedge V_j = 0$. Furthermore, $\mathbf{S} = \sum_{j\in\mathbb{Z}}\partial_t V_j$.

We take $\Phi\in\mathcal{S}(\mathbb{R}^d)$ such that the Fourier transform of Φ has compact support and is equal to 1 on a neighborhood of 0. Then, $\Phi * V_j$ is well-defined and $\nabla\wedge(\Phi * V_j) = 0$. Let us write

$$X_j = \Phi * V_j \text{ and } Y_j = V_j - X_j.$$

Thus,

$$Y_j = \nabla\left(\frac{1}{\Delta}\nabla\cdot Y_j\right)$$

and by Poincaré's Lemma,

$$X_j = \nabla\left(\int_0^1 x\cdot X_j(t,\lambda x)d\lambda\right).$$

Therefore, we conclude that $\mathbf{S} = \nabla p$ with

$$p = \partial_t\sum_{j\in\mathbb{Z}}\left(\int_0^1 x\cdot X_j(t,\lambda x)d\lambda + \frac{1}{\Delta}\nabla\cdot Y_j\right).$$

◇

2.2 The Poisson problem $-\Delta U = \partial_k\partial_i\partial_j h$

In this section we consider a simple but crucial Poisson problem :

Proposition 2.1. *If* $h \in L^1(\mathbb{R}^d, (1+|x|)^{-(d+1)}dx)$, *then*

$$U = U_1 + U_2 = (\partial_k(\varphi\partial_i\partial_j G_d)) * h + \partial_k((1-\varphi)\partial_i\partial_j G_d) * h.$$

is a distribution, U_2 *belongs to* $L^1(\mathbb{R}^d, (1+|x|)^{-(d+1)}dx)$ *and* U *is a solution of the problem*

$$-\Delta U = \partial_k\partial_i\partial_j h.$$

In fact, U *is the unique solution in* $\mathcal{S}'$ *such that* $\lim_{\tau\to 0} e^{\tau\Delta}U = 0$ *in* $\mathcal{S}'$.

Proof. We can verify that $\partial_j G_d$ satisfies

$$\partial_j G_d = \int_0^{+\infty} \partial_j W_t dt$$

where $W_t(x)$ is the heat kernel $W_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. In particular, on $\mathbb{R}^d \setminus \{0\}$, we have

$$\partial_j G_d = c_d \frac{x_j}{|x|^d} \text{ with } c_d = \frac{-1}{2(4\pi)^{d/2}} \int_0^{+\infty} e^{-\frac{1}{4u}} \frac{du}{u^{\frac{d+2}{2}}}.$$

Remark that $U_1 = (\partial_k(\varphi\partial_i\partial_j G_d)) * h$, is well-defined since $\partial_k(\varphi\partial_i\partial_j G_d)$ is a compactly supported distribution. In order to control U_2, let us write

$$\begin{aligned}
&\int\int \frac{1}{(1+|x|)^{d+1}} |\partial_k((1-\varphi)\partial_i\partial_j G_d(x-y))||h(y)|dydx \\
&\leq \int\int \frac{1}{(1+|x|)^{d+1}} \frac{C}{(1+|x-y|)^{d+1}} |h(y)|dydx \\
&\leq C' \int \frac{1}{(1+|y|)^{d+1}} |h(y)|dy
\end{aligned}$$

where the last inequality is justified by the following fact

$$\begin{aligned}
&\int \frac{1}{(1+|x|)^{d+1}} \frac{1}{(1+|x-y|)^{d+1}} dx \\
&\leq \int_{|x|>\frac{|y|}{2}} \frac{1}{(1+|x|)^{d+1}} \frac{1}{(1+|x-y|)^{d+1}} dx + \int_{|x-y|>\frac{|y|}{2}} \frac{1}{(1+|x|)^{d+1}} \frac{1}{(1+|x-y|)^{d+1}} dx \\
&\leq \frac{2^{d+1}}{(1+|y|)^{d+1}} \int \frac{1}{(1+|x-y|)^{d+1}} dx + \frac{2^{d+1}}{(1+|y|)^{d+1}} \int \frac{1}{(1+|x|)^{d+1}} dx \\
&\leq C \frac{1}{(1+|y|)^{d+1}}.
\end{aligned}$$

As U is well-defined, we compute $-\Delta U$. We find $-\Delta U_1$ is equal to

$$(-\Delta\partial_k(\varphi\partial_i\partial_j G_d)) * h = \partial_k(\varphi\partial_i\partial_j h) - \partial_k((\Delta\varphi)\partial_i\partial_j G_d) * h - 2 \sum_{1\leq l\leq d} \partial_k((\partial_l\varphi)\partial_l\partial_i\partial_j G_d) * h.$$

To calculate $-\Delta U_2$, we verify that it is possible to differentiate under the integration sign to obtain

$$-\Delta U_2 = \partial_k((1-\varphi)\partial_i\partial_j h) + \partial_k((\Delta\varphi)\partial_i\partial_j G_d) * h + 2\sum_{1\le l\le d}\partial_k((\partial_l\varphi)\partial_l\partial_i\partial_j G_d) * h.$$

Therefore, U is a solution of the Poisson problem.

A computation for $e^{\tau\Delta}U$ shows that

$$e^{\tau\Delta}U = (e^{\tau\Delta}\partial_k\partial_i\partial_j G_d) * h,$$

hence

$$|e^{\tau\Delta}U(x)| \le C\int \frac{1}{(\sqrt{\tau}+|x-y|)^{d+1}}|h(y)|\,dy.$$

By dominated convergence we find $\lim_{\tau\to+\infty} e^{\tau\Delta}U = 0$ in $L^1(\mathbb{R}^d,(1+|x|)^{-(d+1)}dx)$. We suppose that V is another solution of the Poisson problem with $V \in \mathcal{S}'$ and $\lim_{\tau\to+\infty} e^{\tau\Delta}V = 0$ in $\mathcal{S}'$, then $\Delta(U-V) = 0$ and $U - V \in \mathcal{S}'$, so that $U - V$ is a polynomial; since the assumption $\lim_{\tau\to 0} e^{\tau\Delta}(U-V) = 0$, we deduce that this polynomial is 0. ◇

If we consider better integrability of h, we expect to enhance integrability of U_2. For instance, we find :

Proposition 2.2. *If $h \in L^1(\mathbb{R}^d,(1+|x|)^{-d}dx))$ then*

$$U_2 = \partial_k((1-\varphi)\partial_i\partial_j G_d) * h$$

belongs to $L^1(\mathbb{R}^d,(1+|x|)^{-d})$.

Proof.
We know that

$$\int\int \frac{1}{(1+|x|)^d}|\partial_k((1-\varphi)\partial_i\partial_j G_d(x-y))||h(y)|\,dy\,dx$$
$$\le C\int\int \frac{1}{(1+|x|)^d}\frac{1}{(1+|x-y|)^{d+1}}|h(y)|\,dy\,dx.$$

For $|y| \le 1$, we just observe that

$$\int \frac{1}{(1+|x|)^d}\frac{1}{(1+|x-y|)^{d+1}}dx \le C\int \frac{1}{(1+|x|)^{d+1}}dx \le C'$$

while for $|y| > 1$, as the real number $\int_{|x|<\frac{1}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x-\frac{y}{|y|}|^2} dx$ does not depend on y, we can write

$$\begin{aligned}
&\int \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx \\
&\leq \int_{|x|>\frac{|y|}{2}} \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx + \int_{|x|<\frac{|y|}{2}} \frac{1}{(1+|x|)^d} \frac{1}{(1+|x-y|)^{d+1}} dx \\
&\leq \frac{2^d}{(1+|y|)^d} \int \frac{1}{(1+|x-y|)^{d+1}} dx + \frac{2^{d-1}}{(1+|y|)^{d-1}} \int_{|x|<\frac{|y|}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x-y|^2} dx \\
&\leq C \frac{1}{(1+|y|)^d} + C \frac{1}{(1+|y|)^{d-1}} \frac{1}{|y|} \int_{|x|<\frac{1}{2}} \frac{1}{|x|^{d-1}} \frac{1}{|x-\frac{y}{|y|}|^2} dx \\
&\leq C' \frac{1}{(1+|y|)^d}.
\end{aligned}$$

This fact proves the proposition. ◇

2.3 The Poisson problem $-\Delta V = \partial_i \partial_j h$

Proposition 2.3. *Let $h \in L^1((1+|x|)^{-d-1}dx)$ and let $A_\varphi = (1-\varphi)\partial_i\partial_j G_d$. Then,*

$$V = V_1 + V_2 = (\varphi \partial_i \partial_j G_d) * h + \int (A_\varphi(x-y) - A_\varphi(-y)) h(y) dy$$

is a distribution such that V_2 belongs to $L^1((1+|x|)^{-\gamma})$, for $\gamma > d+1$, and V is a solution of the problem

$$-\Delta V = \partial_i \partial_j h.$$

Proof. We know that V_1 is well defined because $\varphi \partial_i \partial_j G_d$ is a compactly supported distribution. Below we verify that V_2 is well defined.

For $|y| \leq 1$, we have

$$\int \frac{1}{(1+|x|)^\gamma} |A_\varphi(x-y) - A_\varphi(-y)| dx \leq C \|A_\varphi\|_{L^\infty} \int \frac{1}{(1+|x|)^\gamma} dx.$$

For $|y| > 1$, by the mean value inequality we obtain

$$\int_{|x|<\frac{|y|}{2}} \frac{1}{(1+|x|)^\gamma} |A_\varphi(x-y) - A_\varphi(-y)| dx \leq C \frac{1}{|y|^{d+1}} \int_{|x|<\frac{|y|}{2}} \frac{|x|}{(1+|x|)^\gamma} dx$$

and the other part can be controlled in the following way

$$\int_{|x|>\frac{|y|}{2}} \frac{1}{(1+|x|)^\gamma} |A_\varphi(-y)| dx \leq C \frac{1}{|y|^d} \int_{|x|>\frac{|y|}{2}} \frac{1}{|x|^\gamma} \leq C \frac{1}{|y|^\gamma}$$

and for $\varepsilon > 0$ such that $\gamma - \varepsilon \geq d+1$, we have that

$$\begin{aligned}\int_{|x|>\frac{|y|}{2}} \frac{1}{(1+|x|)^\gamma}|A_\varphi(x-y)|dx &\leq C\int_{|x|>\frac{|y|}{2}} \frac{1}{|x|^\gamma}\frac{1}{(1+|x-y|)^d}dx \\ &\leq C\int_{|x|>\frac{|y|}{2}} \frac{1}{|x|^\gamma}\frac{1}{|x-y|^{d-\varepsilon}}dx \\ &\leq C\frac{1}{|y|^{\gamma-\epsilon}}\int_{|x|>\frac{1}{2}} \frac{1}{|x|^\gamma}\frac{1}{|x-\frac{y}{|y|}|^{d-\varepsilon}}dx \\ &\leq C'\frac{1}{|y|^{d+1}}.\end{aligned}$$

$\diamond$

2.4 Characterisation of the pressure term.

We slightly generalize Theorem 1 in order to make it applicable to other classical incompressible equations as the magnetohydrodynamics system or the Euler equations. Remark that for $p \in (1, +\infty)$, $L^p(\mathbb{R}^d, w_{d+1}\,dx) \subset L^1(\mathbb{R}^d, w_{d+1}\,dx)$ but $L^p(\mathbb{R}^d, w_d\,dx)$ is not contained in $L^1(\mathbb{R}^d, w_d\,dx)$. The theorem of characterisation is stated as follows:

Theorem 3. *Consider the dimension $d \in \{2,3\}$. Let $0 < T < +\infty$. Consider a tensor $\mathbb{H}(t,x) = \big(H_{i,j}(t,x)\big)_{1\leq i,j\leq d}$ belonging to $L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\,dx))$. Let $\boldsymbol{u}$ be a solution of*

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla\cdot\mathbb{H} - \mathbf{S} \\ \nabla\cdot\boldsymbol{u} = 0, \quad \nabla\wedge\mathbf{S} = 0, \quad \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \end{cases} \tag{2.5}$$

which satisfies : $\boldsymbol{u}$ belongs to $L^1((0,T), L^1_{w_{d+1}}(\mathbb{R}^d))$, $\mathbf{S}$ belongs to $\mathcal{D}'((0,T)\times\mathbb{R}^d)$, and $\boldsymbol{u}(t)$ converges to $\boldsymbol{u}_0 \in L^1_{w_{d+1}}$ in $\mathcal{D}'$.

We consider $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x) = 1$ on a neighborhood of 0 and we define

$$A_{i,j,\varphi} = (1-\varphi)\partial_i\partial_j G_d.$$

Then, there exist $g(t) \in L^1((0,T))$ such that

$$\mathbf{S} = \nabla p_\varphi + \partial_t g$$

with

$$p_\varphi = \sum_{i,j}(\varphi\partial_i\partial_j G_d) * (H_{i,j}) + \sum_{i,j}\int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(H_{i,j}(t,y))\,dy.$$

Moreover,

- *∇p_φ does not depend on the choice of φ : If we change φ by ψ, we find*

$$p_\varphi(t,x) - p_\psi(t,x) = \sum_{i,j}\int (A_{i,j,\psi}(-y) - A_{i,j,\varphi}(-y))(H_{i,j}(t,y))\,dy.$$

- *∇p_φ is the unique solution of the Poisson problem*

$$\Delta \boldsymbol{w} = -\nabla(\nabla\cdot\mathbb{H})$$

with

$$\lim_{\tau\to+\infty} e^{\tau\Delta}\boldsymbol{w} = 0 \textit{ in } \mathcal{T}'.$$

- *If we assume that* $\mathbb{H}$ *belongs to* $L^1((0,T), L^1_{w_d}(\mathbb{R}^d))$ *and* $\boldsymbol{u} \in L^1((0,T), L^p_{w_d}(\mathbb{R}^d))$, *for some* $p \in [1,+\infty)$, *we have* g *is constant and we can take* $g = 0$, *and* $\nabla p_\varphi = \nabla p_0$ *where*

$$p_0 = \sum_{i,j}(\varphi\partial_i\partial_j G_d) * (H_{i,j}) + \sum_{i,j}((1-\varphi)\partial_i\partial_j G_d) * (H_{i,j}),$$

p_0 *does not depend on* φ *and we could define* $p_0 = \sum_{i,j}(\partial_i\partial_j G_d) * (H_{i,j})$.

Proof. Taking the divergence in the differential equation

$$\partial_t \mathbf{u} = \Delta\mathbf{u} - \nabla\cdot\mathbb{H} - \mathbf{S},$$

we find

$$-\sum_{i,j}\partial_i\partial_j H_{i,j} - \nabla\cdot\mathbf{S} = 0$$

and

$$-\Delta\mathbf{S} = \nabla(\sum_{i,j}\partial_i\partial_j(H_{i,j})).$$

We denote $A_{i,j,\varphi} = (1-\varphi)\partial_i\partial_j G_d$. By Proposition 2.3, we can define

$$p_\varphi = \sum_{i,j}(\varphi\partial_i\partial_j G_d) * h_{i,j} + \sum_{i,j}\int (A_{i,j,\varphi}(x-y) - A_{i,j\varphi}(-y))h_{i,j}(y)dy$$

and

$$U = U_1 + U_2 = \nabla\sum_{i,j}(\varphi\partial_i\partial_j G_d) * h_{i,j} + \nabla\sum_{i,j}((1-\varphi)\partial_i\partial_j G_d) * h_{i,j} = \nabla p_\varphi.$$

We can define $\tilde{U} = \mathbf{S} - U$. First, remark that $\Delta U = \Delta\mathbf{S}$ so that $\Delta\tilde{U} = 0$, thus $\tilde{U}$ is harmonic in space variable.

Now, we consider a test function $\alpha \in \mathcal{D}(\mathbb{R})$ such that $\alpha(t) = 0$ for all $|t| \geq \varepsilon$, and a test function $\beta \in \mathcal{D}(\mathbb{R}^3)$. For $t \in (\varepsilon, T-\varepsilon)$ we have

$$\begin{aligned}\tilde{U}(t) *_{t,x} (\alpha\otimes\beta) =& (\mathbf{u} * (-\partial_t\alpha\otimes\beta + \alpha\otimes\Delta\beta) + (-\mathbb{H})\cdot *(\alpha\otimes\nabla\beta))(t,\cdot) \\ &- \sum_{i,j}((h_{ij}) * (\nabla(\varphi\partial_i\partial_j G_d) * (\alpha\otimes\beta)))(t,\cdot) - (U_2 * (\alpha\otimes\beta))(t,\cdot).\end{aligned}$$

By Proposition 2.1, we conclude that $\tilde{U} * (\alpha\otimes\beta)(t,.)$ belongs to $L^1(\mathbb{R}^d, (1+|x|)^{-d-1})$, and then it is a tempered distribution. Since it is harmonic it must to be a polynomial. Belonging to the set $L^1(\mathbb{R}^d, (1+|x|)^{-d-1})$ this polynomial is constant.

If $\mathbb{H}$ belongs to $L^1((0,T), L^1_{w_d}(\mathbb{R}^d))$ and $\mathbf{u}$ belongs to $L^1((0,T), L^p_{w_d}(\mathbb{R}^d))$, we have that this polynomial is identically equal to a constant $k = (k_1, ..., k_d) \in \mathbb{R}^d$ and belongs to $L^1_{w_d}(\mathbb{R}^d) + L^p_{w_d}(\mathbb{R}^d)$, and now we prove that k is zero.

We suppose that k is non zero. We have $k = \mathbf{x} + \mathbf{y}$, with $\mathbf{x} \in L^1_{w_d}(\mathbb{R}^d)$ and $\mathbf{y} \in L^p_{w_d}(\mathbb{R}^d)$.

As $\mathbf{x} \in L^1_{w_d}(\mathbb{R}^d)$ and k does not belong to $L^1_{w_d}(\mathbb{R}^d)$ we have $\mathbf{y} = k - \mathbf{x}$ does not belong to $L^1_{w_d}(\mathbb{R}^d)$, thus we have two cases : $\int_{|k-\mathbf{x}|\geq\frac{|k|}{2}} \frac{|k-\mathbf{x}|}{(1+|z|)^d}dz = +\infty$ or $\int_{|k-\mathbf{x}|<\frac{|k|}{2}} \frac{|k-\mathbf{x}|}{(1+|z|)^d}dz = +\infty$.

If $\int_{|k-\mathbf{x}|\geq\frac{|k|}{2}} \frac{|k-\mathbf{x}|}{(1+|z|)^d} dz = +\infty$, as for $|k-\mathbf{x}| \geq \frac{|k|}{2}$ we have $|k-\mathbf{x}|^p \geq c_k|k-\mathbf{x}|$, for some constant $c_k > 0$, we deduce $\int_{|k-\mathbf{x}|\geq\frac{|k|}{2}} \frac{|k-\mathbf{x}|^p}{(1+|z|)^d} dz = +\infty$ and then $\mathbf{y} = k - \mathbf{x}$ does not belong to $L^p_{w_d}(\mathbb{R}^d)$ which is a contradiction.

If $\int_{|k-\mathbf{x}|<\frac{|k|}{2}} \frac{|k-\mathbf{x}|}{(1+|z|)^d} dz = +\infty$, as the condition $|k-\mathbf{x}| < \frac{|k|}{2}$ implies $|\mathbf{x}| > |k| - |\mathbf{x}-k| > |\mathbf{x}-k|$, we deduce $\int_{|k-\mathbf{x}|\geq\frac{|k|}{2}} \frac{|\mathbf{x}|}{(1+|z|)^d} dz = +\infty$ and then $\mathbf{x}$ does not belong to $L^1_{w_d}(\mathbb{R}^d)$ which is a contradiction.

Thus, we have $k = 0$.

Now, using an approximation of the identity $\Phi_\varepsilon = \frac{1}{\varepsilon^4}\alpha(\frac{t}{\varepsilon})\beta(\frac{x}{\varepsilon})$ and taking the limit when ϵ goes to 0, we obtain a similar result for $\tilde{U}$. We thus have $\mathbf{S} = \nabla p_\varphi + f(t)$, with $f(t) = 0$ if $\mathbb{H}$ belongs to $L^1((0,T), L^1_{w_d}(\mathbb{R}^d))$ and $\mathbf{u} \in L^1((0,T), L^p_{w_d}(\mathbb{R}^d))$.

As f does not depend on x, for a function $\beta \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \beta\, dx = 1$ we have $f = f *_x \beta$; then we can write

$$f(t) = \partial_t(\mathbf{u}_0 * \beta - \mathbf{u} * \beta + \int_0^t \mathbf{u} * \Delta\beta - (\mathbf{u}\otimes\mathbf{u} - \mathbb{F}) \cdot *\nabla\beta - p_\varphi * \nabla\beta\, ds) = \partial_t g.$$

Since $\partial_t\partial_j g = \partial_j f = 0$ y $\partial_j g(0,.) = 0$, we find that g depends only on t; moreover, the formula above shows that $g \in L^1((0,T))$. ◇

The proof of Theorem 2 is classic, and this property is called the extended Galilean invariance of the Navier–Stokes equations :

Proof. We have assumed that

$$\partial_t\mathbf{u} = \Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p_\varphi - \frac{d}{dt}g(t),$$

with $g \in L^1((0,T))$. We define

$$E(t) = \int_0^t g(\lambda)d\lambda \text{ and } \mathbf{w} = \mathbf{u}(t, x - E(t)) + g(t).$$

Then, we compute $\partial_t\mathbf{w}$ to find

$$\begin{aligned}\partial_t\mathbf{w} =& \partial_t\mathbf{u}(t,x-E(t)) - g(t)\cdot\nabla\mathbf{u}(t,x-E(t)) + \frac{d}{dt}g(t)\\ =& \Delta\mathbf{u}(t,x-E(t)) - [(\mathbf{u}\cdot\nabla)\mathbf{u}](t,x-E(t)) - \nabla p_\varphi(t,x-E(t)) - \frac{d}{dt}g(t)\\ &- g(t)\cdot\nabla\mathbf{u}(t,x-E(t)) + \frac{d}{dt}g(t)\\ =& \Delta\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{w} - \nabla p_\varphi(t,x-E(t)).\end{aligned}$$

If we define $q_\varphi(t,x) = p_\varphi(t,x-E(t))$, then we obtain

$$q_\varphi = \sum_{i,j}(\varphi\partial_i\partial_j G_d) * (w_i w_j) + \sum_{i,j}\int (A_{i,j,\varphi}(x-y) - A_{i,j,\varphi}(-y))(w_i(t,y)w_j(t,y))\, dy,$$

which proves the theorem. ◇

2.5 Some Applications

Proposition 2.1 and 2.3 have a nice implication. We may define the Leray projection operator on the divergence of tensors that belong to $L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\,dx))$:

Definition 2.2. *Let $\mathbb{H} \in L^1((0,T), L^1(\mathbb{R}^d, w_{d+1}\,dx))$ and $\boldsymbol{w} = \nabla \cdot \mathbb{H}$. The Leray projection $\mathbb{P}(\boldsymbol{w})$ of $\boldsymbol{w}$ on solenoidal vector fields is defined by*

$$\mathbb{P}\boldsymbol{w} = \boldsymbol{w} - \nabla p_\varphi$$

where ∇p_φ is the unique solution of

$$-\Delta \nabla p = -\nabla(\nabla \cdot \boldsymbol{w})$$

such that

$$\lim_{\tau \to +\infty} e^{\tau\Delta} \nabla p = 0.$$

Remark *Note that Proposition 2.3 implies $\nabla \cdot \mathbb{P}(\boldsymbol{w}) = 0$.*

After applying the Leray projector, the Navier–Stokes equations take the following special form :

$$(MNS) \quad \partial_t \mathbf{u} = \Delta \mathbf{u} - \mathbb{P}\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F}), \quad \mathbf{u}(0,.) = \mathbf{u}_0.$$

Then the following integro-differential equation arise

$$\mathbf{u} = e^{t\Delta}\mathbf{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (\mathbf{u} \otimes \mathbf{u} - \mathbb{F})\,ds,$$

where a solution $\mathbf{u}$ can be seen as a fixed point of an application on a suitable space.

The study of the convolution operator $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot$ is the core of the method of mild solutions of Kato and Fujita (Fujita and Kato, 1962). The kernel of the operator $e^{\Delta}\mathbb{P}$ is called the Oseen kernel. Thus, we will call equations (MNS) a mild formulation of the Navier–Stokes equations.

The mild formulation together with the local Leray energy inequality has been as well a key tool for extending Leray's theory of weak solutions in L^2 to the setting of weak solutions with infinite energy. We may propose a general definition of suitable Leray-type weak solutions :

Definition 2.3 (Suitable Leray-type solution)**.**
Consider $\mathbb{F} \in L^2((0,T), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}}))$ and $\boldsymbol{u}_0 \in L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}})$ with $\nabla \cdot \boldsymbol{u}_0 = 0$. We consider the Navier–Stokes problem on $(0,T) \times \mathbb{R}^d$:

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - \mathbb{P}(\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{F})) \\ \nabla \cdot \boldsymbol{u} = 0, \quad \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \end{cases} \tag{2.6}$$

A suitable Leray-type solution $\boldsymbol{u}$ of the Navier–Stokes equations is a vector field $\boldsymbol{u}$ defined on $(0,T) \times \mathbb{R}^d$ satisfying the equations above and such that :

- *$\boldsymbol{u}$ is locally $L^2_t H^1_x$ on $(0,T) \times \mathbb{R}^d$*
- $\sup_{0<t<T} \int |\boldsymbol{u}(t,x)|^2 \frac{1}{(1+|x|)^{d+1}}\,dx < +\infty$
- $\iint_{(0,T)\times\mathbb{R}^d} |\nabla \otimes \boldsymbol{u}(t,x)|^2 \frac{1}{(1+|x|)^{d+1}}\,dx\,dt < +\infty$

- *the application* $t \in [0,T) \mapsto \int \boldsymbol{u}(t,x)\cdot \boldsymbol{w}(x)\,dx$ *is continuous for every smooth compactly supported vector field* $\boldsymbol{w}$
- *for every compact subset* K *of* $\mathbb{R}^d$, $\lim_{t\to 0}\int_K |\boldsymbol{u}(t,x)-\boldsymbol{u}_0(x)|^2\,dx = 0$.
- *defining* p_φ *as (the) solution of* $-\Delta p_\varphi = \sum_{i,j}\partial_i\partial_j(u_iu_j - F_{i,j})$ *given by Proposition 2.3,* $\boldsymbol{u}$ *is suitable in the sense of Caffarelli, Kohn and Nirenberg : there exists a non-negative locally bounded Borel measure* μ *on* $(0,T)\times\mathbb{R}^d$ *such that*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2}{2}) - |\nabla\otimes\boldsymbol{u}|^2 - \nabla\cdot((\frac{|\boldsymbol{u}|^2}{2}+p_\varphi)\boldsymbol{u}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu$$

Remarks :

a) Under those hypotheses, p_φ belongs locally to $L^{3/2}_{t,x}$ and **u** belongs locally to $L^3_{t,x}$ so that $(\frac{|\mathbf{u}|^2}{2}+p_\varphi)\mathbf{u}$ is well-defined as a distribution.

b) Caffarelli, Kohn and Nirenberg in 1982 (L. Caffarelli and Nirenberg, 1982) introduced suitability to get estimates on partial regularity for weak Leray solutions. Suitability is a local assumption. When we consider a solution of the Navier–Stokes equations on a small domain with no specifications on the behaviour of **u** at the boundary, the estimates on the pressure (and the Leray projection operator) are no longer available. However, Wolf described in (Wolf, 2017) a local decomposition of the pressure; he could generalize the notion of suitability to this new description of the pressure. We refer the paper (Chamorro, Lemarié-Rieusset, and Mayoufi, 2018) for more information on the equivalence of various notions of suitability.

c) In the case of uniformly locally square integrable solutions, the relationship between the mild formulation (MNS) and the system (NS) described in Theorem 1 has been studied in (Furioli, Lemarié-Rieusset, and Terraneo, 2000; Lemarié-Rieusset, 2002), other paper we refer is (Dubois, 2002). Their results show the equivalence between (NS) and (MNS) in the case when **u** and $\mathbb{F}$ decay at infinity (more precisely, when **u** belongs to the closure of test functions in $(L^2_tL^2_x)_{\text{uloc}}$ and when $\mathbb{F}$ belongs to the closure of test functions in $(L^1_tL^1_x)_{\text{uloc}}$). Earlier results for mild solutions in $L^p_tL^q_x$ with $\frac{2}{p}+\frac{d}{q}\le 1$ and $d<q<+\infty$, were proved in 1972 by Fabes, Jones and Riviere (Fabes, Jones, and Riviere, 1972), it is a simpler case where the theory of singular integrals may be directly applied.

d) Non-decaying solutions has been discussed in (Kukavica, 2003) for the study of the Cauchy problem with initial value in L^∞ and in (Kukavica and Vicol, 2008) for the study of the Cauchy problem with initial value in BMO^{-1}. The systems (NS) and (MNS) are no longer equivalent and if $\mathbb{F}=0$, general solutions are better described through the extended Galilean invariance of the equations. In this paper, we find again such a description in the case of more general weak solutions, for which the integral formulation does not provide any existence or uniqueness results, in contrast to the case of solutions in L^∞ or BMO^{-1} data. More precisely, a suitable Leray-type solution (u,p) arise to the (MNS) formulation if and only if $\nabla p = \nabla p_\varphi$ while for general weak solutions we may have solutions such that $\nabla p = \nabla p_\varphi + f(t)$.

We list here a few examples to be found in the literature :

1. Solutions in L^2 : in 1934, Leray studied the Navier–Stokes problem (NS) with an initial data $\mathbf{u}_0 \in L^2$ and a forcing tensor $\mathbb{F} \in L^2_t L^2_x$ (Leray, 1934). He found solutions belonging to $\mathbf{u} \in L^\infty L^2 \cap L^2 \dot{H}^1$. Leray's construction by mollification provides suitable solutions which are automatically a solution of the mild formulation of the Navier–Stokes equations (MNS).

2. Solutions in L^2_{uloc} : in 1999, Lemarié-Rieusset studied the Navier–Stokes problem (MNS) with an initial data $\mathbf{u}_0 \in L^2_{\mathrm{uloc}}$ (Lemarié-Rieusset, 1999; Lemarié-Rieusset, 2002). He obtained (local in time) existence of a suitable solution $\mathbf{u}$ on $(0, T_0) \times \mathbb{R}^d$ satisfying

$$\sup_{x_0 \in \mathbb{R}^d} \sup_{0<t<T_0} \int_{B(x_0,1)} |\mathbf{u}(t,x)|^2 \, dx < +\infty$$

and

$$\sup_{x_0 \in \mathbb{R}^d} \int_0^{T_0} \int_{B(x_0,1)} |\nabla \otimes \mathbf{u}(t,x)|^2 \, dx < +\infty.$$

Remark that we have $\mathbf{u} \in L^2((0,T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^{d+1}} \, dx))$ but $\mathbf{u}$ does not belong to $L^2((0,T_0), L^2(\mathbb{R}^d, \frac{1}{(1+|x|)^d} \, dx))$; thus, in this setting, problems (NS) and (MNS) are not equivalent.

Various reformulations of local Leray solutions in L^2_{uloc} have been provided, two examples are those of (Kikuchi and Seregin, 2007) and (Bradshaw and Tsai, 2019a). However, as we have seen the formulas proposed for the pressure are actually equivalent, as they all imply that $\mathbf{u}$ is solution to the (MNS) problem.

Basson proved in 2006 that, for $d = 2$ the solution $\mathbf{u}$ is indeed global (i.e. $T_0 = T$) and is unique (Basson, 2006b) .

3. Solutions in weighted L^2 spaces will be studied in the following chapters. For such solutions, (NS) and (MNS) are equivalents, and the fact that $\nabla p = \nabla p_\varphi$ can be deduced from the weighted integrability of the solution. In particular we will expose the results in (Fernández-Dalgo and Lemarié-Rieusset, 2020b) where the data satisfy $\mathbf{u}_0 \in L^2(\mathbb{R}^3, w_\gamma \, dx)$ and $\mathbb{F} \in L^2((0,+\infty), L^2(\mathbb{R}^3, w_\gamma \, dx))$ with $0 < \gamma \leq 2$. They proved (global in time) existence of a suitable solution $\mathbf{u}$ such that, for all $T_0 < +\infty$,

$$\sup_{0<t<T_0} \int |\mathbf{u}(t,x)|^2 \, w_\gamma(x) \, dx < +\infty$$

and

$$\int_0^{T_0} \int |\nabla \otimes \mathbf{u}(t,x)|^2 \, w_\gamma(x), dx < +\infty.$$

By Theorem 1, we can see that the formula proposed by Bradshaw and Tsai for the pressure can be derived from the weighted integrability of the solution and $\mathbf{u}$ is a solution of the (MNS).

4. Homogeneous Statistical Solutions : in 1977, Vishik and Fursikov considered the (MNS) problem with a random initial value $\mathbf{u}_0(\omega)$ (Vishik and Fursikov, 1977). The statistics of the initial distributions were supposed to be invariant through translation of the arguments of $\mathbf{u}_0$, more precisely : for every Borel

subset B of $L^2_{\text{loc}}(\mathbb{R}^3)$ and every $x_0 \in \mathbb{R}^3$,

$$Pr(\mathbf{u}_0(\cdot - x_0) \in B) = Pr(\mathbf{u}_0 \in B).$$

Another crucial assumption was that $\mathbf{u}_0$ has a bounded mean energy density :

$$e_0 = \mathbb{E}\left(\frac{\int_{|x|\leq 1} |\mathbf{u}_0|^2 \, dx}{\int_{|x|\leq 1} \, dx}\right) < +\infty.$$

Then, we have

$$Pr(\mathbf{u}_0 \in L^2 \text{ and } \mathbf{u} \neq 0) = 0$$

while, for any $\epsilon > 0$,

$$Pr(\int |\mathbf{u}_0|^2 \frac{1}{(1+|x|)^{3+\epsilon}} \, dx < +\infty) = 1.$$

In (Vishik and Fursikov, 1988), they solved the Navier–Stokes equation for almost every initial value $\mathbf{u}_0(\omega)$, and the solution belonged almost surely to the space $L^\infty_t L^2_x(\frac{1}{(1+|x|)^{3+\epsilon}} \, dx)$ with $\nabla \otimes \mathbf{u} \in L^2_t L^2_x(\frac{1}{(1+|x|)^{3+\epsilon}} \, dx)$.

Basson gave a description of the pressure in those equations (which is equivalent to our description through the Leray projection operator). He proved the suitability of these solutions as well (Basson, 2006a).

Chapter 3

Solutions in weighted L^2 spaces

We place ourselves in the context of the weighted L^2 setting, in dimension d with $d \in \{2,3,4\}$, when the weight Φ satisfies some basic hypothesis that allow us to use the Leray's projection operator and energy controls.

First, we recall the definition of Muckenhoupt weights : for $1 < q < +\infty$, a positive weight w belongs to $\mathcal{A}_q(\mathbb{R}^d)$ if and only if

$$\sup_{x\in\mathbb{R}^d,\rho>0} \left(\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} \Phi\, dy\right)^{\frac{1}{q}} \left(\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} \Phi^{-\frac{1}{q-1}}\, dy\right)^{1-\frac{1}{q}} < +\infty. \tag{3.1}$$

Definition 3.1. *Let Φ be a function on $\mathbb{R}^d$ ($2 \le d \le 4$). We say that Φ is an adapted weight function if it is a continuous Lipschitz function such that :*

- $(\boldsymbol{H}1)$ $0 < \Phi \le 1$.
- $(\boldsymbol{H}2)$ *There exists $C_1 > 0$ such that $|\nabla\Phi| \le C_1 \Phi^{\frac{3}{2}}$*
- $(\boldsymbol{H}3)$ *There exists $r \in (1,2]$ such that $\Phi^r \in \mathcal{A}_r$ (where $\mathcal{A}_r$ is the Muckenhoupt class of weights). For $d = 4$, we require $r < 2$ as well.*
- $(\boldsymbol{H}4)$ *There exists $C_2 > 0$ such that $\Phi(x) \le \Phi(\frac{x}{\lambda}) \le C_2\lambda^2\Phi(x)$, for all $\lambda \ge 1$.*

We can verify that $w_\gamma = \frac{1}{(1+|x|)^\gamma}$ belongs to $\mathcal{A}_q(\mathbb{R}^d)$ if and only if

$$-d(q-1) < \gamma < d.$$

Then, $\Phi = w_\gamma$ is an adapted weight if $0 \le \gamma \le 2$ and $\gamma < d$.

To analyze weights that depend only on some variable, it is convenient replace in inequality (3.1) the balls $B(x,\rho)$ by the cubes $Q(x,\rho) =]x_1-\rho, x_1+\rho[\times\cdots\times]x_d-\rho, x_d+\rho[$.

Thus, it is easy to verify that if $\Phi(x) = \Psi(x_1,x_2)$ and $1 < q < +\infty$, we have $\Phi \in \mathcal{A}_q(\mathbb{R}^3)$ if and only if $\Psi \in \mathcal{A}_q(\mathbb{R}^2)$. In fact, for $I_1 =]x_1-\rho, x_1+\rho[$, $I_2 =]x_2-\rho, x_2+\rho[$, $I_3 =]x_3-\rho, x_3+\rho[$ we have

$$\left(\int_{Q(x,\rho)} \Phi\, dy\right)^{\frac{1}{q}} \left(\int_{Q(x,\rho)} \Phi^{-\frac{1}{q-1}}\, dy\right)^{1-\frac{1}{q}} = 2\rho\left(\int_{I_1\times I_2} \Psi\, dy\right)^{\frac{1}{q}} \left(\int_{I_1\times I_2} \Psi^{-\frac{1}{q-1}} dy\right)^{1-\frac{1}{q}}.$$

As corollary, $\Phi(x) = \frac{1}{(1+r)^\gamma}$, where $r = \sqrt{x_1^2+x_2^2}$, is an adapted weight on $\mathbb{R}^3$ if and only if $0 \le \gamma < 2$.

In summary, the following slowly decaying functions are adapted weights :

- $d = 2$, $\Phi(x) = \frac{1}{(1+|x|)^\gamma}$ where $0 \le \gamma < 2$
- $d = 3$ or $d = 4$, $\Phi(x) = \frac{1}{(1+|x|)^\gamma}$ where $0 \le \gamma \le 2$
- $d = 3$, $\Phi(x) = \frac{1}{(1+r)^\gamma}$ where $r = \sqrt{x_1^2 + x_2^2}$ and $0 \le \gamma < 2$.

Our main result in this chapter concerns the existence of weak suitable solutions belonging to a weighted L^2 space. Observe that the weight allows us to consider initial data with a weak decay at infinity.

Theorem 4. *Consider the dimension $d \in \{2,3,4\}$ and an adapted weight Φ. Let $\boldsymbol{u}_0$ be a divergence free vector field which belongs to $L^2(\Phi\, dx, \mathbb{R}^d)$. Let $\mathbb{F} = (F_{i,j}(t,x))_{i,j}$ be a tensor such that $\mathbb{F} \in L^2((0,+\infty), L^2(\Phi\, dx, \mathbb{R}^d))$.*

Then, there exists a global solution $\boldsymbol{u}$ of the problem

$$(NS)\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p + \nabla\cdot\mathbb{F} \\ \\ \nabla\cdot\boldsymbol{u} = 0, \qquad\qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

which satisfies

- *$\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Phi dx))$ and $\nabla\boldsymbol{u}$ belongs to $L^2((0,T), L^2(\Phi dx))$, for all $T > 0$,*
- *$p = \sum_{1\le i,j\le d} \mathcal{R}_i\mathcal{R}_j(u_iu_j - F_{i,j})$,*
- *the map $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2(\Phi\, dx)$, and strongly continuous at $t = 0$,*
- *the local energy inequality for $d \in \{2,3\}$: there exists a locally finite non-negative measure μ such that*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - \nabla\cdot\left(\frac{|\boldsymbol{u}|^2}{2}\boldsymbol{u} - p\boldsymbol{u}\right) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu,$$

and in the case of dimension $d = 2$, we have $\mu = 0$. Moreover, we find

$$\begin{aligned} &\|\boldsymbol{u}(t,.)\|^2_{L^2(\Phi dx)} + 2\int_0^t \|\nabla\otimes\boldsymbol{u}(s,.)\|^2_{L^2(\Phi dx)}\, ds \\ \le&\|\boldsymbol{u}_0\|^2_{L^2(\Phi dx)} - \int_0^t\int \nabla(|\boldsymbol{u}|^2)\cdot\nabla\Phi\, dx\, ds + \int_0^t\int(|\boldsymbol{u}|^2\boldsymbol{u} + 2p\boldsymbol{u})\cdot\nabla\Phi\, dx\, ds \\ &- 2\sum_i\sum_j\int_0^t\int F_{i,j}u_j\partial_i\Phi + F_{i,j}\partial_iu_j\Phi\, dx\, ds. \end{aligned}$$

3.1 Some lemmas on weights.

As a consequence of the Hölder inequality, we have $\mathcal{A}_q(\mathbb{R}^d) \subset \mathcal{A}_r(\mathbb{R}^d)$ if $q \le r$.

Lemma 3.1. *If $\Phi \in \mathcal{A}_s$ then for all $\theta \in (0,1)$, we have $\Phi^\theta \in \mathcal{A}_p$ where $\theta = \frac{p-1}{s-1}$. In particular, if we suppose Φ to satisfy $(\boldsymbol{H}3)$, we obtain $\Phi \in \mathcal{A}_p$ with $p = 1 + \frac{r-1}{r} = 2 - \frac{1}{r} < 2$, and then $\Phi \in \mathcal{A}_2$.*

Proof. We use the Hölder inequality, as $\frac{1}{p} = \frac{1}{s} + \frac{s-p}{ps}$ we have

$$
\begin{aligned}
&(\int_Q \Phi^{\frac{p-1}{s-1}}\, dx)^{\frac{1}{p}} (\int_Q \Phi^{-(\frac{p-1}{s-1})(\frac{1}{p-1})} dx)^{1-\frac{1}{p}} \\
&= (\int_Q (\Phi^{\frac{1}{s}} (\Phi^{-\frac{1}{s-1}})^{\frac{s-p}{ps}})^p\, dx)^{\frac{1}{p}} (\int_Q \Phi^{-(\frac{p-1}{s-1})(\frac{1}{p-1})} dx)^{1-\frac{1}{p}} \\
&\leq (\int_Q \Phi\, dx)^{\frac{1}{s}} (\int_Q \Phi^{-\frac{1}{s-1}}\, dx)^{\frac{1}{p}-\frac{1}{s}+1-\frac{1}{p}}
\end{aligned}
$$

◇

The behavior of adapted weights in the Sobolev espaces $W^{1,r}$ is favorable :

Lemma 3.2. *Let Φ be a weight satisfying $(\boldsymbol{H}1)$ and $(\boldsymbol{H}2)$ and let $1 \leq r < +\infty$. Then :*
a) $\sqrt{\Phi} f \in H^1$ if and only if $f \in L^2(\Phi\, dx)$ and $\nabla f \in L^2(\Phi\, dx)$; and we have

$$\|\sqrt{\Phi} f\|_{H^1} \approx \left(\int \Phi(|f|^2 + |\nabla f|^2)\, dx\right)^{1/2}$$

b) $\Phi f \in W^{1,r}$ if and only if $f \in L^r(\Phi^r\, dx)$ and $\nabla f \in L^r(\Phi^r\, dx)$; and

$$\|\Phi f\|_{W^{1,r}} \approx \left(\int \Phi^r(|f|^r + |\nabla f|^r)\, dx\right)^{1/r}$$

Proof. We just need to observe that $|\nabla \Phi| \leq C_1 \Phi^{3/2} \leq C_1 \Phi$ and $|\nabla(\sqrt{\Phi})| = \frac{1}{2}\frac{|\nabla \Phi|}{\Phi}\sqrt{\Phi} \leq \frac{1}{2} C_1 \sqrt{\Phi}$. ◇

It is well known that for a weight $w \in \mathcal{A}_q$ $(1 < q < +\infty)$, the Riesz transforms and the Hardy–Littlewood maximal function are bounded on $L^q(w\, dx)$, we refer to (Stein, 1993). In our setting the following inequalities will be useful :

Lemma 3.3. *Consider the dimension $d \in \{2, 3, 4\}$ and let Φ be a weight satisfying $(\boldsymbol{H}1)$, $(\boldsymbol{H}2)$ and $(\boldsymbol{H}3)$. Then :*
a) for $j = 1, \ldots, d$, the Riesz transforms $\mathcal{R}_j$ satisfy that $\|\sqrt{\Phi}\mathcal{R}_j f\|_2 \leq C\|\sqrt{\Phi} f\|_2$ and $\|\sqrt{\Phi}\mathcal{R}_j f\|_{H^1} \leq C\|\sqrt{\Phi} f\|_{H^1}$;
b) for $j = 1, \ldots, d$ the Riesz transforms $\mathcal{R}_j$ fulfills $\|\Phi\mathcal{R}_j f\|_r \leq C\|\Phi f\|_r$ and $\|\Phi\mathcal{R}_j f\|_{W^{1,r}} \leq C\|\Phi f\|_{W^{1,r}}$;
c) if $\mathbb{P}$ is the Leray projection operator then for a vector field $\boldsymbol{u}$ we have $\|\sqrt{\Phi}\mathbb{P}\boldsymbol{u}\|_2 \leq C\|\sqrt{\Phi}\boldsymbol{u}\|_2$ and $\|\sqrt{\Phi}\mathbb{P}\boldsymbol{u}\|_{H^1} \leq C\|\sqrt{\Phi}\boldsymbol{u}\|_{H^1}$;
d) for a vector field $\boldsymbol{u}$ we have

$$\|\sqrt{\Phi}\,\boldsymbol{u}\|_{H^1} \approx \|\sqrt{\Phi}\,\boldsymbol{u}\|_2 + \|\sqrt{\Phi}\nabla \cdot \boldsymbol{u}\|_2 + \|\sqrt{\Phi}\nabla \wedge \boldsymbol{u}\|_2.$$

e) Let $\theta_\epsilon(x) = \frac{1}{\epsilon^d}\theta(\frac{x}{\epsilon})$, where $\theta \in \mathcal{D}(\mathbb{R}^d)$, θ is non-negative and radially decreasing and $\int \theta\, dx = 1$. Then, there exists a constant C, which does not depend on ϵ nor f, such that $\|\sqrt{\Phi}\,(\theta_\epsilon * f)\|_2 \leq C\|\sqrt{\Phi}\, f\|_2$ and $\|\sqrt{\Phi}\,(\theta_\epsilon * f)\|_{H^1} \leq C(\|\sqrt{\Phi}\, f\|_{L^2} + \|\sqrt{\Phi}\,\nabla f\|_{L^2})$.

Proof. a) Since $\partial_k(\mathcal{R}_j f) = \mathcal{R}_j(\partial_k f)$ and using $\Phi \in \mathcal{A}_2$ and Lemma 3.2 we obtain the inequality.

b) Using $\Phi^r \in \mathcal{A}_r$ and Lemma 3.2 we can conclude the inequality.

c) Can be obtained from a) since for $\mathbf{v} = \mathbb{P}\mathbf{u}$, we have $v_j = \sum_{k=1}^d \mathcal{R}_j\mathcal{R}_k(u_k)$.

d) If $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_d)$, we have the identity

$$-\Delta\mathbf{u} = \nabla \wedge (\nabla \wedge \mathbf{u}) - \nabla(\nabla \cdot \mathbf{u})$$

so that

$$\partial_k \mathbf{u} = \mathcal{R}_k \mathcal{R} \wedge (\nabla \wedge \mathbf{u}) - \mathcal{R}_k \mathcal{R}(\nabla \cdot \mathbf{u}),$$

then applying a) we obtain d)

e) We recall the classical inequality (see Theorem 2.1.10 in Chapter 2 of (Grafakos, 2008)) which states that we have $|\theta_\epsilon * f| \leq \mathcal{M}_f$, where $\mathcal{M}_f$ is the Hardy–Littlewood maximal function of f, and, similarly, $|\partial_k(\theta_\epsilon * f)| \leq \mathcal{M}_{\partial_k f}$. Then, as $\Phi \in \mathcal{A}_2$ and using Lemma 3.2 we conclude e). ◇

Our assumptions on adapted weights, also imply slowly decaying at infinity :

Lemma 3.4. *Let Φ be a weight which satisfy $(\boldsymbol{H}1)$ and $(\boldsymbol{H}2)$. Then there exists a constant C_3 such that*

$$\frac{1}{(1+|x|)^2} \leq C_3 \Phi.$$

Moreover, if $d = 3$ and Φ depends only on $r = \sqrt{x_1^2 + x_2^2}$, then we obtain

$$\frac{1}{(1+|r|)^2} \leq C_3 \Phi.$$

Proof. Let us denote $x_0 = \frac{1}{|x|} x$ and $g(\lambda) = \Phi(\lambda x_0)$. We remark that

$$g'(\lambda) = x_0 \cdot \nabla \Phi(\lambda x_0) \geq -C_1 (\Phi(\lambda x_0))^{3/2} = -C_1 g(\lambda)^{3/2}.$$

Therefore,

$$C_1 \lambda \geq -\int_0^\lambda g'(\mu) g(\mu)^{-3/2} \, d\mu = 2(g(\lambda)^{-1/2} - g(0)^{-1/2})$$

and hence

$$\Phi(x)^{-1/2} \leq \Phi(0) + \frac{C_1}{2}|x| \leq \sqrt{C_3}(1+|x|).$$

If Φ depends only on r, we just need to observe that

$$\frac{1}{(1+|r|)^2} \leq C_3 \Phi(x_1, x_2, 0) = C_3 \Phi(x).$$

◇

Proof of Theorem 4

3.2 A priori estimates

Consider $\phi \in \mathcal{D}(\mathbb{R}^d)$ a real-valued test function which is equal to 1 on a neighborhood of 0 and let $\phi_\epsilon(x) = \phi(\epsilon x)$. Let

$$\mathbf{u}_{0,\epsilon} = \mathbb{P}(\phi_\epsilon \mathbf{u}_0) \qquad \text{and} \qquad \mathbb{F}_\epsilon = \phi_\epsilon \mathbb{F}.$$

Then, $\mathbf{u}_{0,\epsilon}$ is divergence free and since $\Phi \in \mathcal{A}_2$, $\mathbf{u}_{0,\epsilon}$ converges to $\mathbf{u}_0$ in $L^2(\Phi \, dx)$. We also check, by dominated convergence, that $\mathbb{F}_\epsilon$ converges to the tensor $\mathbb{F}$ in $L^2((0,T), L^2(\Phi \, dx))$.

Let $\theta_\epsilon(x) = \frac{1}{\epsilon^d}\theta(\frac{x}{\epsilon})$, where $\theta \in \mathcal{D}(\mathbb{R}^d)$, θ is non-negative and radially decreasing and $\int \theta \, dx = 1$.

Observe that $\mathbf{u}_{0,\epsilon}$ belongs to L^2. Then, a classical result for the mollified Navier–Stokes equations gives the existence of a unique global solution $\mathbf{u}_\epsilon$ of the problem

$$(NS_\epsilon)\begin{cases}\partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - ((\mathbf{u}_\epsilon * \theta_\epsilon)\cdot\nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon + \nabla\cdot\mathbb{F}_\epsilon \\ \\ \nabla\cdot\mathbf{u}_\epsilon = 0, \qquad\qquad \mathbf{u}_\epsilon(0,.) = \mathbf{u}_{0,\epsilon}\end{cases}$$

which belongs to $\mathcal{C}([0,+\infty), L^2(\mathbb{R}^d)) \cap L^2((0,+\infty), \dot H^1(\mathbb{R}^d))$.

For the dimension $d\in\{2,3\}$, we look to demonstrate that

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{u}_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2\,ds \\ &\le \|\sqrt{\Phi}\mathbf{u}_{0,\epsilon}\|_{L^2}^2 + c\|\sqrt{\Phi}\mathbb{F}_\epsilon\|^2_{L^2((0,+\infty),L^2)} + C_\Phi\int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2 + \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^{2d}\,ds,\end{aligned} \tag{3.2}$$

where C_Φ does not depend on ϵ nor on $\mathbf{u}_0$. When $d=4$, we will obtain the next useful variant of this inequality, which holds under the assumption $\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_2 < \epsilon_0$ with $C\epsilon_0 < \frac{1}{8}$ where $C>0$ is a fixed constant; the variant is

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{u}_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2\,ds \\ &\le \|\sqrt{\Phi}\mathbf{u}_{0,\epsilon}\|_{L^2}^2 + c\|\sqrt{\Phi}\mathbb{F}_\epsilon\|^2_{L^2((0,+\infty),L^2)} + C_\Phi\int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2\,ds.\end{aligned} \tag{3.3}$$

Let us denote $\mathbf{b}_\epsilon = \mathbf{u}_\epsilon * \theta_\epsilon$. As we have seen $\sqrt{\Phi}, \nabla\sqrt{\Phi}\in L^\infty$, then pointwise multiplication by $\sqrt{\Phi}$ maps boundedly H^1 to H^1 and H^{-1} to H^{-1}. Thus, we know that $\sqrt{\Phi}\mathbf{u}_\epsilon \in L^2H^1$ and $\sqrt{\Phi}\partial_t\mathbf{u}_\epsilon \in L^2H^{-1}$, so that we can calculate $\int \partial_t\mathbf{u}_\epsilon\cdot\mathbf{u}_\epsilon\Phi\,dx$ and more precisely we find :

$$\begin{aligned}\int \frac{|\mathbf{u}_\epsilon(t,x)|^2}{2}\Phi\,dx &+ \int_0^t\int |\nabla\otimes\mathbf{u}_\epsilon|^2\ \Phi dx\,ds \\ &= \int \frac{|\mathbf{u}_{0,\epsilon}(x)|^2}{2}\Phi\,dx - \int_0^t\int (\nabla\otimes\mathbf{u}_\epsilon)\cdot(\nabla\Phi\otimes\mathbf{u}_\epsilon)\,dx\,ds \\ &+ \int_0^t\int (\frac{|\mathbf{u}_\epsilon|^2}{2}\mathbf{b}_\epsilon + p_\epsilon\mathbf{u}_\epsilon)\cdot\nabla\Phi\,dx\,ds \\ &- \sum_i\sum_j\int_0^t\int F_{\epsilon,i,j}u_{\epsilon,j}\partial_i\Phi + F_{\epsilon,i,j}\partial_i u_{\epsilon,j}\Phi\,dx\,ds.\end{aligned}$$

We use the fact that $|\nabla\Phi| \le C_0\Phi^{\frac{3}{2}} \le C_0\Phi$, in order to control the following term

$$\begin{aligned}&|-\int_0^t\int(\nabla\otimes\mathbf{u}_\epsilon)\cdot(\nabla\Phi\otimes\mathbf{u}_\epsilon)dx\,ds| \\ &\le \frac{1}{8}\int_0^t \|\sqrt{\Phi}\,\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2 + C\int_0^t\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_{L^2}^2,\end{aligned}$$

and also we can easily control the term

$$\begin{aligned}&|-\sum_i\sum_j\int_0^t\int F_{\epsilon,i,j}u_{\epsilon,j}\partial_i\Phi + F_{\epsilon,i,j}\partial_i u_{\epsilon,j}\Phi\,dx\,ds| \\ &\le \frac{1}{8}\int_0^t\|\sqrt{\Phi}\,\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2 + c\int_0^t\|\sqrt{\Phi}\,\mathbb{F}_\epsilon\|_{L^2}^2 + C_\Phi\int_0^t\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_{L^2}^2.\end{aligned}$$

Now, we analyze the integrals containing the pressure term. Strategically, we distinguish two cases :

- **Case 1:** $d = 2$ and $r \in (1, 2]$, or $d = 3$ and $r \in [\frac{6}{5}, 2]$, or $d = 4$ and $r \in [\frac{4}{3}, 2)$. We observe that for those values of d and r we have

$$0 \leq \frac{d}{2} - \frac{d}{2r} \leq 1 \text{ and } \dot{H}^{\frac{d}{2}-\frac{d}{2r}} \subset L^{2r}$$

and

$$0 \leq \frac{d}{r} - \frac{d}{2} \leq 1 \text{ and } \dot{H}^{\frac{d}{r}-\frac{d}{2}} \subset L^{\frac{r}{r-1}}.$$

Using the continuity of the Riesz transforms on $L^r(\Phi^r dx)$, we observe that

$$\begin{aligned}\int_0^t \int \Big(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2} + |p_\epsilon||\mathbf{u}_\epsilon|\Big) |\nabla\Phi| \, dx \, ds &\leq \int_0^t \|\Phi(|\mathbf{u}_\epsilon| \, |\mathbf{b}_\epsilon| + |p_\epsilon|)\|_r \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{r}{r-1}} \\ \leq C \int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{2r} \|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{2r} \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{r}{r-1}} ds\end{aligned}$$

Using the Sobolev embedding $\dot{H}^{\frac{d}{2}-\frac{d}{2r}} \subset L^{2r}$, the fact that $|\nabla\sqrt{\Phi}| \leq C\sqrt{\Phi}$, and the continuity of the maximal function operator on $L^2(\Phi dx)$, we obtain

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{2r} \\ &\leq C\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2^{1-(\frac{d}{2}-\frac{d}{2r})} \|\nabla \otimes (\sqrt{\Phi}\mathbf{b}_\epsilon)\|_2^{\frac{d}{2}-\frac{d}{2r}} \\ &\leq C'\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2^{1-(\frac{d}{2}-\frac{d}{2r})} (\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla \otimes \mathbf{b}_\epsilon\|_2)^{\frac{d}{2}-\frac{d}{2r}} \\ &\leq C''\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(\frac{d}{2}-\frac{d}{2r})} (\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_\epsilon\|_2)^{\frac{d}{2}-\frac{d}{2r}},\end{aligned}$$

and

$$\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{2r} \leq C\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(\frac{d}{2}-\frac{d}{2r})} (\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_\epsilon\|_2)^{\frac{d}{2}-\frac{d}{2r}}.$$

Using the embedding $\dot{H}^{\frac{d}{r}-\frac{d}{2}} \subset L^{\frac{r}{r-1}}$, we also have

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{r}{r-1}} \\ &\leq C\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(\frac{d}{r}-\frac{d}{2})} \|\nabla \otimes (\sqrt{\Phi}\mathbf{u}_\epsilon)\|_{L^2}^{\frac{d}{r}-\frac{d}{2}} \\ &\leq C\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(\frac{d}{r}-\frac{d}{2})} (\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_\epsilon\|_{L^2})^{\frac{d}{r}-\frac{d}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^t \int \Big(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2} + |p_\epsilon||\mathbf{u}_\epsilon|\Big) |\nabla\Phi| \, dx \, ds \\ &\qquad \leq C \int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{3-\frac{d}{2}} (\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_\epsilon\|_{L^2})^{\frac{d}{2}} \, ds.\end{aligned}$$

Using the Young inequality, we then find for $d = 2$ or $d = 3$

$$\begin{aligned}&\int_0^t \int \Big(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2} + |p_\epsilon||\mathbf{u}_\epsilon|\Big) |\nabla\Phi| \, dx \, ds \\ &\quad \leq \frac{1}{8} \int_0^t \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_\epsilon\|_{L^2}^2 \, ds + C_\Phi \int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2 + \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^{\frac{12-2d}{4-d}} \, ds,\end{aligned}$$

where, we have $\frac{12-2d}{4-d} = 2d$ since $d \in \{2,3\}$.
When we consider $d = 4$, provided that $\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_2 < \epsilon_0$ with $C\epsilon_0 < \frac{1}{8}$ we find

$$\int_0^t \int (\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2} + |p_\epsilon||\mathbf{u}_\epsilon|)\,|\nabla\Phi|\,dx\,ds$$
$$\leq \frac{1}{8}\int_0^t \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2\,ds + \frac{1}{8}\int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2\,ds,$$

- **Case 2:** $d = 3$ and $r \in (1, \frac{6}{5})$, or $d = 4$ and $r \in (1, \frac{4}{3})$. Let $q = \frac{dr}{d-r}$; for those values of d, r and q, we have

$$W^{1,r} \subset L^q$$
$$0 \leq d - \frac{d}{r} \leq 1 \text{ and } \dot{H}^{d(1-\frac{1}{r})} \subset L^{\frac{2r}{2-r}}.$$

and

$$0 \leq \frac{d}{r} - \frac{d}{2} - 1 \leq 1 \text{ and } \dot{H}^{\frac{d}{r}-\frac{d}{2}-1} \subset L^{\frac{q}{q-1}}.$$

Using the continuity of the Riesz transforms on $L^r(\Phi^r dx)$, we can observe that

$$\int_0^t \int (\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2} + |p||\mathbf{u}_\epsilon|)\,|\nabla\Phi|\,dx\,ds$$
$$\leq \int_0^t \|\Phi|\mathbf{u}_\epsilon|^2\|_q\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\frac{q}{q-1}}\,ds + \int_0^t \|\Phi p\|_q\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{q}{q-1}}\,ds$$
$$\leq C\int_0^t \|\Phi|\mathbf{u}_\epsilon|^2\|_{W^{1,r}}\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\frac{q}{q-1}}\,ds + \sum_{ij}\int_0^t \|\Phi b_{\epsilon,i}u_{\epsilon,j}\|_{W^{1,r}}\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{q}{q-1}}\,ds.$$

We remark that

$$\|\Phi b_{\epsilon,i}u_{\epsilon,j}\|_{W^{1,r}}$$
$$\leq \|\Phi b_{\epsilon,i}u_{\epsilon,j}\|_r + \sum_k(\|b_{\epsilon,i}u_{\epsilon,j}\,\partial_k\Phi\|_{L^r} + \|\Phi\,b_{\epsilon,i}\,\partial_k u_{\epsilon,j}\|_{L^r} + \|\Phi\,u_{\epsilon,i}\,\partial_k b_{\epsilon,j}\|_{L^r})$$
$$\leq C(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{2r}{2-r}}\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2 + \|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\frac{2r}{2-r}}\|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_2 + \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\frac{2r}{2-r}}\|\sqrt{\Phi}\nabla\otimes\mathbf{b}_\epsilon\|_2),$$
$$\leq C'(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2} + \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2})(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{\dot{H}^{d(1-\frac{1}{r})}} + \|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\dot{H}^{d(1-\frac{1}{r})}}).$$

and

$$\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\dot{H}^{d(1-\frac{1}{r})}}$$
$$\leq C\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2^{1-(d-\frac{d}{r})}\|\nabla\otimes(\sqrt{\Phi}\mathbf{b}_\epsilon)\|_2^{d-\frac{d}{r}}$$
$$\leq C'\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2^{1-(d-\frac{d}{r})}(\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2 + \|\sqrt{\Phi}\nabla\otimes\mathbf{b}_\epsilon\|_2)^{d-\frac{d}{r}}$$
$$\leq C''\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(d-\frac{d}{r})}(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2} + \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2})^{d-\frac{d}{r}}.$$

Therefore, we get

$$\sum_{i,j}\|\Phi b_{\epsilon,i}u_{\epsilon,j}\|_{W^{1,r}} + \|\Phi|u_\epsilon|^2\|_{W^{1,r}}$$
$$\leq C\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{1-(d-\frac{d}{r})}(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2} + \|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2})^{1+d-\frac{d}{r}}.$$

On the other hand, we have

$$\begin{aligned}
&\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_{\frac{q}{q-1}} \\
&\leq C\|\sqrt{\Phi}\mathbf{b}_\epsilon\|_2^{2-(\frac{d}{r}-\frac{d}{2})}\|\nabla\otimes(\sqrt{\Phi}\mathbf{b}_\epsilon)\|_2^{\frac{d}{r}-\frac{d}{2}-1} \\
&\leq C'\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{2-(\frac{d}{r}-\frac{d}{2})}(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}+\|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2})^{\frac{d}{r}-\frac{d}{2}-1}.
\end{aligned}$$

Hence, we find again

$$\begin{aligned}
&\int_0^t\int(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2}+|p_\epsilon||\mathbf{u}_\epsilon|)\,|\nabla\Phi|\,dx\,ds \\
&\qquad\leq C\int_0^t\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{3-\frac{d}{2}}(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2+\|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2})^{\frac{d}{2}}\,ds.
\end{aligned}$$

and we conclude in the same way as for the first case.

In the Case 1 and Case 2, we have found for $d\in\{2,3\}$,

$$\begin{aligned}
&\int_0^t\int(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2}+|p_\epsilon||\mathbf{u}_\epsilon|)\,|\nabla\Phi|\,dx\,ds \\
&\quad\leq\frac{1}{8}\int_0^t\|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2ds+\frac{1}{8}\int_0^t\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^{2d}\,ds
\end{aligned}$$

and for $d=4$, under the assumption $\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_2<\epsilon_0$ with $C\epsilon_0<\frac{1}{8}$ where $C>0$ is a fixed constant, we find

$$\begin{aligned}
&\int_0^t\int(\frac{|\mathbf{u}_\epsilon|^2|\mathbf{b}_\epsilon|}{2}+|p_\epsilon||\mathbf{u}_\epsilon|)\,|\nabla\Phi|\,dx\,ds \\
&\qquad\leq\frac{1}{8}\int_0^t\|\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}^2ds+C_\Phi\int_0^t\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2\,ds.
\end{aligned}$$

From these controls, we get inequalities (3.2) and (3.3). Next, we will see that these inequalities give us a control on the size of $\|\sqrt{\Phi}\,\mathbf{u}_\epsilon\|_2$ on an interval of time that does not depend on ϵ :

Lemma 3.5. *Suppose that α is a continuous non-negative function on $[0,T)$ which satisfies, for three constants $A,B\in(0,+\infty)$ and $b\in[1,\infty)$,*

$$\alpha(t)\leq A+B\int_0^t\alpha(s)+\alpha(s)^b\,ds.$$

- *For $b>1$, let $0<T_1<T$ and $T_0=\min(T_1,\frac{1}{3^bB(A^{b-1}+(BT_1)^{b-1})})$. Then, we have, for every $t\in[0,T_0]$, $\alpha(t)\leq 3A$.*
- *We suppose $b=1$. Then, we have, for every $t\in[0,\frac{1}{4B}]$, the estimate $\alpha(t)\leq 2A$.*

Proof. First, consider $b>1$. We seek to estimate the first time $T^*<T_1$ (if it exists) for which we have

$$\alpha(T^*)=3A.$$

We have

$$\alpha\leq\frac{A}{BT_1}+(\frac{BT_1}{A})^{b-1}\alpha^b.$$

We thus find

$$\alpha(T^*) \leq 2A + BT^*(3A)^b(1 + (\frac{BT_1}{A})^{b-1})$$

which implies

$$BT^* 3^b (A^{b-1} + (BT_1)^{b-1}) \geq 1.$$

Now, consider $b = 1$, so that

$$\alpha(t) \leq A + 2B \int_0^t \alpha(s).$$

We look to estimate the first time $T^* < T$ (if it exists) for which we have

$$\alpha(T^*) = 2A.$$

We have

$$2A = \alpha(T^*) \leq A + 2BT^* 2A$$

which implies

$$T^* \geq \frac{1}{4B}$$

◇

Applying Lemma 3.5 to the inequalities (3.2) and (3.3), we thus find that there exists a constant $\mathbf{C}_\Phi \geq 1$ such that if T_0 satisfies

- if $d = 2$, $\mathbf{C}_\Phi \left(1 + \|\mathbf{u}_0\|^2_{L^2(\Phi dx)} + \|\mathbb{F}\|^2_{L^2((0,+\infty),L^2(\Phi dx))}\right) T_0 = 1$
- if $d = 3$, $\mathbf{C}_\Phi \left(1 + \|\mathbf{u}_0\|^2_{L^2(\Phi dx)} + \|\mathbb{F}\|^2_{L^2((0,+\infty),L^2(\Phi dx))}\right)^2 T_0 = 1$
- if $d = 4$ and

  $$c_\Phi(\|\mathbf{u}_0\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}\|^2_{L^2((0,+\infty),L^2(\Phi dx))}) \leq \frac{1}{\mathbf{C}_\Phi},$$

  with $\mathbf{C}_\Phi\, T_0 = 1$, and where $c_\Phi > 0$ is a constant such that for all $\epsilon > 0$,

  $$\|\mathbf{u}_{0,\epsilon}\|^2_{L^2(\Phi\, dx)} + c\|\mathbb{F}_\epsilon\|^2_{L^2((0,+\infty),L^2(\Phi dx))} \leq c_\Phi(\|\mathbf{u}_0\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}\|^2_{L^2((0,+\infty),L^2(\Phi dx))}),$$

then

$$\sup_{0\leq t\leq T_0} \|\mathbf{u}_\epsilon(t,.)\|^2_{L^2(\Phi dx)} + \int_0^{T_0} \|\nabla \otimes \mathbf{u}_\epsilon\|^2_{L^2(\Phi\, dx)}\, ds \leq \mathbf{C}_\Phi(\|\mathbf{u}_0\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}\|^2_{L^2((0,+\infty),L^2(\Phi dx))}). \tag{3.4}$$

3.3 Passage to the limit and local existence

We know that $\mathbf{u}_\epsilon$ is bounded in $L^\infty((0,T_0),L^2(\Phi\, dx))$ and $\nabla \otimes \mathbf{u}_\epsilon$ is bounded in $L^2((0,T_0),L^2(\Phi\, dx))$. This will alow us to use the following version of the Aubin–Lions theorem :

Lemma 3.6 (Aubin–Lions compactness theorem)**.** *Let $s > 0$, $q > 1$ and $\sigma < 0$. Let $(f_n)_n$ be a sequence of functions on $(0,T) \times \mathbb{R}^d$ such that, for all $T_0 \in (0,T)$ and all $\varphi \in \mathcal{D}(\mathbb{R}^d)$,*

- *φf_n is bounded in $L^2((0,T_0),H^s)$*
- *$\varphi\partial_t f_n$ is bounded in $L^q((0,T_0),H^\sigma)$.*

Then, there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^d)$, more precisely : if we denote f_∞ the limit, then for all $T_0\in(0,T)$ and for all $R_0>0$,

$$\lim_{n_k\to+\infty}\int_0^{T_0}\int_{|x|\le R_0}|f_{n_k}-f_\infty|^2\,dx\,dt=0.$$

For a proof of the Lemma, we refer the books (Boyer and Fabrie, 2012; Lemarié-Rieusset, 2016).

We want to verify that $\varphi\partial_t\mathbf{u}_\epsilon$ is bounded in $L^\alpha((0,T_0),H^{-s})$ for some $s\in(-\infty,0)$ and some $\alpha>1$.

In Case 1, we have seen that $\Phi\mathbf{b}_\epsilon\otimes\mathbf{u}_\epsilon$ and $\Phi p_\epsilon=\sum_{i=1}^3\sum_{j=1}^3\mathcal{R}_i\mathcal{R}_j(b_{\epsilon,i}u_{\epsilon,j})$ are bounded in $L^{\alpha_1}((0,T_0),L^r)$, where $\alpha_1=\frac{2r}{dr-d}$, so that $\alpha_1\in[2,\infty)$ if $d=2$, $\alpha_1\in[\frac{4}{3},4]$ if $d=3$ and $\alpha_1\in(1,2]$ if $d=4$.

In Case 2, we have seen that $\Phi\mathbf{b}_\epsilon\otimes\mathbf{u}_\epsilon$ and Φp_ϵ are bounded in $L^{\alpha_2}((0,T_0),W^{1,r})$, where $\alpha_2=\frac{2r}{r+dr-d}$ and thus it is bounded in $L^{\alpha_2}L^q$, with $q=\frac{dr}{d-r}$. We observe that if $d=3$ then $\alpha_2\in(\frac{4}{3},2)$ and if $d=4$ then $\alpha_2\in(1,2)$.

Let $\varphi\in\mathcal{D}(\mathbb{R}^d)$. We have that $\varphi\mathbf{u}_\epsilon$ is bounded in $L^2((0,T_0),H^1)$, it is the first assumption to apply the Aubin-Lions theorem. Now, writing

$$\partial_t\mathbf{u}_\epsilon=\Delta\mathbf{u}_\epsilon-(\sum_{j=1}^3\partial_j(b_{\epsilon,j}\mathbf{u}_\epsilon)+\nabla p_\epsilon)+\nabla\cdot\mathbb{F}_\epsilon$$

and using the embeddings $L^r\subset\dot H^{\frac{d}{2}-\frac{d}{r}}\subset H^{-1}$ (in Case 1) or $L^{\frac{dr}{d-r}}\subset H^{-(\frac{d}{r}-\frac{d}{2}-1)}\subset H^{-1}$ (in Case 2) we see that $\varphi\partial_t\mathbf{u}_\epsilon$ is bounded in $L^{\alpha_i}((0,T_0),H^{-2})$.

Thus, by the Aubin–Lions theorem, there exist $\mathbf{u}$ and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging to 0 such that $\mathbf{u}_{\epsilon_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T_0)\times\mathbb{R}^3)$: for every $\tilde T\in(0,T_0)$ and every $R>0$, we have

$$\lim_{k\to+\infty}\int_0^{\tilde T}\int_{|y|<R}|\mathbf{u}_{\epsilon_k}-\mathbf{u}|^2\,dx\,ds=0.$$

Then, we have that $\mathbf{u}_{\epsilon_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T_0),L^2(\Phi dx))$, $\nabla\otimes\mathbf{u}_{\epsilon_k}$ converges weakly to $\nabla\otimes\mathbf{u}$ in $L^2((0,T_0),L^2(\Phi dx))$, and $\mathbf{u}_{\epsilon_k}$ converges weakly to $\mathbf{u}$ in $L^3((0,T_0),L^3(\Phi^{\frac{3}{2}}dx))$. We deduce that $\mathbf{b}_{\epsilon_k}\otimes\mathbf{u}_{\epsilon_k}$ is weakly convergent in $(L^{6/5}L^{6/5})_{\mathrm{loc}}$ to $\mathbf{b}\otimes\mathbf{u}$ and thus in $\mathcal{D}'((0,T_0)\times\mathbb{R}^d)$; as in Case 1, it is bounded in $L^{\alpha_1}((0,T_0),L^r)$, and in Case 2 it is bounded in $L^{\alpha_2}((0,T_0),W^{1,r})$, it is weakly convergent in these spaces respectively (as $\mathcal{D}$ is dense in their dual spaces).

By the continuity of the Riesz transforms on $L^r(\Phi^r dx)$ and on $W^{1,r}(\Phi^r dx)$, we find that in Case 1 and Case 2, p_{ϵ_k} is convergent to the distribution

$$p=\sum_{i=1}^3\sum_{j=1}^3\mathcal{R}_i\mathcal{R}_j(u_iu_j).$$

We thus have obtained

$$\partial_t\mathbf{u}=\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla p+\nabla\cdot\mathbb{F}.$$

Moreover, we have seen that $\partial_t \mathbf{u}$ is locally in $L^1 H^{-2}$, and thus $\mathbf{u}$ has representative such that $t \mapsto \mathbf{u}(t,.)$ is continuous from $[0,T_0)$ to $\mathcal{D}'(\mathbb{R}^d)$ and coincides with $\mathbf{u}(0,.) + \int_0^t \partial_t \mathbf{u}\, ds$.

In the sense of distributions, we have

$$\mathbf{u}(0,.) + \int_0^t \partial_t \mathbf{u}\, ds = \mathbf{u} = \lim_{k\to+\infty} \mathbf{u}_{\epsilon_k} = \lim_{k\to+\infty} \mathbf{u}_{0,\epsilon_k} + \int_0^t \partial_t \mathbf{u}_{n_k}\, ds = \mathbf{u}_0 + \int_0^t \partial_t \mathbf{u}\, ds,$$

hence, $\mathbf{u}(0,.) = \mathbf{u}_0$, and thus $\mathbf{u}$ is a solution of (NS).

Now, we want to prove the energy balance. In the case of dimension 2, we remark that, since $\sqrt{\Phi}\mathbf{u} \in L^\infty L^2 \cap L^2 H^1$, we have by interpolation that $\sqrt{\Phi}\mathbf{u} \in L^4 L^4$, and then we can define $((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{u}$. Then the equality

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{u} - p\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F})$$

can be proved.

Let us consider the case $d = 3$. We define

$$A_\epsilon = -\partial_t(\frac{|\mathbf{u}_\epsilon|^2}{2}) + \Delta(\frac{|\mathbf{u}_\epsilon|^2}{2}) - \nabla\cdot\left(\frac{|\mathbf{u}_\epsilon|^2}{2}\mathbf{u}_\epsilon - p_\epsilon\mathbf{u}_\epsilon\right) + \mathbf{u}_\epsilon\cdot(\nabla\cdot\mathbb{F}_\epsilon) = |\nabla\otimes\mathbf{u}_\epsilon|^2.$$

Remark that, as u_{ϵ_k} is locally strongly convergent in L^2L^2; and locally bounded in $L^\infty L^2$, it is then locally strongly convergent in $L^{p'}L^2$, for all $p' \in [2,+\infty)$. Then, as $\sqrt{\Phi}\nabla\otimes\mathbf{u}_\epsilon$ is bounded in $L^2((0,T),L^2)$, by the Gagliardo-Nirenberg interpolation inequalities we obtain $\mathbf{u}_{\epsilon_k}$ is locally strongly convergent in $L^{p'}L^{q'}$ with $\frac{2}{p'} + \frac{3}{q'} > \frac{d}{2}$.

In Case 1, we know that p_{ϵ_k} is locally weakly convergent in $L^{\alpha_1}L^r$ and we know by the remark above that $\mathbf{u}_{\epsilon_k}$ is locally strongly convergent in $L^{\frac{\alpha_1}{\alpha_1-1}}L^{\frac{r}{r-1}}$, and hence $p_{\epsilon_k}\mathbf{u}_{\epsilon_k}$ converges in the sense of distributions.

In Case 2, we know that p_{ϵ_k} is locally weakly convergent in $L^{\alpha_2}L^q$ and by the observation above, $\mathbf{u}_{\epsilon_k}$ is locally strongly convergent in $L^{\frac{\alpha_2}{\alpha_2-1}}L^{\frac{q}{q-1}}$, and hence $p_{\epsilon_k}\mathbf{u}_{\epsilon_k}$ converges in the sense of distributions.

Thus, A_{ϵ_k} is convergent in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ to

$$A = -\partial_t(\frac{|\mathbf{u}|^2}{2}) + \Delta(\frac{|\mathbf{u}|^2}{2}) - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{u} - p\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}),$$

and $A = \lim_{k\to+\infty} |\nabla\otimes\mathbf{u}_{\epsilon_k}|^2$. If $\theta \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ is non-negative, we have that $\sqrt{\theta}\nabla\otimes\mathbf{u}_{\epsilon_k}$ is weakly convergent in L^2L^2 to $\sqrt{\theta}\nabla\otimes\mathbf{u}$, so that

$$\iint A\theta\, dx\, ds = \lim_{\epsilon_k\to+\infty} \iint A_{\epsilon_k}\theta\, dx\, ds = \lim_{k\to+\infty} \iint |\nabla\otimes\mathbf{u}_{\epsilon_k}|^2\theta\, dx\, ds \geq \iint |\nabla\otimes\mathbf{u}|^2\theta\, dx\, ds.$$

Hence, in the case $d = 3$, there exists a non-negative locally finite measure μ on $(0,T)\times\mathbb{R}^3$ such that $A = |\nabla\mathbf{u}|^2 + \mu$, i.e. such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla\mathbf{u}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{u} - p\mathbf{u}\right) + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

3.4 Convergence to the initial data

Let us consider inequalities (3.2) and (3.3), together with the inequality (3.4). On $(0, T_0)$ we have a uniform control, in ϵ and t, of $\|\mathbf{u}_\epsilon(t,.)\|^2_{L^2(\Phi dx)}$. Thus, from (3.2) and (3.3) we get

$$\|\mathbf{u}_\epsilon(t,.)\|^2_{L^2(\Phi\, dx)} \le \|\mathbf{u}_{0,\epsilon}\|^2_{L^2(\Phi\, dx)} + C_\Phi t(1 + \|\mathbf{u}_0\|^{2d}_{L^2(\Phi\, dx)} + \|\mathbb{F}\|^{2d}_{L^2((0,+\infty),L^2(\Phi dx))}).$$

We remember that $\mathbf{u}_{0,\epsilon_k}$ is strongly convergent to $\mathbf{u}_0$ in $L^2(\Phi\, dx)$. Moreover, as $\mathbf{u}_{\epsilon_k} = \mathbf{u}_{0,\epsilon_k} + \int_0^t \partial_t \mathbf{u}_{\epsilon_k}\, ds$, we observe that $\mathbf{u}_{\epsilon_k}(t,.)$ is convergent to $\mathbf{u}(t,.)$ in $\mathcal{D}'(\mathbb{R}^d)$, and we can deduce that it is weakly convergent in $L^2(\Phi\, dx)$ (as it is bounded in $L^2(\Phi dx)$), so that

$$\|\mathbf{u}(t,.)\|^2_{L^2(\Phi\, dx)} \le \|\mathbf{u}_0\|^2_{L^2(\Phi\, dx)} + C_\Phi t(1 + \|\mathbf{u}_0\|^{2d}_{L^2(\Phi\, dx)} + \|\mathbb{F}\|^{2d}_{L^2((0,+\infty),L^2(\Phi dx))}).$$

This remark implies

$$\limsup_{t\to 0} \|\mathbf{u}(t,.)\|^2_{L^2(\Phi dx)} \le \|\mathbf{u}_0\|^2_{L^2(\Phi dx)}.$$

For reciprocal inequality, we recall that $\mathbf{u}$ is weakly continuous in $L^2(\Phi dx)$, therefore

$$\|\mathbf{u}_0\|^2_{L^2(\Phi dx)} \le \liminf_{t\to 0} \|\mathbf{u}(t,.)\|^2_{L^2(\Phi dx)}.$$

We conclude that $\|\mathbf{u}_0\|^2_{L^2(\Phi dx)} = \lim_{t\to 0} \|\mathbf{u}(t,.)\|^2_{L^2(\Phi dx)}$, It allows us to turn the weak convergence into a strong convergence in the Hilbert space. ◇

3.5 Global existence using a scaling argument

We fix $\lambda > 0$. Then $\mathbf{u}_\epsilon$ is a solution of the Cauchy initial value problem for the approximated Navier–Stokes system (NS_ϵ) on $(0, T)$ with initial value $\mathbf{u}_{0,\epsilon}$ and forcing tensor $\mathbb{F}_\epsilon$ if and only if $\mathbf{u}_{\epsilon,\lambda}(t,x) = \lambda \mathbf{u}_\epsilon(\lambda^2 t, \lambda x)$ is a solution for the approximated Navier–Stokes equations $(NS_{\lambda\epsilon})$ on $(0, T/\lambda^2)$ with initial value $\mathbf{u}_{0,\epsilon,\lambda}(x) = \lambda \mathbf{u}_{0,\epsilon}(\lambda x)$ and forcing tensor $\mathbb{F}_{\epsilon,\lambda}(x) = \lambda^2 \mathbb{F}_\epsilon(\lambda^2 t, \lambda x)$. We shall denote $\mathbf{u}_{0,\lambda} = \lambda \mathbf{u}_0(\lambda x)$ and $\mathbb{F}_\lambda(x) = \lambda^2 \mathbb{F}(\lambda^2 t, \lambda x)$.

We have proved that for $d \in \{2, 3\}$,

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{u}_{\epsilon,\lambda}(t)\|^2_{L^2} + \int_0^t \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_{\epsilon,\lambda}\|^2_{L^2} \\ &\le \|\sqrt{\Phi}\mathbf{u}_{0,\epsilon,\lambda}\|^2_{L^2} + C\|\sqrt{\Phi}\mathbb{F}_{\epsilon,\lambda}\|^2_{L^2((0,+\infty),L^2)} + C_\Phi \int_0^t \|\sqrt{\Phi}\mathbf{u}_{\epsilon,\lambda}\|^2_{L^2} + \|\sqrt{\Phi}\mathbf{u}_{\epsilon,\lambda}\|^{2d}_{L^2}\, ds\end{aligned}$$

where C_Φ does not depend on ϵ, λ nor on $\mathbf{u}_0$. When $d = 4$, under the assumption $\|\sqrt{\Phi}\, \mathbf{u}_\epsilon\|_2 < \epsilon_0$ with $C\epsilon_0 < \frac{1}{8}$ where $C > 0$ is a fixed constant, we have found

$$\begin{aligned}&\|\sqrt{\Phi}\mathbf{u}_{\epsilon,\lambda}(t)\|^2_{L^2} + \int_0^t \|\sqrt{\Phi}\nabla \otimes \mathbf{u}_{\epsilon,\lambda}\|^2_{L^2}\, ds \\ &\le \|\sqrt{\Phi}\mathbf{u}_{0,\epsilon,\lambda}\|^2_{L^2} + c\|\sqrt{\Phi}\mathbb{F}_{\epsilon,\lambda}\|^2_{L^2((0,+\infty),L^2)} + C_\Phi \int_0^t \|\sqrt{\Phi}\mathbf{u}_{\epsilon,\lambda}\|^2_{L^2}\, ds.\end{aligned}$$

By Lemma 3.5, we thus found that there exists a constant $\mathbf{C}_\Phi \ge 1$ such that if T_λ satisfies

- if $d = 2$, $\mathbf{C}_\Phi \left(1 + \|\mathbf{u}_{0,\lambda}\|^2_{L^2(\Phi dx)} + \|\mathbb{F}_\lambda\|^2_{L^2((0,+\infty),L^2(\Phi dx))}\right) T_\lambda = 1$
- if $d = 3$, $\mathbf{C}_\Phi \left(1 + \|\mathbf{u}_{0,\lambda}\|^2_{L^2(\Phi dx)} + \|\mathbb{F}_\lambda\|^2_{L^2((0,+\infty),L^2(\Phi dx))}\right)^2 T_\lambda = 1$
- if $d = 4$ and

$$c_\Phi(\|u_{0,\lambda}\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}_\lambda\|^2_{L^2((0,+\infty),L^2(\Phi dx))}) \leq \frac{1}{\mathbf{C}_\Phi} \qquad \text{with } \mathbf{C}_\Phi\, T_\lambda = 1$$

then

$$\sup_{0\leq t\leq T_\lambda} \|\mathbf{u}_{\epsilon,\lambda}(t,.)\|^2_{L^2(\Phi dx)} + \int_0^{T_\lambda} \|\nabla \otimes \mathbf{u}_{\epsilon,\lambda}\|^2_{L^2(\Phi\, dx)}\, ds$$
$$\leq \mathbf{C}_\Phi(\|\mathbf{u}_{0,\lambda}\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}_\lambda\|^2_{L^2((0,+\infty),L^2(\Phi dx))}).$$

It permits to deduce that the solutions $\mathbf{u}_\epsilon$ are controlled, uniformly in ϵ, on $(0, \lambda^2 T_\lambda)$. In fact, we get for $t \in (0, T_\lambda)$,

$$\int |\mathbf{u}_{\epsilon,\lambda}(t,x)|^2\Phi(x)\, dx = \int |\mathbf{u}_\epsilon(\lambda^2 t, y)|^2 \Phi(\frac{y}{\lambda})\lambda^{2-d}\, dy \geq \lambda^{2-d} \int |\mathbf{u}_\epsilon(\lambda^2 t, x)|^2 \Phi(x)\, dx$$

and

$$\int_0^{T_\lambda} \int |\nabla \otimes \mathbf{u}_{\epsilon,\lambda}(t,x)|^2 \Phi(x)\, dx\, dt = \int_0^{\lambda^2 T_\lambda} \int |\nabla \otimes \mathbf{u}_{\epsilon,\lambda}(s,y)|^2 \Phi(\frac{y}{\lambda})\lambda^{2-d}\, dy\, ds$$
$$\geq \lambda^{2-d} \int_0^{\lambda^2 T_\lambda} \int |\nabla \otimes \mathbf{u}_\epsilon(t,x)|^2 \Phi(x)\, dx\, dt.$$

Moreover, we get

$$\lim_{\lambda\to+\infty} \lambda^2 T_\lambda = +\infty$$

when $2 \leq d \leq 4$, and if $d = 4$,

$$\lim_{\lambda\to+\infty} \|\mathbf{u}_{0,\lambda}\|^2_{L^2(\Phi\, dx)} + \|\mathbb{F}_\lambda\|^2_{L^2((0,+\infty),L^2(\Phi dx))} = 0.$$

Indeed, we observe that

$$\int \lambda^2 |\mathbf{u}_0(\lambda x)|^2 \Phi(x)\, dx = \lambda^{2-d} \int |\mathbf{u}_0(x)|^2 \Phi(\frac{x}{\lambda})\, dx = \lambda^{4-d} \int |\mathbf{u}_0(x)|^2 \frac{\Phi(\frac{x}{\lambda})}{\lambda^2 \Phi(x)} \Phi(x)\, dx$$

and

$$\int_0^{+\infty} \int \lambda^4 |\mathbb{F}(\lambda^2 x, \lambda x)|^2 \Phi(x)\, dx = \lambda^{2-d} \int_0^{+\infty} \int |\mathbb{F}(t,x)|^2 \Phi(\frac{x}{\lambda})\, dx$$
$$= \lambda^{4-d} \int_0^{+\infty} \int |\mathbb{F}(t,x)|^2 \frac{\Phi(\frac{x}{\lambda})}{\lambda^2 \Phi(x)} \Phi(x)\, dx.$$

Since $\frac{\Phi(\frac{x}{\lambda})}{\lambda^2\Phi(x)} \leq \min\{C_2, \frac{1}{\lambda^2 \Phi(x)}\}$ by hypothesis (**H**4), we find by dominated convergence that $\|\mathbf{u}_{0,\lambda}\|_{L^2(\Phi\, dx)} = o(\lambda^{\frac{4-d}{2}})$ and also $\|\mathbb{F}_\lambda\|_{L^2((0,+\infty),L^2(\Phi\, dx))} = o(\lambda^{\frac{4-d}{2}})$, so that $\lim_{\lambda\to+\infty} \lambda^2 T_\lambda = +\infty$.

Therefore, if we consider a finite time T and a sequence ϵ_k converging to 0, we may choose λ such that $\lambda^2 T_\lambda > T$ (if $d = 4$ we demand λ to satisfy $c_\Phi(\|\mathbf{u}_{0,\lambda}\|_{L^2(\Phi\, dx)} + \|\mathbb{F}_\lambda\|_{L^2((0,+\infty),L^2(\Phi\, dx))}) < \epsilon_0$ as well). As we have seen, we obtain a uniform control of $\mathbf{u}_\epsilon$ and of $\nabla \otimes \mathbf{u}_\epsilon$ on $(0, \lambda^2 T_\lambda)$, hence a uniform control of $\mathbf{u}_\epsilon$ and of $\nabla \otimes \mathbf{u}_\epsilon$ on $(0, T)$. We thus may exhibit a solution on $(0, T)$ using again the Rellich–Lions theorem, by extracting a subsequence ϵ_{k_n}. A diagonal argument conducts then to exhibit a global solution.

Thus we have proved Theorem 4. $\diamond$

3.6 Global Regular Solutions in dimension 2

In dimension $d = 2$, we can treat the existence problem in higher regularity, while, in the case $d = 3$, one will restrict the analysis to the case of axisymmetric flows with no swirl which will be treated in the following chapter.

Theorem 5. *Consider $d = 2$ and a weight Φ satisfying $(\boldsymbol{H}1) - (\boldsymbol{H}4)$. Let $\boldsymbol{u}_0$ be a divergence free vector field, such that $\boldsymbol{u}_0, \nabla \otimes \boldsymbol{u}_0$ belong to $L^2(\Phi dx)$. Let $\mathbb{F} = (F_{i,j}(t,x))$ be a tensor such that $\mathbb{F}, \nabla \cdot \mathbb{F}$ belong to $L^2((0,+\infty), L^2(\Phi dx))$. Then, there exists a global solution $\boldsymbol{u}$ of the problem*

$$(NS) \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \\ \nabla \cdot \boldsymbol{u} = 0, \qquad\qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

such that

- *$\boldsymbol{u}$ and $\nabla \boldsymbol{u}$ belong to $L^\infty((0,T), L^2(\Phi\, dx))$ and $\Delta \boldsymbol{u}$ belongs to $L^2((0,T), L^2(\Phi\, dx))$, for all $T > 0$*
- *the maps $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ and $t \in [0,+\infty) \mapsto \nabla \boldsymbol{u}(t,.)$ are weakly continuous from $[0,+\infty)$ to $L^2(\Phi dx)$, and are strongly continuous at $t = 0$.*

In the case of dimension $d = 2$, the Navier–Stokes equations are well-posed in H^1 and we don't need to mollify the equations. More precisely, we may approximate the Navier–Stokes equations with the system

$$(NS_\epsilon) \begin{cases} \partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon + \nabla \cdot \mathbb{F}_\epsilon \\ \\ \nabla \cdot \mathbf{u}_\epsilon = 0, \qquad\qquad \mathbf{u}_\epsilon(0,.) = \mathbf{u}_{0,\epsilon} \end{cases}$$

where

$$\mathbf{u}_{0,\epsilon} = \mathbb{P}(\phi_\epsilon \mathbf{u}_0) \qquad \text{and} \qquad \mathbb{F}_\epsilon = \phi_\epsilon \mathbb{F}.$$

Observe that the vorticity $\omega_\epsilon = \nabla \wedge \mathbf{u}_\epsilon$ is solution of

$$\begin{cases} \partial_t \omega_\epsilon = \Delta \omega_\epsilon - (\mathbf{u}_\epsilon \cdot \nabla)\omega_\epsilon + \nabla \wedge (\nabla \cdot \mathbb{F}_\epsilon) \\ \\ \nabla \cdot \omega_\epsilon = 0, \qquad\qquad \omega_\epsilon(0,.) = \omega_{0,\epsilon} \end{cases}$$

where

$$\omega_{0,\epsilon} = \nabla \wedge (\phi_\epsilon \mathbf{u}_0).$$

Since $\mathbf{u}_{0,\epsilon}$ belongs to H^1, we know that we have a global solution $\mathbf{u}_\epsilon$. We then just have to prove that, for every finite time T_0, the norms $\|\omega_\epsilon\|_{L^\infty((0,T_0),L^2(\Phi\, dx))}$ and $\|\nabla \omega_\epsilon\|_{L^2((0,T_0),L^2(\Phi\, dx))}$ are uniformly controlled.

We can compute $\int \partial_t \omega_\epsilon \cdot \omega_\epsilon \Phi \, dx$, which gives

$$\begin{aligned}
&\int \frac{|\omega_\epsilon(t,x)|^2}{2}\Phi \, dx + \int_0^t \int |\nabla \omega_\epsilon|^2 \; \Phi dx\, ds \\
&\quad = \int \frac{|\omega_{0,\epsilon}(x)|^2}{2}\Phi \, dx - \int_0^t \int \nabla(\frac{|\omega_\epsilon|^2}{2}) \;\cdot \nabla\Phi dx\, ds \\
&\quad + \int_0^t \int \frac{|\omega_\epsilon|^2}{2}\mathbf{u}_\epsilon \;\cdot \nabla \Phi \, dx\, ds \\
&\quad + \int_0^t \int (\nabla\cdot \mathbb{F}_\epsilon)_2(-\partial_1\omega_\epsilon \, \Phi - \partial_1\Phi\, \omega_\epsilon)) + (\nabla \cdot \mathbb{F}_\epsilon)_1(\partial_2 \omega_\epsilon\, \Phi + \partial_2\Phi\, \omega_\epsilon))\, dx\, ds,
\end{aligned}$$

where $(\nabla\cdot\mathbb{F}_\epsilon)_j$ is the j-coordinate of $\nabla\cdot\mathbb{F}_\epsilon$, more precisely $(\nabla\cdot\mathbb{F}_\epsilon)_j = \sum_i \partial_i F_{\epsilon,j}$.

As

$$\begin{aligned}
\int_0^t \int \frac{|\omega_\epsilon|^2}{2}\mathbf{u}_\epsilon \;\cdot \nabla \Phi \, dx\, ds &\leq \int_0^t \|\sqrt{\Phi}\omega_\epsilon\|^2_{L^{\frac{8}{3}}} \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^4} \\
&\leq \int_0^t (\|\sqrt{\Phi}\omega_\epsilon\|_{L^2}^{3/4} \|\nabla(\sqrt{\Phi}\omega_\epsilon)\|_{L^2}^{1/4})^2 \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^4}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\|\sqrt{\Phi}\omega_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Phi}\nabla\omega_\epsilon\|_{L^2}^2 \\
&\leq \|\sqrt{\Phi}\omega_{0,\epsilon}\|_{L^2}^2 + C\|\sqrt{\Phi}(\nabla\cdot\mathbb{F}_\epsilon)\|^2_{L^2((0,+\infty),L^2)} + C_\Phi \int_0^t \|\sqrt{\Phi}\omega_\epsilon\|^2_{L^2}(1+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^4}^{\frac{4}{3}})\, ds
\end{aligned}$$

We can conclude that, for all $T>0$ and for all $t\in(0,T)$,

$$\begin{aligned}
&\|\sqrt{\Phi}\omega_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Phi}\nabla\omega_\epsilon\|_{L^2}^2 \\
&\quad \leq (\|\sqrt{\Phi}\omega_{0,\epsilon}\|_{L^2}^2 + C\|\sqrt{\Phi}(\nabla\cdot\mathbb{F}_\epsilon)\|^2_{L^2((0,+\infty),L^2)})e^{C_\Phi \sup_{\epsilon>0}\int_0^t (1+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^4})^{\frac{4}{3}}\, ds}
\end{aligned}$$

Thus, we have uniform controls on $(0,T)$. ◇

Chapter 4

Regular Axisymmetric Solutions without swirl

4.1 Axisymmetry.

We consider the usual coordinates (x_1, x_2, x_3) in the Euclidean space $\mathbb{R}^3$, and the cylindrical coordinates (r, θ, z) provided by the formulas $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and $x_3 = z$.

Let us denote $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the usual canonical basis

$$\mathbf{e}_1 = (1,0,0),\ \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1).$$

We attach to each point x (with $r \neq 0$) another orthonormal basis

$$\mathbf{e}_r = \frac{\partial x}{\partial r} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \quad \mathbf{e}_\theta = \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \quad \mathbf{e}_z = \frac{\partial x}{\partial z} = \mathbf{e}_3.$$

Let us consider a vector field $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$. Then, we find their coordinates in the new basis

$$\mathbf{u} = (u_1\cos\theta + u_2\sin\theta)\,\mathbf{e}_r + (-u_1\sin\theta + u_2\cos\theta)\,\mathbf{e}_\theta + u_3\,\mathbf{e}_z.$$

We denote $(u_r, u_\theta, u_z)_p$ the coordinates of $\mathbf{u}$ in the basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$. In what follows in this chapter we will consider axially symmetric (axisymmetric) vector fields $\mathbf{u}$ without swirl and axisymmetric scalar functions a, which means that

$$\mathbf{u} = u_r(r,z)\,\mathbf{e}_r + u_z(r,z)\,\mathbf{e}_z \quad \text{and} \quad a = a(r,z).$$

Our goal is to adapt the energy estimates in weighted spaces to the framework of axisymmetric solutions of the Navier–Stokes equations. We look to generalize the following well known result is due to Ladyzhenskaya, we refer to (Ladyzhenskaya, 1968; Lemarié-Rieusset, 2016).

Proposition 4.1. *Consider a divergence free axisymmetric vector field $\boldsymbol{u}_0$ without swirl which belongs to H^1. Then, the problem*

$$(NS)\begin{cases} \partial_t \boldsymbol{u} = \Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p \\ \nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

has a unique solution $\boldsymbol{u} \in \mathcal{C}([0,+\infty), H^1)$. This solution is axisymmetric without swirl. Moreover, $\boldsymbol{u}, \nabla\otimes\boldsymbol{u}$ belong to $L^\infty((0,+\infty), L^2)$, and $\nabla\otimes\boldsymbol{u}, \Delta\boldsymbol{u}$ belong to $L^2((0,+\infty), L^2)$.

If $\boldsymbol{u}_0 \in H^2$ then the following inequality is fulfilled

$$\int \frac{|\omega(t)|^2}{r^2} dx \leq \int \frac{|\omega_0|^2}{r^2} \leq \|\nabla \otimes \omega_0\|_2^2.$$

This result is our starting point.

The uniqueness of the solutions in the statement above can be seen as a consequence of the Serrin's Weak-Strong uniqueness criterion. The problem of the uniqueness of solutions, when we treat with weighted spaces, is non-trivial and we leave it as an open problem.

Below, we present our main results with respect to axisymetric without swirl solutions :

Theorem 6. *Consider a weight Φ satisfying $(\boldsymbol{H}1) - (\boldsymbol{H}4)$. Assume that Φ depends only on $r = \sqrt{x_1^2 + x_2^2}$ and there exists a continuous weight Ψ (that depends only on r) such that $\Phi \leq \Psi \leq 1$, $\Psi \in \mathcal{A}_2$ and there exists $C_1 > 0$ such that*

$$|\nabla \Psi| \leq C_1 \sqrt{\Phi} \Psi$$

Let $\boldsymbol{u}_0$ be a divergence free axisymmetric vector field without swirl, such that $\boldsymbol{u}_0$ belongs to $L^2(\Phi\, dx)$ and $\nabla \otimes \boldsymbol{u}_0$ belongs to $L^2(\Psi\, dx)$. Then there exists a time $T > 0$, and a local solution $\boldsymbol{u}$ on $(0, T)$ of the problem

$$(NS) \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} - \nabla p \\ \nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{u}(0, .) = \boldsymbol{u}_0 \end{cases}$$

such that

- *$\boldsymbol{u}$ is axisymmetric without swirl, $\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Phi\, dx))$, $\nabla \otimes \boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Psi\, dx))$ and $\Delta \boldsymbol{u}$ belongs to $L^2((0,T), L^2(\Psi\, dx))$,*
- *the maps $t \mapsto \boldsymbol{u}(t, .)$ and $t \mapsto \nabla \boldsymbol{u}(t, .)$ are weakly continuous from $[0, T)$ to $L^2(\Phi\, dx)$, and are strongly continuous at $t = 0$,*

Remark: If the vorticity is more integrable at time $t = 0$, it will remain in positive times as well.

Example : The weights $\Phi(x) = \frac{1}{(1+r)^\gamma}$ and $\Psi(x) = \frac{1}{(1+r)^\delta}$ with $0 \leq \delta \leq \gamma < 2$ fulfill these hypothesis.

If we suppose the weight Ψ to be a little more regular, we obtain a global existence result. The next theorem precise these conditions on the weight.

Theorem 7. *Consider a weight Φ satisfying $(\boldsymbol{H}1) - (\boldsymbol{H}4)$. Assume that Φ depends only on $r = \sqrt{x_1^2 + x_2^2}$ and there exists a continuous weight $\Psi \in W^{2,\infty}$ (that depends only on r) such that $\Phi \leq \Psi \leq 1$, $\Psi \in \mathcal{A}_2$ and there exists $C_1 > 0$ such that*

$$|\nabla \Psi| \leq C_1 \sqrt{\Phi} \Psi \text{ and } |\Delta(\sqrt{\Psi})| \leq C_1 \sqrt{\Phi}.$$

Let $\boldsymbol{u}_0$ be a divergence free axisymmetric vector field without swirl, such that $\boldsymbol{u}_0$ belongs to $L^2(\Phi dx)$ and $\nabla \otimes \boldsymbol{u}_0$ belongs to $L^2(\Psi dx)$. Then there exists a global solution $\boldsymbol{u}$ of the problem

$$(NS) \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} - \nabla p \\ \nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{u}(0, .) = \boldsymbol{u}_0 \end{cases}$$

such that

- *$\boldsymbol{u}$ is axisymmetric without swirl, $\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Phi\, dx))$, $\nabla \otimes \boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Psi\, dx))$ and $\Delta \boldsymbol{u}$ belongs to $L^2((0,T), L^2(\Psi\, dx))$, for all $T > 0$,*
- *the maps $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ and $t \in [0,+\infty) \mapsto \nabla \otimes \boldsymbol{u}(t,.)$ are weakly continuous from $[0,+\infty)$ to $L^2(\Phi\, dx)$ and to $L^2(\Psi\, dx)$ respectively, and are strongly continuous at $t = 0$.*

Example : The weights $\Phi(x) = \frac{1}{(1+r)^\gamma}$ and $\Psi(x) = \frac{1}{(1+r^2)^{\delta/2}}$ with $0 \le \delta \le \gamma < 2$ fulfill the hypothesis of the last theorem. Remark also that $\frac{1}{(1+r^2)^{\delta/2}}$ is equivalent to $\frac{1}{(1+r)^\delta}$.

4.2 A priori controls and local existence

Let us take a real-valued radial function $\phi \in \mathcal{D}(\mathbb{R}^2)$ which is equal to 1 on a neighborhood of 0 and let $\phi_\epsilon(x) = \phi(\epsilon(x_1,x_2))$. For $\epsilon \in (0,1]$, we denote

$$\mathbf{u}_{0,\epsilon} = \mathbb{P}(\phi_\epsilon \mathbf{u}_0).$$

Thus, $\mathbf{u}_{0,\epsilon}$ is a divergence free axisymmetric without swirl vector field and $\mathbf{u}_{0,\epsilon}$ belongs to H^1. We have

$$\omega_{0,\epsilon} = \nabla \wedge \mathbf{u}_{0,\epsilon} = \nabla \wedge (\phi_\epsilon \mathbf{u}_0) = \phi_\epsilon \omega_0 + \epsilon (\nabla\phi)(\epsilon x) \wedge \mathbf{u}_0.$$

Then, using the fact that $\Phi \in \mathcal{A}_2$ and $|\epsilon \nabla \phi(\epsilon x)| \le C\frac{1}{r}\mathbb{1}_{r \ge \frac{1}{C\epsilon}} \le C' \mathbb{1}_{r\ge \frac{1}{C\epsilon}} \sqrt{\Phi}$, we can conclude that

$$\lim_{\varepsilon \to 0} \|\mathbf{u}_0 - \mathbf{u}_{0,\epsilon}\|_{L^2(\Phi\, dx)} + \|\omega_0 - \omega_{0,\epsilon}\|_{L^2(\Psi\, dx)} = 0.$$

Let $\mathbf{u}_\epsilon$ be the global solution of the following approximated problem

$$(NS_\epsilon) \begin{cases} \partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon \\ \nabla \cdot \mathbf{u}_\epsilon = 0, \qquad \mathbf{u}_\epsilon(0,.) = \mathbf{u}_{0,\epsilon} \end{cases}$$

given by the Proposition 4.1. We denote $\omega_\epsilon = \nabla \wedge \mathbf{u}_\epsilon$, therefore

$$\partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon$$

and

$$\partial_t \omega_\epsilon = \Delta \omega_\epsilon + (\omega_\epsilon \cdot \nabla)\mathbf{u}_\epsilon - (\mathbf{u}_\epsilon \cdot \nabla)\omega_\epsilon. \tag{4.1}$$

Since $\sqrt{\Psi}\omega_\epsilon \in L^2H^1$ (because $\sqrt{\Psi}, \nabla\sqrt{\Psi} \in L^\infty$) and $\sqrt{\Psi}\partial_t\omega_\epsilon \in L^2H^{-1}$, we can compute $\int \partial_t\omega_\epsilon \cdot \omega_\epsilon \Psi\, dx$. More precisely, using (4.1) we find

$$\begin{aligned}
\int \frac{|\omega_\epsilon(t,x)|^2}{2}\Psi\, dx &+ \int_0^t \int |\nabla\otimes\omega_\epsilon|^2\ \Psi dx\, ds \\
&= \int \frac{|\omega_{0,\epsilon}(x)|^2}{2}\Psi\, dx - \int_0^t \int \nabla(\frac{|\omega_\epsilon|^2}{2})\ \cdot\nabla\Psi dx\, ds \\
&+ \int_0^t \int \frac{|\omega_\epsilon|^2}{2}\mathbf{u}_\epsilon\ \cdot\nabla\Psi\ - (\omega_\epsilon\cdot\mathbf{u}_\epsilon)\omega_\epsilon\cdot\nabla\Psi\, dx \\
&- \int_0^t \int ((\omega_\epsilon\cdot\nabla)\omega_\epsilon)\cdot\mathbf{u}_\epsilon\ \Psi\, dx\, ds \\
&\le \int \frac{|\omega_{0,\epsilon}(x)|^2}{2}\Psi\, dx + \frac{1}{8}\int_0^t \int |\nabla\otimes\omega_\epsilon|^2\ \Psi dx\, ds + C\int_0^t \|\sqrt{\Psi}\,\omega_\epsilon\|_2^2\, ds \\
&+ C\int_0^t \|\sqrt{\Psi}\,\omega_\epsilon\|_2\|\sqrt{\Psi}\,\omega_\epsilon\|_6\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_3\, ds \\
&- \int_0^t \int ((\omega_\epsilon\cdot\nabla)\omega_\epsilon)\cdot\mathbf{u}_\epsilon\ \Psi\, dx\, ds \\
&\le \int \frac{|\omega_{0,\epsilon}(x)|^2}{2}\Psi\, dx + \frac{1}{4}\int_0^t \int |\nabla\otimes\omega_\epsilon|^2\ \Psi dx\, ds + C\int_0^t \|\sqrt{\Psi}\,\omega_\epsilon\|_2^2\, ds \\
&+ C'\int_0^t \|\sqrt{\Psi}\,\omega_\epsilon\|_2^2(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_3 + (\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_3^{4/3})\, ds \\
&- \int_0^t \int ((\omega_\epsilon\cdot\nabla)\omega_\epsilon)\cdot\mathbf{u}_\epsilon\ \Psi\, dx\, ds
\end{aligned}$$

As $\omega_\epsilon = \omega_{\epsilon,\theta}\,\mathbf{e}_\theta$, we obtain

$$\omega_\epsilon\cdot\nabla\omega_\epsilon = -\frac{\omega_{\epsilon,\theta}^2}{r}\,\mathbf{e}_r.$$

Then, in order to control $\mathbf{u}_\epsilon\cdot(\omega_\epsilon\cdot\nabla\omega_\epsilon)$, we split the domain of integration in two parts, one where r is small and the other one where r is large. The support of ϕ_1 is contained in $\{x\ /\ r < R\}$ for some $R > 0$, and the support of $1-\phi_1$ is contained in the set $\{x\ /\ r > R_0\}$ for some $R_0 > 0$. We observe that

$$\inf_{r<R}\Phi(x) = \inf_{\sqrt{x_1^2+x_2^2}<R}\Phi(x_1,x_2,0) > 0$$

and

$$\inf_{r<R}\Psi(x) = \inf_{\sqrt{x_1^2+x_2^2}<R}\Psi(x_1,x_2,0) > 0.$$

On the other hand, we see that

$$\inf_{r>R_0} r^2\Phi(x) = \inf_{\sqrt{x_1^2+x_2^2}>R_0}(x_1^2+x_2^2)\Phi(x_1,x_2,0) \ge \inf_{|x|>R_0}|x|^2\Phi(x) > 0.$$

Then, we write :

$$
\begin{aligned}
-\int_0^t &\int ((\omega_\epsilon\cdot\nabla)\omega_\epsilon)\cdot \mathbf{u}_\epsilon\ \Psi\, dx\, ds\\
&= \int_0^t \int \phi_1(\,(\omega_\epsilon\cdot\nabla)\mathbf{u}_\epsilon)\cdot\omega_\epsilon\,)\,\Psi\, dx\, ds + \int_0^t\int(\omega_\epsilon\cdot\mathbf{u}_\epsilon)(\omega_\epsilon\cdot\nabla\phi_1)\Psi\, dx\, ds\\
&\quad + \int_0^t\int \phi_1(\omega_\epsilon\cdot\mathbf{u}_\epsilon)\omega_\epsilon\cdot\nabla\Psi dx\, ds\\
&\quad - \int_0^t\int(1-\phi_1)(\mathbf{u}_\epsilon\cdot(\omega_\epsilon\cdot\nabla\omega_\epsilon))\Psi dx\, ds\\
&\le C\int_0^t\int |\omega_\epsilon|^2|\nabla\otimes\mathbf{u}_\epsilon|\,\Psi^{3/2}\, dx\, ds + C\int_0^t\int|\omega_\epsilon|^2|\mathbf{u}_\epsilon|\,\sqrt{\Phi}\,\Psi\, dx\, ds.
\end{aligned}
$$

Since $\Psi\in\mathcal{A}_2$ we have $\|\sqrt{\Psi}\nabla\otimes\mathbf{u}_\epsilon\|_2\approx\|\sqrt{\Psi}\omega_\epsilon\|_2$; moreover, we know that

$$
\|\nabla\otimes(\sqrt{\Phi}\mathbf{u}_\epsilon)\|_2\le C(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2+\|\sqrt{\Psi}\omega_\epsilon\|_2)
$$

and

$$
\|\nabla\otimes(\sqrt{\Psi}\omega_\epsilon)\|_2\le C(\|\sqrt{\Psi}\omega_\epsilon\|_2+\|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_2).
$$

Therefore, we get

$$
\begin{aligned}
-\int_0^t &\int((\omega_\epsilon\cdot\nabla)\omega_\epsilon)\cdot\mathbf{u}_\epsilon\ \Psi\, dx\, ds\\
&\le C\int_0^t\|\sqrt{\Psi}\nabla\otimes\mathbf{u}_\epsilon\|_{L^2}\|\sqrt{\Psi}\omega_\epsilon\|_{L^3}\|\sqrt{\Psi}\omega_\epsilon\|_{L^6}\, ds\\
&\quad + C\int_0^t\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^6}\|\sqrt{\Psi}\omega_\epsilon\|_{L^3}\|\sqrt{\Psi}\omega_\epsilon\|_{L^2}\, ds\\
&\le C'\int_0^t\|\sqrt{\Psi}\omega_\epsilon\|_{L^2}^{\frac{3}{2}}(\|\sqrt{\Psi}\omega_\epsilon\|_{L^2}+\|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_{L^2})^{\frac{3}{2}}\, ds\\
&\quad + C'\int_0^t\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}\|\sqrt{\Psi}\omega_\epsilon\|_{L^2}^{\frac{3}{2}}(\|\sqrt{\Psi}\omega_\epsilon\|_{L^2}+\|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_{L^2})^{\frac{1}{2}}\, ds\\
&\le C''\int_0^t(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{4/3})\|\sqrt{\Psi}\omega_\epsilon\|_2^2+\|\sqrt{\Psi}\omega_\epsilon\|_2^3+\|\sqrt{\Psi}\omega_\epsilon\|_2^6\, ds\\
&\quad + \frac{1}{8}\int_0^t\|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_2^2\, ds
\end{aligned}
$$

and finally,

$$
\begin{aligned}
&\|\sqrt{\Psi}\omega_\epsilon(t)\|_{L^2}^2+\int_0^t\|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_{L^2}^2\, ds\\
&\le\|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2}^2+C\int(1+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_3+(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_3^{4/3})\|\sqrt{\Psi}\omega_\epsilon\|_2^2\, ds\\
&+C\int_0^t(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2\ +\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{4/3})\|\sqrt{\Psi}\omega_\epsilon\|_2^2+\|\sqrt{\Psi}\omega_\epsilon\|_2^3+\|\sqrt{\Psi}\omega_\epsilon\|_2^6\, ds\\
&\le\|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2}^2\\
&+C'\int_0^t(1+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2\ +\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_2^{4/3})\|\sqrt{\Psi}\omega_\epsilon\|_2^2+\|\sqrt{\Psi}\omega_\epsilon\|_2^6\, ds
\end{aligned}
$$

We already know that the quantity $\|\sqrt{\Phi}\mathbf{u}_\epsilon(t)\|_{L^2}$ remains bounded (independently of ϵ) on every bounded interval. Therefore, we may again use Lemma 3.5

to control

$$\sup_{0\leq t\leq T_0} \| \omega_\epsilon(t,.)\|^2_{L^2(\Psi dx)} + \int_0^{T_0} \|\nabla\omega_\epsilon\|^2_{L^2(\Psi dx)}\, ds$$

for some T_0, independently of ϵ. After, this control is transferred to the limit ω since $\omega = \lim \omega_{\epsilon_k} = \lim \nabla \wedge \mathbf{u}_{\epsilon_k}$. We thus have proved local existence of a regular solution and Theorem 6 is proved.

4.3 Very regular initial value.

Next, we present an apparently more restrictive result than our main Theorem (Theorem 7), however we will see that it implies almost directly our main Theorem.

Proposition 4.2. *Consider a weight Φ satisfying $(\boldsymbol{H}1) - (\boldsymbol{H}4)$. Assume that Φ depends only on $r = \sqrt{x_1^2 + x_2^2}$ and there exists a continuous weight Ψ (that depends only on r) such that $\Phi \leq \Psi \leq 1$, $\Psi \in \mathcal{A}_2$ and there exists $C_1 > 0$ such that*

$$|\nabla\Psi| \leq C_1\sqrt{\Phi}\Psi \text{ and } |\Delta(\sqrt{\Psi})| \leq C_1\sqrt{\Phi}.$$

Consider a divergence free axisymmetric vector field $\boldsymbol{u}_0$ without swirl, such that $\boldsymbol{u}_0$ belongs to $L^2(\Phi dx)$, and $\nabla \otimes \boldsymbol{u}_0$, $\Delta\boldsymbol{u}_0$ belong to $L^2(\Psi dx)$. Then there exists a global solution $\boldsymbol{u}$ of the problem

$$(NS)\begin{cases} \partial_t \boldsymbol{u} = \Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p \\ \nabla\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0 \end{cases}$$

such that

- *$\boldsymbol{u}$ is axisymmetric without swirl, $\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Phi\, dx))$, $\nabla\otimes\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2(\Psi\, dx))$ and $\Delta\boldsymbol{u}$ belongs to $L^2((0,T), L^2(\Psi\, dx))$, for all $T > 0$,*
- *the maps $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ and $t \in [0,+\infty) \mapsto \nabla\otimes\boldsymbol{u}(t,.)$ are weakly continuous from $[0,+\infty)$ to $L^2(\Phi\, dx)$ and to $L^2(\Psi\, dx)$ respectively, and are strongly continuous at $t = 0$,*

Proof. We retake the notation used in the proof of the last theorem. Ladyzhenskaya's inequality for axisymmetric fields with no swirl gives

$$\int \frac{|\omega_\epsilon(t)|^2}{r^2} dx \leq \int \frac{|\omega_{0,\epsilon}|^2}{r^2} dx.$$

Since

$$\partial_i\omega_{0,\epsilon} = \phi_\epsilon\partial_i\omega_0 + \epsilon\partial_i\phi(\epsilon x)\partial_i\omega_0 + \epsilon(\nabla\phi)(\epsilon x)\wedge\partial_i\mathbf{u}_0 + \epsilon^2(\nabla\partial_i\phi)(\epsilon x)\wedge\mathbf{u}_0,$$

we obtain that

$$\lim_{\epsilon\to 0}\|\nabla\otimes\omega_{0,\epsilon} - \nabla\otimes\omega_0\|_{L^2(\Psi\, dx)} = 0.$$

As

$$\int \frac{|\omega_{0,\epsilon} - \omega_0|^2}{r^2} dx \leq C(\int_{0<r<1} |\nabla\otimes\omega_{0,\epsilon} - \nabla\otimes\omega_0|^2\Psi\, dx + \int_{1<r<+\infty} |\omega_{0,\epsilon} - \omega_0|^2\Psi\, dx),$$

we also find

$$\lim_{\epsilon\to 0}\int \frac{|\omega_{0,\epsilon} - \omega_0|^2}{r^2} dx = 0.$$

We have observed that

$$\begin{aligned}&\int \frac{|\omega_\epsilon(t,x)|^2}{2}\Psi\,dx + \int_0^t\int |\nabla\otimes\omega_\epsilon|^2\ \Psi dx\,ds\\ &= \int \frac{|\omega_{0,\epsilon}(x)|^2}{2}\Psi\,dx - \int_0^t\int \nabla(\frac{|\omega_\epsilon|^2}{2})\ \cdot\nabla\Psi dx\,ds\\ &+ \int_0^t\int\frac{|\omega_\epsilon|^2}{2}\mathbf{u}_\epsilon\ \cdot\nabla\Psi\,dx\,ds\\ &- \int_0^t\int(\omega_\epsilon\cdot\mathbf{u}_\epsilon)\omega_\epsilon\ \cdot\nabla\Psi\,dx\,ds - \int_0^t\int \mathbf{u}_\epsilon(\omega_\epsilon\cdot\nabla\omega_\epsilon)\,\Psi\,dx\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\sqrt{\Psi}\omega_\epsilon(t)\|_{L^2}^2 + 2\int_0^t \|\sqrt{\Psi}\nabla\omega_\epsilon\|_{L^2}^2\\ &\leq \|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2}^2 + 2\int_0^t \|\sqrt{\Psi}\omega_\epsilon\|_{L^2}\|\sqrt{\Psi}\nabla\omega_\epsilon\|_{L^2}\\ &+ \int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3}\|\sqrt{\Psi}\omega_\epsilon\|_{L^3}^2\\ &+ \int_0^t \frac{1}{r}|\mathbf{u}_{r,\epsilon}||\omega_\epsilon|^2\Psi\,dx\,ds\end{aligned}$$

Now, we have

$$\int_0^t\int\frac{1-\phi_1(x)}{r}|\mathbf{u}_{r,\epsilon}||\omega_\epsilon|^2\Psi\,dx\,ds \leq \int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3}\|\sqrt{\Psi}\omega_\epsilon\|_{L^3}^2$$

and

$$\int_0^t\int\frac{\phi_1(x)}{r}|\mathbf{u}_{\epsilon,r}||\omega_\epsilon|^2\Psi\,dx\,ds \leq C\int_0^t \|\frac{\omega_\epsilon}{r}\|_{L^2}\|\sqrt{\Psi}\mathbf{u}_\epsilon\|_{L^\infty}\|\sqrt{\Psi}\omega_\epsilon\|_{L^2},$$

Moreover,

$$\begin{aligned}\|\frac{\omega_\epsilon}{r}\|_{L^2} \leq C\|\frac{\omega_{0,\epsilon}}{r}\|_{L^2} &\leq C(\|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2} + \|\sqrt{\Psi}\nabla\otimes\omega_{0,\epsilon}\|_{L^2})\\ &\leq C'(\|\sqrt{\Phi}\mathbf{u}_0\|_{L^2} + \|\sqrt{\Psi}\omega_0\|_{L^2} + \|\sqrt{\Psi}\nabla\otimes\omega_0\|_{L^2})\end{aligned}$$

and

$$\begin{aligned}\|\sqrt{\Psi}\mathbf{u}_\epsilon\|_{L^\infty}^2 &\leq C\|\nabla\otimes(\sqrt{\Psi}\mathbf{u}_\epsilon)\|_2\|\Delta(\sqrt{\Psi}\mathbf{u}_\epsilon)\|_2\\ &\leq C'(\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2} + \|\sqrt{\Psi}\omega_\epsilon\|_{L^2} + \|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_{L^2})^2.\end{aligned}$$

Let us denote $A_0 = \|\sqrt{\Phi}\mathbf{u}_0\|_{L^2} + \|\sqrt{\Psi}\omega_0\|_{L^2} + \|\sqrt{\Psi}\nabla\otimes\omega_0\|_{L^2}$. Then, we find

$$\begin{aligned}&\|\sqrt{\Psi}\omega_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Psi}\nabla\otimes\omega_\epsilon\|_{L^2}^2\\ &\leq \|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2}^2 + C\int_0^t \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2\\ &+ C_{\Phi,\Psi}\int_0^t \|\sqrt{\Psi}\omega_\epsilon\|_{L^2}^2(1 + A_0 + A_0^2 + \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3} + \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3}^2)\,ds\end{aligned}$$

We can thus conclude that, for all $T > 0$ and for all $t \in (0, T)$,

$$\|\sqrt{\Psi}\omega_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\Psi}\nabla \otimes \omega_\epsilon\|_{L^2}^2$$
$$\leq (\|\sqrt{\Psi}\omega_{0,\epsilon}\|_{L^2}^2 + C \sup_{\epsilon>0} \int_0^T \|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^2}^2) e^{C_{\Phi,\Psi} \sup_{\epsilon>0} \int_0^t (1+A_0^2+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3}+\|\sqrt{\Phi}\mathbf{u}_\epsilon\|_{L^3}^2)\, ds}$$

Therefore, we can obtain a solution on $(0, T)$ using the Aubin–Lions Theorem. A diagonal argument allows to get a global solution. ◇

4.4 Existence of global regular axisymmetric solutions

Proof of Theorem 7

Theorem 6 permits to find a local solution $\mathbf{v}$ on $(0, T_0)$ with initial value $\mathbf{u}_0$, which is continuous from $(0, T_0)$ to $\mathcal{D}'$. Then, we take $T_1 \in (0, T_0)$ such that $\nabla \otimes (\nabla \wedge \mathbf{v})(T_1, .) \in L^2(\Phi dx)$. Using Proposition 4.2, we get a solution $\mathbf{w}$ on $(T_1, +\infty)$ and initial value $\mathbf{v}(T_1)$. Our global solution is then defined as $\mathbf{u} = \mathbf{v}$ on $(0, T_1)$ and $\mathbf{u} = \mathbf{w}$ on $(T_1, +\infty)$.

In fact, if ϕ is a test function of $\mathbb{R}^3$, then $\partial_t(\phi\mathbf{v})$ belongs to $L^1 H^{-1}$ and $\phi\mathbf{v}$ is continuous from $[0, T_1]$ to H^{-1}. Similarly, for the solution $\mathbf{w}$ on $t > T_1$, with $\mathbf{v}(T_1) = \mathbf{w}(T_1)$, we have that $\phi\mathbf{w}$ is continuous from $[T_1, +\infty[$ to H^{-1}.

Then, for a test function $\psi(t, x)$ we find

$$\iint_{t<T_1} \mathbf{v}\partial_t\psi \, dtdx = -\iint_{t<T_1} \psi\partial_t\mathbf{v} \, dtdx + \int \mathbf{v}(T_1, x)\psi(T_1, x)\, dx$$

and

$$\iint_{t>T_1} \mathbf{w}\partial_t\psi \, dtdx = -\iint_{t>T_1} \psi\partial_t\mathbf{w} \, dtdx - \int \mathbf{v}(T_1, x)\psi(T_1, x)\, dx,$$

so that for $\mathbf{u} = 1_{t<T_1}\mathbf{v} + 1_{t>T_1}\mathbf{w}$, we have

$$\partial_t\mathbf{u} = 1_{t<T_1}\partial_t\mathbf{v} + 1_{t>T_1}\partial_t\mathbf{w}.$$

Thus, Theorem 7 is proved. ◇

In summary, we have generalized the Ladyzhenskaya's result (Proposition 4.1), on axisymmetric solutions with initial data in H^1, and we preserve the local regularity properties. However, note that we do not address the uniqueness problem, despite having good local regularity properties for these new axisymmetric solutions. We leave it as a nice *open problem*.

The difficulty of this problem is that the weight is no longer derived in the estimation of the bilinear term and thus we cannot conclude directly without additional hypothesis.

Chapter 5

Discretely Selfsimilar Solutions

In this section, we propose a new simple proof of the results of Chae and Wolf (Chae and Wolf, 2018) and Bradshaw and Tsai (Bradshaw and Tsai, 2019a) on the existence of discretely selfsimilar solutions of the Navier–Stokes problem (and of Jia and Šverák (Jia and Šverák, 2014) for self-similar solutions)

We specify the concepts related to the notion of self-similarity.

Definition 5.1. *We consider $\lambda > 1$.*

We say that $\boldsymbol{u}_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is a λ-discretely selfsimilar (λ-DSS) vector field if

$$\lambda \boldsymbol{u}_0(\lambda x) = \boldsymbol{u}_0(x).$$

A time dependent vector field $\boldsymbol{u} \in L^2_{\mathrm{loc}}([0,+\infty) \times \mathbb{R}^3)$ is called λ-DSS if

$$\lambda \boldsymbol{u}(\lambda^2 t, \lambda x) = \boldsymbol{u}(t,x).$$

A forcing tensor $\mathbb{F} \in L^2_{\mathrm{loc}}([0,+\infty) \times \mathbb{R}^3)$ is called λ-DSS if

$$\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t,x).$$

We speak of selfsimilarity if $\mathbf{u}_0$, $\mathbf{u}$ or $\mathbb{F}$ are λ-DSS for all real $\lambda > 1$.

The next examples permit us to link the discretely selfsimilarity notion with the weighted spaces $L^2_{w_\gamma} = L^2((1+|x|)^{-\gamma} dx)$. We observe that a very important feature of the weights $w_\gamma(x) = (1+|x|)^{-\gamma}$ is the control of their gradients : if $\gamma > 0$ then

$$|\nabla w_\gamma(x)| = \gamma \frac{w_\gamma(x)}{1+|x|}.$$

In particular, $|\nabla w_\gamma(x)| \le C_\gamma w_\gamma(x)$, and if $\gamma \le 2$ then we have $|\nabla w_\gamma(x)| \le C_\gamma w_{\frac{3\gamma}{2}}(x)$.

Examples :

- Let $\gamma > 1$ and $\lambda > 1$. Then, if $\mathbf{u}_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is λ-DSS, then $\mathbf{u}_0 \in L^2_{w_\gamma}$ and there exist two positive constants, $A_{\gamma,\lambda}$ and $B_{\gamma,\lambda}$, such that

$$A_{\gamma,\lambda} \int_{1<|x|\le\lambda} |\mathbf{u}_0(x)|^2\, dx \le \int |\mathbf{u}_0(x)|^2 w_\gamma(x)\, dx \le B_{\gamma,\lambda} \int_{1<|x|\le\lambda} |\mathbf{u}_0(x)|^2\, dx.$$

- $\mathbf{u}_0 \in L^2_{\mathrm{loc}}$ is self-similar if and only if it is of the form $\mathbf{u}_0 = \frac{\mathbf{w}_0(\frac{x}{|x|})}{|x|}$ with $\mathbf{w}_0 \in L^2(S^2)$.

- Consider $\gamma > 1$. $\mathbb{F}$ belongs to $L^2((0,+\infty), L^2_{w_\gamma})$ is self-similar if and only if it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$ with $\int |\mathbb{F}_0(x)|^2 \frac{1}{|x|}\, dx < +\infty$.

Proof.

- If $\mathbf{u}_0$ is λ-DSS, we have for $k \in \mathbb{Z}$

$$\int_{\lambda^k<|x|<\lambda^{k+1}} |\mathbf{u}_0(x)|^2 w_\gamma(x)\,dx \le \frac{\lambda^k}{(1+\lambda^k)^\gamma}\int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2\,dx$$

and we know that $\sum_{k\in\mathbb{Z}} \frac{\lambda^k}{(1+\lambda^k)^\gamma} < +\infty$ for $\gamma > 1$.

- If $\mathbf{u}_0$ is self-similar, we find $\mathbf{u}_0(x) = \frac{1}{|x|}\mathbf{u}_0(\frac{x}{|x|})$. Thus, for all $\lambda > 1$

$$\int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2\,dx = (\lambda-1)\int_{S^2} |\mathbf{u}_0(\sigma)|^2\,d\sigma.$$

- If $\mathbb{F}$ is self-similar, we find $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$. Then,

$$\begin{aligned}\int_0^{+\infty}\int |\mathbb{F}(t,x)|^2\, w_\gamma(x)\,dx\,ds &= \int_0^{+\infty}\int |\mathbb{F}_0(x)|^2 w_\gamma(\sqrt{t}\,x)\,dx\,\frac{dt}{\sqrt{t}}\\ &= C_\gamma \int |\mathbb{F}_0(x)|^2\,\frac{dx}{|x|}\end{aligned}$$

where $C_\gamma = \int_0^{+\infty} \frac{1}{(1+\sqrt{\theta})^\gamma}\frac{d\theta}{\sqrt{\theta}} < +\infty$. ◇

Our main result in the context of discretely selfsimilar initial data is the following one.

Theorem 8. *Let $\gamma \in (1,2)$ and $\lambda \in (1,+\infty)$. If $\boldsymbol{u}_0$ is a divergence free λ-DSS vector field which belongs to $L^2_{w_\gamma}(\mathbb{R}^3)$ and if $\mathbb{F}$ is a λ-DSS forcing tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\le i,j\le 3}$ belonging to $L^2_{loc}((0,+\infty), L^2_{w_\gamma})$, then the Navier–Stokes equations with initial data $\boldsymbol{u}_0$,*

$$(NS)\begin{cases}\partial_t \boldsymbol{u} = \Delta\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p + \nabla\cdot\mathbb{F}\\ \nabla\cdot\boldsymbol{u} = 0, \qquad\qquad \boldsymbol{u}(0,.) = \boldsymbol{u}_0\end{cases}$$

have a weak global solution $\boldsymbol{u}$ which satisfy the following properties :

- *$\boldsymbol{u}$ is a λ-DSS vector field*
- *For all $0 < T < +\infty$, $\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\boldsymbol{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$*
- *The function $t \in [0,+\infty) \mapsto \boldsymbol{u}(t,.)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and strongly continuous in $t = 0$*
- *$\boldsymbol{u}$ is suitable : there exist a locally finite non-negative measure μ on $(0,+\infty)\times\mathbb{R}^3$ such that*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - \nabla\cdot\left((\frac{|\boldsymbol{u}|^2}{2}+p)\boldsymbol{u}\right) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.$$

5.1 A priori estimates for the linearized problem

In this section, we investigate a priori controls for the linearized Navier-Stokes problem, also known as advection-diffusion problem.

Lemma 5.1. *Let $0 \leq \gamma \leq 2$ and $0 < T < +\infty$. Let $\boldsymbol{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ be a divergence-free vector field and let $\mathbb{F}$ be a tensor $\mathbb{F}(t,x) = \big(F_{i,j}(t,x)\big)_{1\leq i,j\leq 3} \in L^2((0,T), L^2_{w_\gamma})$. Let $\alpha \in [3, \frac{10}{3}]$ and let $\boldsymbol{v} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$ be a time-dependent divergence free vector-field.*

Let $(\boldsymbol{u}, p)$ be a solution of the following advection-diffusion problem

$$(LNS) \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} - \nabla p + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\boldsymbol{u} = 0, \ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0 \end{cases}$$

which satisfies :

- *$\boldsymbol{u}$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\boldsymbol{u}$ belongs to $L^2((0,T), L^2_{w_\gamma})$ and $p \in \mathcal{D}'((0,T)\times\mathbb{R}^3)$*
- *the map $t \in [0,+\infty) \mapsto \boldsymbol{u}(t)$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t=0$*
- *the solution $(\boldsymbol{u}, p)$ is suitable : there exist a non-negative locally finite measure μ on $(0,T)\times\mathbb{R}^3$ such that*

$$\partial_t(\frac{|\boldsymbol{u}|^2}{2}) = \Delta(\frac{|\boldsymbol{u}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - \nabla\cdot(\frac{|\boldsymbol{u}|^2}{2}\boldsymbol{v}) - \nabla\cdot(p\boldsymbol{u}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu. \quad (5.1)$$

Then we have the following controls:

- *If $0 < \gamma \leq 2$, for almost every $\tau \geq 0$, and for $\tau = 0$, we have for all $t \geq \tau$,*

$$\begin{aligned} &\|\boldsymbol{u}(t)\|^2_{L^2_{w_\gamma}} + 2\int_\tau^t \|\nabla\boldsymbol{u}(s)\|^2_{L^2_{w_\gamma}}\,ds \\ &\leq \|\boldsymbol{u}(\tau)\|^2_{L^2_{w_\gamma}} - \int_\tau^t\int \nabla(|\boldsymbol{u}|^2)\cdot\nabla w_\gamma\,dx\,ds \\ &+ \int_\tau^t\int (|\boldsymbol{u}|^2\boldsymbol{v} + 2p\boldsymbol{u})\cdot\nabla w_\gamma dx\,ds \\ &- 2\sum_{1\leq i,j\leq 3}(\int_\tau^t\int F_{i,j}(\partial_i u_j)w_\gamma\,dx\,ds + \int_\tau^t\int F_{i,j}u_j\partial_i(w_\gamma)\,dx\,ds), \end{aligned} \quad (5.2)$$

  and the map $t \mapsto \boldsymbol{u}(t)$ from $[0,+\infty)$ to $L^2_{w_\gamma}$ is right strongly continuous almost everywhere and

$$\begin{aligned} &\|\boldsymbol{u}(t)\|^2_{L^2_{w_\gamma}} + \int_\tau^t \|\nabla\boldsymbol{u}(s)\|^2_{L^2_{w_\gamma}}\,ds \\ &\leq \|\boldsymbol{u}(\tau)\|^2_{L^2_{w_\gamma}} + C_\gamma\int_\tau^t \|\mathbb{F}(s)\|^2_{L^2_{w_\gamma}}\,ds + C_{\alpha,\gamma}\int_\tau^t \|\boldsymbol{v}(s)\|^{\frac{2\alpha}{2\alpha-3}}_{L^\alpha_{w_{\alpha\gamma/2}}}\|\boldsymbol{u}(s)\|^2_{L^2_{w_\gamma}}\,ds. \end{aligned} \quad (5.3)$$

- *If $\gamma = 0$, for almost all $\tau \geq 0$ and for $\tau = 0$, we have for all $t \geq \tau$,*

$$\|\boldsymbol{u}(t)\|^2_{L^2} + 2\int_\tau^t \|\nabla\boldsymbol{u}(s)\|^2_{L^2}ds \leq \|\boldsymbol{u}(\tau)\|^2_{L^2} - 2\sum_{1\leq i,j\leq 3}\int_\tau^t\int F_{i,j}\partial_i u_j\ dx\,ds,$$

  and the map $t \mapsto \boldsymbol{u}(t)$ from $[0,+\infty)$ to L^2 is right strongly continuous almost everywhere.

Proof. We consider the case $0 < \gamma \leq 2$ (the changes required for the case $\gamma = 0$ are obvious).

First, remark that we know a formula for the pressure. And by the continuity of the Riesz transforms we even can write $p = \sum_i \sum_j \mathcal{R}_i \mathcal{R}_j (u_i u_j - F_{i,j})$. In fact, we prove below that : for $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha}) \subset (\frac{5}{4}, \frac{3}{2})$ and a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$ (hence $a \in (2, \frac{20\alpha}{50-9\alpha})$),

$$\sum_{1\le i,j\le 3} \mathcal{R}_i \mathcal{R}_j (u_i v_j) \in L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^d)) \tag{5.4}$$

and

$$\sum_{1\le i,j\le 3} \mathcal{R}_i \mathcal{R}_j F_{i,j} \in L^2((0,T), L^2_{w_\gamma}), \tag{5.5}$$

so that the pressure given by Theorem 1 is necessarily $p = \sum_i \sum_j \mathcal{R}_i \mathcal{R}_j (u_i u_j - F_{i,j})$.

As discussed in Section 3.1, we know that the Riesz transforms are continuous on $L^q(w_\delta dx)$ for $0 \le \delta < 3$ and $q \in (1, +\infty)$. Then, as $\gamma\beta < 3$ we have

$$\Big\| \mathcal{R}_i \mathcal{R}_j (u_i v_j) \Big\|_{L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^3))} \le C_{\gamma,\beta} \| u_i v_j \|_{L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^3))},$$

so, taking $\tilde{a}$ and $\tilde{b}$ such that $\frac{1}{a} = \frac{1}{\tilde{a}} + \frac{1}{\alpha}$ and $\frac{1}{\beta} = \frac{1}{\tilde{b}} + \frac{1}{\alpha}$, by the Hölder inequalities, and moreover, as we also have $\frac{2}{\tilde{a}} + \frac{3}{\tilde{b}} = \frac{3}{2}$ (and thus $\tilde{b} \in (\frac{5\alpha}{4\alpha-5}, \frac{6\alpha}{4+\alpha}) \subset (2,6)$), we find

$$\| u_i v_j \|_{L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^3))} \le C_{\gamma,\beta} \| \sqrt{w_\gamma}\, u_i \|_{L^{\tilde{a}}\left((0,T), L^{\tilde{b}}(\mathbb{R}^3)\right)} \cdot \| \sqrt{w_\gamma}\, v_j \|_{L^\alpha((0,T), L^\alpha(\mathbb{R}^3))} < +\infty,$$

To justify that the right side is finite, remark that we have $\sqrt{w_\gamma}\mathbf{u} \in L^\infty((0,T), L^2)$ and $\sqrt{w_\gamma}\nabla\mathbf{u} \in L^2((0,T), L^2)$, then by interpolation in the space variable, for $\tilde{a}$ and $\tilde{b}$, such that $\tilde{b} \in [2,6]$ and $\frac{2}{\tilde{a}} + \frac{3}{\tilde{b}} = \frac{3}{2}$ we find

$$w_{\frac{\gamma}{2}} \mathbf{u} \in L^{\tilde{a}} L^{\tilde{b}}, \tag{5.6}$$

in particular

$$w_{\frac{\gamma}{2}} \mathbf{u} \in L^4 L^3 \cap L^{\frac{10}{3}} L^{\frac{10}{3}} \subset L^\alpha L^\alpha.$$

Indeed, we have

$$\begin{aligned} &\| \sqrt{w_\gamma}\, u_j \|_{L^{\tilde{a}}\left((0,T), L^{\tilde{b}}(\mathbb{R}^3)\right)} \\ &\le \tilde{C}_{\gamma,\tilde{a},\tilde{b}}\, \| \mathbf{u} \|^{\frac{\tilde{a}-2}{\tilde{a}}}_{L^\infty((0,T), L^2_{w_\gamma}(\mathbb{R}^3))} \left(\int_0^T (\| \mathbf{u}(s) \|_{L^2_{w_\gamma}(\mathbb{R}^3)} + \| \nabla \mathbf{u}(s) \|_{L^2_{w_\gamma}(\mathbb{R}^3)})^2 ds \right)^{\frac{1}{\tilde{a}}} < +\infty. \end{aligned}$$

For the remaining term, we directly observe that

$$\| \mathcal{R}_i \mathcal{R}_j F_{ij} \|_{L^2((0,T), L^2_{w_\gamma}(\mathbb{R}^3))} \le C \| F_{ij} \|_{L^2((0,T), L^2_{w_\gamma}(\mathbb{R}^3))}.$$

Remark that under the hypothesis in Theorem 5.1, all the terms in the local energy balance are well defined. In (5.10) below we will justify that $p\mathbf{u} \in L^1_{\mathrm{loc}}$.

Let $0 < t_0 < t_1 < T$, we take a non-decreasing function $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ equal to 0 on $(-\infty, \frac{1}{2})$ and equal to 1 on $(1, +\infty)$. For $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$, let

$$\alpha_{\eta,t_0,t_1}(t) = \alpha(\frac{t-t_0}{\eta}) - \alpha(\frac{t-t_1}{\eta}). \tag{5.7}$$

Observe that α_{η,t_0,t_1} converges almost everywhere to $\mathbb{1}_{(t_0,t_1)}$ when $\eta \to 0$ and $\partial_t \alpha_{\eta,t_0,t_1}$ is the difference between two identity approximations, the first one in t_0 and the second one in t_1.

Consider a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. We define

$$\phi_R(x) = \phi(\frac{x}{R}). \tag{5.8}$$

For $\epsilon > 0$, we let $w_{\gamma,\epsilon} = \frac{1}{(1+\sqrt{\epsilon^2+|x|^2})^\gamma}$ (if $\gamma = 0$, we let $w_{\gamma,\epsilon} = 1$).

We have $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ and $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \geq 0$. Thus, using the local energy balance (5.1), we find

$$\begin{aligned}
&-\int\int \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon}\, dx\, ds + \iint |\nabla \mathbf{u}|^2\, \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon} dx\, ds \\
&\leq -\sum_{i=1}^3 \iint \partial_i \mathbf{u} \cdot \mathbf{u}\, \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad + \sum_{i=1}^3 \iint [(\frac{|\mathbf{u}|^2}{2}) v_i + p u_i] \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad - \sum_{1 \leq i,j \leq 3} (\iint F_{i,j} u_j \alpha_{\eta,t_0,t_1} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds + \iint F_{i,j} \partial_i u_j\, \alpha_{\eta,t_0,t_1} \phi_R\, dx\, ds).
\end{aligned}$$

We denote

$$A_{R,\epsilon}(t) = \int |\mathbf{u}(t,x)|^2 \phi_R(x) w_{\gamma,\epsilon}(x)\, dx.$$

Since

$$-\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon}\, dx\, ds = -\frac{1}{2} \int \partial_t \alpha_{\eta,t_0,t_1} A_{R,\epsilon}(s)\, ds,$$

we find for all t_0 and t_1 Lebesgue points of the measurable functions $A_{R,\epsilon}$,

$$\lim_{\eta \to 0} -\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon}\, dx\, ds = \frac{1}{2}(A_{R,\epsilon}(t_1) - A_{R,\epsilon}(t_0)).$$

Thus, letting η goes to 0 to find

$$\begin{aligned}
&-\lim_{\eta \to 0} \iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta,t_0,t_1} \phi_R w_{\gamma,\epsilon}\, dx\, ds + \int_{t_0}^{t_1} \int |\nabla \mathbf{u}|^2\, \phi_R w_{\gamma,\epsilon} dx\, ds \\
&\quad \leq -\sum_{i=1}^3 \int_{t_0}^{t_1} \int (\partial_i \mathbf{u} \cdot \mathbf{u})\, (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad + \sum_{i=1}^3 \int_{t_0}^{t_1} \int [(\frac{|\mathbf{u}|^2}{2}) v_i + p u_i] (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad - \sum_{1 \leq i,j \leq 3} (\int_{t_0}^{t_1} \int F_{i,j} u_j (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds + \int_{t_0}^{t_1} \int F_{i,j} \partial_i u_j\, \phi_R\, dx\, ds).
\end{aligned}$$

By continuity, we can let t_0 goes to 0 and thus replace t_0 by 0 in the inequality. Moreover, if we let t_1 goes to t, then by weak continuity, we find that

$$A_{R,\epsilon}(t) \leq \liminf_{t_1 \to t} A_{R,\epsilon}(t_1),$$

so that we may as well replace t_1 by $t \in (t_0, T)$. Hence, for almost every $\tau \in (0, T)$, also for $\tau = 0$, and for all $t \in (\tau, T)$:

$$\begin{aligned} &\frac{1}{2}(A_{R,\epsilon}(t) - A_{R,\epsilon}(\tau)) + \int_\tau^t \int |\nabla \mathbf{u}|^2 \phi_R w_{\gamma,\epsilon} dx\, ds \\ &\leq -\sum_{i=1}^{3} \int_\tau^t \int (\partial_i \mathbf{u} \cdot \mathbf{u})\, (w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\ &+ \sum_{i=1}^{3} \int_\tau^t \int [(\frac{|\mathbf{u}|^2}{2}) v_i + p u_i](w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\ &- \sum_{1 \leq i,j \leq 3} (\int_\tau^t \int F_{i,j} u_j (w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds - \int_\tau^t \int F_{i,j} \partial_i u_j\, \phi_R\, dx\, ds), \end{aligned} \tag{5.9}$$

Note that all terms in the right side are dominated. In fact, we will treat the most difficult term. Note that as $\gamma \leq 2$, there exists $C_\gamma > 0$, which does not depend on $R > 1$ nor on $\epsilon > 0$, such that

$$|w_{\gamma,\epsilon}\partial_i \phi_R| + |\phi_R \partial_i w_{\gamma,\epsilon}| \leq C_\gamma \frac{w_\gamma(x)}{1 + |x|} \leq C_\gamma w_{3\gamma/2}(x),$$

thus we have

$$|[(\frac{|\mathbf{u}|^2}{2}) v_i + p u_i](w_{\gamma,\epsilon}\partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon})| \leq C(|\mathbf{u}|^2 |\mathbf{v}| + |p|\, |u|) w_{\frac{3\gamma}{2}},$$

where we must study the expression in the right side. For the term $(|p|\, |u|) w_{\frac{3\gamma}{2}}$, by (5.4) and by (5.5) (for $\beta \in \left(\frac{5}{4}, \frac{6\alpha}{10+\alpha}\right)$ and a such that $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$) we have

$$w_\gamma p \in L^a L^\beta + L^2 L^2.$$

Thereafter, by (5.6) taking $\tilde{b}$ such that $\frac{1}{\beta} + \frac{1}{\tilde{b}} = 1$ and $\tilde{a}$ such that $\frac{2}{\tilde{a}} + \frac{3}{\tilde{b}} = \frac{3}{2}$ we get $w_{\frac{\gamma}{2}} |\mathbf{u}| \in L^{\tilde{a}} L^{\tilde{b}} \cap L^2 L^2 \subset L^{\frac{a}{a-1}} L^{\frac{\beta}{\beta-1}} \cap L^2 L^2$. Thus,

$$|p||u| w_{\frac{3\gamma}{2}} \in L^1. \tag{5.10}$$

Similarly, for the term $|\mathbf{u}|^2 |\mathbf{v}| w_{\frac{3\gamma}{2}}$, as $|\mathbf{v}|\, |\mathbf{u}| w_\gamma \in L^a L^\beta$, and $w_{\frac{\gamma}{2}} |\mathbf{u}| \in L^{\frac{a}{a-1}} L^{\frac{\beta}{\beta-1}}$, we conclude $|\mathbf{u}|^2 |\mathbf{v}| w_{\frac{3\gamma}{2}} \in L^1$.

Thus, by dominated convergence, letting R goes to $+\infty$ and then ϵ goes to 0, we find the energy control (5.2). We let t goes to τ in (5.2), so that

$$\limsup_{t \to \tau} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} \leq \|\mathbf{u}(\tau)\|^2_{L^2_{w_\gamma}}.$$

Also, as $\mathbf{u}$ is weakly continuous in $L^2_{w_\gamma}$,

$$\|\mathbf{u}(\tau)\|^2_{L^2_{w_\gamma}} \leq \liminf_{t\to\tau} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}}.$$

Thus $\|\mathbf{u}(\tau)\|^2_{L^2_{w_\gamma}} = \lim_{t\to\tau} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}}$.

It is known that in a Hilbert space X, if x_n converges weakly to x, and $\|x_n\|_n$ converges to $\|x\|$, then x_n converges strongly to x, in effect we can take the limit when n goes to ∞ in the equality

$$\|x_n - x\|^2_X = \|x_n\|^2_X - 2\langle x_n, x\rangle_X + \|x\|^2_X.$$

By the weak continuity of the map $t \mapsto \mathbf{u}(t) \in L^2_{w_\gamma}$ and as $\lim_{t\to\tau^+} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} = \|\mathbf{u}(\tau)\|^2_{L^2_{w_\gamma}}$, for almost every $\tau \in [0, T)$, and for $\tau = 0$, we conclude the right strongly continuity almost everywhere, and continuity at the initial time 0.

Now, in order to obtain (5.3). Consider the energy control (5.2) and the following estimates:

$$\begin{aligned}\left|\int_\tau^t \int \nabla|\mathbf{u}|^2 \cdot \nabla w_\gamma \, ds\, ds\right| \leq & 2\gamma \int_\tau^t \int |\mathbf{u}||\nabla \mathbf{u}|\, w_\gamma \, dx\, ds \\ \leq & \frac{1}{4}\int_\tau^t \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}} ds + 4\gamma^2 \int_\tau^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} ds,\end{aligned}$$

For the pressure, we write $p = p_1 + p_2$ with

$$p_1 = \sum_{i=1}^3 \sum_{j=1}^3 \mathcal{R}_i\mathcal{R}_j(v_i u_j - c_i b_j), \quad p_2 = -\sum_{i=1}^3 \sum_{j=1}^3 \mathcal{R}_i\mathcal{R}_j(F_{i,j}).$$

Since $w_{\frac{2\gamma\alpha}{2+\alpha}} \in \mathcal{A}_{\frac{2\alpha}{2+\alpha}}$ the continuity of the Riesz transforms gives the following control

$$\begin{aligned}\left|\int_\tau^t \int (|\mathbf{u}|^2\mathbf{v} + 2p_1\mathbf{u}) \cdot \nabla w_\gamma \, dx\, ds\right| & C_\gamma \int_\tau^t \int (|\mathbf{u}|^2|\mathbf{v}| + 2|p_1||\mathbf{u}|)\, w_\gamma^{3/2}\, dx\, ds \\ \leq & C_{\alpha,\gamma}\int_\tau^t \|w_\gamma^{1/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} \|w_\gamma|\mathbf{v}||\mathbf{u}|\|_{\frac{2\alpha}{2+\alpha}}\, ds \\ \leq & C_{\alpha,\gamma}\int_\tau^t \|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} \|w_{\gamma/2}\mathbf{u}\|_2 \|w_{\gamma/2}\mathbf{v}\|_\alpha\, ds.\end{aligned}$$

We remark that $\frac{2\alpha}{\alpha-2} \in [5, 6]$. Thus, using the Gagliardo-Nirenberg inequality we obtain

$$\begin{aligned}\|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} \leq & C_{\alpha,\gamma}\|\nabla(w_{\gamma/2}\mathbf{u})\|_2^{\frac{3}{\alpha}} \|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}} \\ \leq & C_{\alpha,\gamma}(\|w_{\gamma/2}\mathbf{u}\|_2 + \|w_{\gamma/2}\nabla\mathbf{u}\|_2)^{\frac{3}{\alpha}} \|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}},\end{aligned}$$

and by the Young inequality with $\frac{3}{2\alpha}+\frac{2\alpha-3}{2\alpha}=1$,

$$
\begin{aligned}
&C_{\alpha,\gamma}\int_\tau^t \|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}}\|w_{\gamma/2}\mathbf{u}\|_2\|w_{\gamma/2}\mathbf{v}\|_\alpha ds\\
&\leq C_{\alpha,\gamma}\int_\tau^t (\|w_{\gamma/2}\mathbf{u}\|_2+\|w_{\gamma/2}\nabla\mathbf{u}\|_2)^{\frac{3}{\alpha}}\|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}}\|w_{\gamma/2}\mathbf{u}\|_2\|w_{\gamma/2}\mathbf{v}\|_\alpha ds\\
&\leq \frac{1}{16}\int_\tau^t (\|w_{\gamma/2}\nabla\mathbf{u}\|_2^2+\|w_{\gamma/2}\mathbf{u}\|_2^2)\,ds + C_{\alpha,\gamma}\int_\tau^t (\|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}}\|w_{\gamma/2}\mathbf{u}\|_2\|w_{\gamma/2}\mathbf{v}\|_\alpha)^{\frac{2\alpha}{2\alpha-3}}\,ds\\
&\leq \frac{1}{16}\int_\tau^t (\|w_{\gamma/2}\nabla\mathbf{u}\|_2^2+\|w_{\gamma/2}\mathbf{u}\|_2^2)\,ds + C_{\alpha,\gamma}\int_\tau^t \|\mathbf{v}\|_{L^\alpha_{w_{\alpha\gamma/2}}}^{\frac{2\alpha}{2\alpha-3}}\|\mathbf{u}\|_{L^2_{w_\gamma}}^2\,ds.
\end{aligned}
$$

We have found

$$
\begin{aligned}
&\Big|\int_\tau^t\int (|\mathbf{u}|^2\mathbf{v}+2p_1\mathbf{u})\cdot\nabla(w_\gamma)\,dx\,ds\Big|\\
&\qquad\leq \frac{1}{4}\int_\tau^t (\|\nabla\mathbf{u}(s)\|_{L^2_{w_\gamma}}^2+\|\mathbf{u}(s)\|_{L^2_{w_\gamma}}^2)\,ds + C_{\alpha,\gamma}\int_\tau^t \|\mathbf{v}(s)\|_{L^\alpha_{w_{\alpha\gamma/2}}}^{\frac{2\alpha}{2\alpha-3}}\|\mathbf{u}(s)\|_{L^2_{w_\gamma}}^2\,ds.
\end{aligned}
$$

On the other hand, since $w_\gamma\in\mathcal{A}_2$, we can write

$$
\begin{aligned}
\left|\int_\tau^t\int p_2\mathbf{u}\cdot\nabla w_\gamma dx\,ds\right| &\leq C_\gamma\int_\tau^t\int |p_2||\mathbf{u}|w_\gamma\,dx\,ds\\
&\leq C_\gamma\int_\tau^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2+\|p_2\|_{L^2_{w_\gamma}}^2\,ds\\
&\leq C_\gamma\int_\tau^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2+\|\mathbb{F}\|_{L^2_{w_\gamma}}^2\,ds.
\end{aligned}
$$

Finally, for the other terms, we have

$$
\begin{aligned}
\left|2\sum_{1\leq i,j\leq 3}\int_\tau^t\int (F_{i,j}(\partial_i u_j)w_\gamma + F_{i,j}u_j\partial_i(w_\gamma))\,dx\,ds\right| &\leq C_\gamma\int_\tau^t\int |\mathbb{F}|(|\nabla\mathbf{u}|+|\mathbf{u}|)w_\gamma\,dx\,ds\\
&\leq \frac{1}{4}\int_\tau^t \|\nabla\mathbf{u}\|_{L^2_{w_\gamma}}^2\,ds + C_\gamma\int_\tau^t \|\mathbf{u}\|_{L^2_{w_\gamma}}^2\,ds + C_\gamma\int_\tau^t \|\mathbb{F}\|_{L^2_{w_\gamma}}^2\,ds.
\end{aligned}
$$

Hence we have found the estimate (5.3) and Lemma 5.1 is proven. ◇

Using the Grönwall inequality, the following corollary is a direct consequence of Theorem 5.1:

Corollary 5.1. *Under the assumptions of Theorem 5.1, we have*

$$
\begin{aligned}
&\sup_{0<t<T}\|\boldsymbol{u}\|_{L^2_{w_\gamma}}^2+\|\nabla\boldsymbol{u}\|_{L^2((0,T),L^2_{w_\gamma})}^2\\
&\leq \left(\|\boldsymbol{u}_0\|_{L^2_{w_\gamma}}^2+C_\gamma\|\mathbb{F}\|_{L^2((0,T),L^2_{w_\gamma})}\right)\, e^{C_{\alpha,\gamma}(T+T^{\frac{2\alpha-5}{2\alpha-3}}\|\boldsymbol{v}\|_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})}^{\frac{2}{2\alpha-3}})}
\end{aligned}
$$

Another important consequence is the following uniqueness result for the advection-diffusion problem.

Corollary 5.2. *. Let $0\leq\gamma\leq 2$. Let $0<T<+\infty$. Let $\boldsymbol{u}_0\in L^2_{w_\gamma}(\mathbb{R}^3)$ be a divergence-free vector field and $\mathbb{F}(t,x)=\big(F_{i,j}(t,x)\big)_{1\leq i,j\leq 3}$ be a tensor such that $\mathbb{F}(t,x)\in L^2((0,T),L^2_{w_\gamma})$. Let $\alpha\in[3,\frac{10}{3}]$ and let $\boldsymbol{v}\in L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})$ be a time-dependent divergence free vector-fields. Assume moreover that $\boldsymbol{v}\in L^2_tL^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$.*

Let $(\boldsymbol{u}_1, p_1)$ and $(\boldsymbol{u}_2, p_2)$ be two solutions of the advection-diffusion problem

$$\begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - \nabla p + \nabla \cdot \mathbb{F}, \\ \nabla \cdot \boldsymbol{u} = 0, \ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \end{cases}$$

which satisfies for $k \in \{1, 2\}$:

- *$\boldsymbol{u}_k$ belongs to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \boldsymbol{u}_k$ belongs to $L^2((0,T), L^2_{w_\gamma})$ and p_k is a distribution on $(0,T) \times \mathbb{R}^3$*
- *the map $t \in [0, +\infty) \mapsto \boldsymbol{u}_k(t)$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*

Then $(\boldsymbol{u}_1, p_1) = (\boldsymbol{u}_2, p_2)$.

Proof.
We know that p_k satisfy

$$p_k = \sum_{1 \le i,j \le 3} \mathcal{R}_i \mathcal{R}_j (u_{k,i} v_j - F_{i,j}).$$

Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, $p = p_1 - p_2$. Then we have

$$\begin{cases} \partial_t \mathbf{w} = \Delta \mathbf{w} - (\mathbf{v} \cdot \nabla)\mathbf{w} - \nabla p, \\ \nabla \cdot \mathbf{w} = 0, \ \mathbf{w}(0, \cdot) = 0, \end{cases}$$

Consider $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha})$ and a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$. For all compact subset K of $(0,T) \times \mathbb{R}^3$, we have $\mathbf{w} \otimes \mathbf{v}$ to $L^2_t L^2_x(K)$, and to $L^a((0,T), L^\beta_{w_{\beta\gamma}})$ by (5.4).

We will verify that $\partial_t \mathbf{w}$ is locally $L^2 H^{-1}$. Let $\varphi, \psi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ such that $\psi = 1$ on the neighborhood of the support of φ. We write

$$\varphi p = \varphi \mathcal{R} \otimes \mathcal{R}(\psi(\mathbf{v} \otimes \mathbf{w})) + \varphi \mathcal{R} \otimes \mathcal{R}((1-\psi)(\mathbf{v} \otimes \mathbf{w})).$$

We get

$$\|\varphi \mathcal{R} \otimes \mathcal{R}(\psi(\mathbf{v} \otimes \mathbf{w}))\|_{L^2 L^2} \le C_{\varphi,\psi} \|\psi(\mathbf{v} \otimes \mathbf{w})\|_{L^2 L^2(\mathrm{Supp}(\psi))} < +\infty$$

and

$$\|\varphi \mathcal{R} \otimes \mathcal{R}((1-\psi)(\mathbf{v} \otimes \mathbf{w}))\|_{L^a L^\infty} \le C_{\varphi,\psi} \|(\mathbf{v} \otimes \mathbf{w})\|_{L^a L^\beta_{w_{\beta\gamma}}} < +\infty$$

with

$$C_{\varphi,\psi} \le C \|\varphi\|_\infty \|1-\psi\|_\infty \sup_{x \in \mathrm{Supp}\, \varphi} \left(\int_{y \in \mathrm{Supp}\,(1-\psi)} \left(\frac{(1+|y|)^\gamma}{|x-y|^3} \right)^{\frac{\beta}{\beta-1}} \right)^{1-\frac{1}{\beta}} < +\infty,$$

where we have used the fact that $(3-\gamma)\frac{\beta}{\beta-1} > 3$. Thus, we may take the scalar product of $\partial_t \mathbf{w}$ with $\mathbf{w}$ in order to find that

$$\partial_t (\frac{|\mathbf{w}|^2}{2}) = \Delta (\frac{|\mathbf{w}|^2}{2}) - |\nabla \mathbf{w}|^2 - \nabla \cdot \left((\frac{|\mathbf{w}|^2}{2}) \mathbf{v} \right) - \nabla \cdot (p\mathbf{w}).$$

The assumptions of Theorem 5.1 are satisfied then we use Corollary 5.1 to find that $\mathbf{w} = 0$ and consequently $p = 0$. ◇

We will use the following control.

Corollary 5.3. *Assume the hypothesis of Theorem 5.1. If* $\boldsymbol{v}$ *is controlled by* $\mathbf{u}$ *in the following sense: for every* $t \in (0, T)$,

$$\|\boldsymbol{v}(t)\|^2_{L^\alpha_{w_{\alpha\gamma/2}}} \leq C_{0,\alpha,\gamma} \|\boldsymbol{u}(t)\|^{2(\frac{3}{2}-\frac{3}{\alpha})}_{L^2_{w_\gamma}} (\|\boldsymbol{u}(t)\|^2_{L^2_{w_\gamma}} + \|\nabla \boldsymbol{u}(t)\|^2_{L^2_{w_\gamma}})^{(\frac{3}{\alpha}-\frac{1}{2})},$$

then there exists a constant $\mathbf{C}_{\alpha,\gamma} \geq 1$ *such that if* $T_0 < T$ *satisfies*

$$\mathbf{C}_{\alpha,\gamma} \left(1 + \|\boldsymbol{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, ds\right)^2 T_0 \leq 1$$

we have

$$\sup_{0 \leq t \leq T_0} \|\boldsymbol{u}(t)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\nabla \boldsymbol{u}(s)\|^2_{L^2_{w_\gamma}} \, ds \leq \mathbf{C}_{\alpha,\gamma}(1 + \|\boldsymbol{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, ds).$$

Proof. Since (5.3),

$$\begin{aligned} &\|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} + \int_0^t \|\nabla \mathbf{u}(s)\|^2_{L^2_{w_\gamma}} \, ds \\ \leq &\|\mathbf{u}(0)\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^t \|\mathbb{F}(s)\|^2_{L^2_{w_\gamma}} \, ds + C_{\alpha,\gamma} \int_0^t \|\mathbf{v}(s)\|^{\frac{2\alpha}{2\alpha-3}}_{L^\alpha_{w_{\alpha\gamma/2}}} \|\mathbf{u}(s)\|^2_{L^2_{w_\gamma}} \, ds. \end{aligned}$$

As we have

$$\|\mathbf{v}(s)\|^{\frac{2\alpha}{2\alpha-3}}_{L^\alpha_{w_{\alpha\gamma/2}}} \leq C_{0,\alpha,\gamma} \|\mathbf{u}\|^{\frac{3(\alpha-2)}{2\alpha-3}}_{L^2_{w_\gamma}} (\|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}})^{\frac{6-\alpha}{2(2\alpha-3)}},$$

and using again the Young inequalities with $\frac{6-\alpha}{2(2\alpha-3)} + \frac{5\alpha-12}{2(2\alpha-3)} = 1$ we obtain

$$\begin{aligned} \|\mathbf{v}(s)\|^{\frac{2\alpha}{2\alpha-3}}_{L^\alpha_{w_{\alpha\gamma/2}}} \|\mathbf{u}(s)\|^2_{L^2_{w_\gamma}} \leq &C_{\alpha,\gamma} \|\mathbf{u}\|^{\frac{7\alpha-12}{2\alpha-3}}_{L^2_{w_\gamma}} (\|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}})^{\frac{6-\alpha}{2(2\alpha-3)}} \\ \leq &\frac{1}{16} (\|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}}) + C_{\alpha,\gamma} \|\mathbf{u}\|^{\frac{2(7\alpha-12)}{5\alpha-12}}_{L^2_{w_\gamma}}. \end{aligned}$$

Thus, we find

$$\begin{aligned} &\|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} + \frac{1}{2} \int_0^t \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}} \, ds \\ \leq &\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + C_\gamma \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, ds + C_{\alpha,\gamma} \int_0^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|\mathbf{u}\|^{\frac{2(7\alpha-12)}{5\alpha-12}}_{L^2_{w_\gamma}} \, ds. \end{aligned}$$

As $\frac{2(7\alpha-12)}{5\alpha-12} \leq 6$, and moreover, using Lemma 3.5 with the function $\alpha(t) = \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} + \frac{1}{2} \int_0^t \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}} \, ds$, we get for a constant $\mathbf{C}_{\alpha,\gamma} > 0$, if

$$T_0 \leq \frac{1}{\mathbf{C}_{\alpha,\gamma} \left(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, ds\right)^2}$$

then

$$\sup_{0\leq t\leq T_0} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\nabla \mathbf{u}(s)\|^2_{L^2_{w_\gamma}}\, ds \leq \mathbf{C}_{\alpha,\gamma}(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds).$$

$\diamond$

5.2 Solutions to the advection-diffusion problem

In this section, we establish the existence of solutions for the linearized problem. First, we consider the following stability result:

Theorem 9. *Let $0 \leq \gamma \leq 2$. Let $0 < T < +\infty$. Let $\boldsymbol{u}_{0,n} \in L^2_{w_\gamma}(\mathbb{R}^3)$ be divergence-free vector fields. Let $\mathbb{F}_n \in L^2((0,T), L^2_{w_\gamma})$ be tensors. Let $\alpha \in [3, \frac{10}{3}]$ and let $\boldsymbol{v}_n$ be time-dependent divergence free vector-fields wich belong to $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$.*

Let $(\boldsymbol{u}_n, p_n)$ be solutions of the following advection-diffusion problems

$$(LNS_n)\begin{cases} \partial_t \boldsymbol{u}_n = \Delta \boldsymbol{u}_n - (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n, \\ \nabla \cdot \boldsymbol{u}_n = 0, \ \boldsymbol{u}_n(0,\cdot) = \boldsymbol{u}_{0,n} \end{cases}$$

verifying the same hypothesis of Theorem 5.1.

If $\boldsymbol{u}_{0,n}$ is strongly convergent to $\boldsymbol{u}_{0,\infty}$ in $L^2_{w_\gamma}$, if the sequence $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$, and moreover, if the sequence $\boldsymbol{v}_n$ is bounded in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$, then there exists $\boldsymbol{u}_\infty, \boldsymbol{v}_\infty, p_\infty$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that

- *$\boldsymbol{u}_{n_k}$ converges *-weakly to $\boldsymbol{u}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla \boldsymbol{u}_{n_k}$ converges weakly to $\nabla \boldsymbol{u}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$.*
- *$\boldsymbol{v}_{n_k}$ converges weakly to $\boldsymbol{v}_\infty$ in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$, p_{n_k} converges weakly to p_∞ in $L^a((0,T_0), L^\beta_{w_{\gamma\beta}}) + L^2((0,T_0), L^2_{w_\gamma})$, for a parameter $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha})$ and for a parameter a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$.*
- *$\boldsymbol{u}_{n_k}$ converges strongly to $\boldsymbol{u}_\infty$ in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have*

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\boldsymbol{u}_{n_k}(s,y) - \boldsymbol{u}_\infty(s,y)|^2\, ds\, dy = 0.$$

Moreover, $(\boldsymbol{u}_\infty, p_\infty)$ is a solution of the problem

$$(LNS_\infty)\begin{cases} \partial_t \boldsymbol{u}_\infty = \Delta \boldsymbol{u}_\infty - (\boldsymbol{v}_\infty \cdot \nabla)\boldsymbol{u}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \\ \nabla \cdot \boldsymbol{u}_\infty = 0, \ \boldsymbol{u}_\infty(0,\cdot) = \boldsymbol{u}_{0,\infty}, \end{cases}$$

and verifies the hypothesis of Theorem 5.1.

Proof.

By Theorem 5.1 and Corollary 5.1, we know that $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$. In particular, writing $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{n,i}u_{n,j}), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(F_{n,i,j}),$$

we obtain, by (5.4) and (5.5), that $p_{n,1}$ is bounded in $L^a((0,T), L^\beta_{w_{\beta\gamma}})$ and $p_{n,2}$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. We have that the sequence $\varphi \mathbf{u}_n$ is bounded in $L^2((0,T), H^1)$. Moreover, by the controls for p_n, we get that $\varphi \partial_t \mathbf{u}_n$ is bounded in $L^2L^2 + L^2W^{-1,\beta} + L^2H^{-1} \subset L^2((0,T), H^{\frac{1}{2}-\frac{3}{\beta}}) \subset L^2((0,T), H^{-2})$. Thus, by a Rellich-Lions lemma there exists $\mathbf{u}_\infty \in L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$ and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}_\infty$ in $L^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2 \, dy \, ds = 0.$$

As $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$ we have that $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$ and we have that $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$.

Using the Banach–Alaoglu's theorem, there exists $\mathbf{v}_\infty$ such that $\mathbf{v}_{n_k}$ converges weakly to $\mathbf{v}_\infty$ in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$. In particular, we have that the term $v_{n_k,i}u_{n_k,j}$ is weakly convergent in $(L^{6/5}L^{6/5})_{\mathrm{loc}}$ and thus in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$. As by (5.4), this term is bounded in $L^a((0,T), L^\beta_{w_{\beta\gamma}})$, it is weakly convergent in $L^a((0,T), L^\beta_{w_{\beta\gamma}})$.

Let us write $p_\infty = p_{\infty,1} + p_{\infty,2}$ with

$$p_{\infty,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{\infty,i}u_{\infty,j}), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(F_{\infty,i,j}).$$

As the Riesz transforms are bounded on the spaces $L^\beta_{w_{\beta\gamma}}$ and $L^2_{w_\gamma}$, we find that $p_{n_k,1}$ is weakly convergent in $L^a((0,T), L^\beta_{w_{\beta\gamma}})$ to $p_{\infty,1}$, and moreover, we find that $p_{n_k,2}$ is strongly convergent in $L^2((0,T), L^2_{w_\gamma})$ to $p_{\infty,2}$.

With those facts, we obtain that $(\mathbf{u}_\infty, p_\infty)$ verifies in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$:

$$\partial_t \mathbf{u}_\infty = \Delta \mathbf{u}_\infty - (\mathbf{v}_\infty \cdot \nabla)\mathbf{u}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \nabla \cdot \mathbf{u}_\infty = 0,$$

In particular, $\partial_t \mathbf{u}_\infty$ belongs locally to the space $L^2_t H^{-2}_x$, and then this function has a representative such that $t \mapsto \mathbf{u}_\infty(t,\cdot)$ is continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ and coincides with $\mathbf{u}_\infty(0,\cdot) + \int_0^t \partial_t \mathbf{u}_\infty \, ds$. We have necessarily $\mathbf{u}_\infty(0,\cdot) = \mathbf{u}_{0,\infty}$ and thus $\mathbf{u}_\infty$ is a solution of (LNS_∞).

Following, we define

$$\begin{aligned} A_{n_k} =& -\partial_t(\frac{|\mathbf{u}_{n_k}|^2}{2}) + \Delta(\frac{|\mathbf{u}_{n_k}|^2}{2}) - \nabla \cdot \left((\frac{|\mathbf{u}_{n_k}|^2}{2})\mathbf{v}_{n_k} \right) \\ & - \nabla \cdot (p_{n_k}\mathbf{u}_{n_k}) + \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}), \end{aligned}$$

and in order to take the limit when n_k goes to ∞, we remark that, as u_{ϵ_k} is locally strongly convergent in L^2L^2; and locally bounded in $L^\infty L^2$, it is locally strongly convergent in $L^{p'}L^2$, for $p' < \infty$. Then, since $w_{\frac{\gamma}{2}}\nabla \otimes \mathbf{u}_{n_k}$ is bounded in $L^2((0,T), L^2)$, by the Gagliardo-Nirenberg interpolation inequalities we obtain $\mathbf{u}_{n_k}$ is locally strongly convergent in $L^{p'}L^{q'}$ with $\frac{2}{p'} + \frac{3}{q'} > \frac{3}{2}$.

We know that p_{n_k} is locally weakly convergent in L^aL^β and $\mathbf{u}_{n_k}$ is locally strongly convergent in $L^{\frac{a}{a-1}}L^{\frac{\beta}{\beta-1}}$ since $2(1-\frac{1}{a}) + 3(1-\frac{1}{\beta}) = \frac{13}{2} - \frac{5}{\alpha} > \frac{3}{2}$, then $p_{n_k}\mathbf{u}_{n_k}$ converges in the sense of distributions.

With these remarks we conclude that A_{n_k} converges to A_∞ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ where

$$A_\infty = -\partial_t(\frac{|\mathbf{u}_\infty|^2}{2}) + \Delta(\frac{|\mathbf{u}_\infty|^2}{2}) - \nabla \cdot \left((\frac{|\mathbf{u}_\infty|^2}{2})\mathbf{v}_\infty \right) - \nabla \cdot (p_\infty \mathbf{u}_\infty) + \mathbf{u}_\infty \cdot (\nabla \cdot \mathbb{F}_\infty).$$

Moreover, by hypothesis there exists μ_{n_k} a non-negative locally finite measure on $(0,T)\times\mathbb{R}^3$ such that

$$\begin{aligned} \partial_t(\frac{|\mathbf{u}_{n_k}|^2}{2}) =& \Delta(\frac{|\mathbf{u}_{n_k}|^2}{2}) - |\nabla\mathbf{u}_{n_k}|^2 - \nabla \cdot \left((\frac{|\mathbf{u}_{n_k}|^2}{2})\mathbf{v}_{n_k} \right) \\ & - \nabla \cdot (p_{n_k}\mathbf{u}_{n_k}) + \mathbf{u}_{n_k} \cdot (\nabla \cdot \mathbb{F}_{n_k}) - \mu_{n_k}. \end{aligned}$$

Since the definition of A_{n_k} we can write $A_{n_k} = |\nabla\mathbf{u}_{n_k}|^2 + \mu_{n_k}$, and thus

$$A_\infty = \lim_{n_k\to+\infty} |\nabla\mathbf{u}_{n_k}|^2 + \mu_{n_k}.$$

By weak convergence, we have for a non-negative function $\Phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$

$$\begin{aligned} \iint A_\infty\Phi\,dx\,ds =& \lim_{n_k\to+\infty}\iint A_{n_k}\Phi\,dx\,ds \geq \limsup_{n_k\to+\infty}\iint |\nabla\mathbf{u}_{n_k}|^2\Phi\,dx\,ds \\ \geq& \iint |\nabla\mathbf{u}_\infty|^2\Phi\,dx\,ds. \end{aligned}$$

Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$ such that $A_\infty = |\nabla\mathbf{u}_\infty|^2 + \mu_\infty$, and then we obtain

$$\begin{aligned} \partial_t(\frac{|\mathbf{u}_\infty|^2}{2}) =& \Delta(\frac{|\mathbf{u}_\infty|^2}{2}) - |\nabla\mathbf{u}_\infty|^2 - \nabla \cdot \left((\frac{|\mathbf{u}_\infty|^2}{2})\mathbf{v}_\infty \right) \\ & - \nabla \cdot (p_\infty\mathbf{u}_\infty) + \mathbf{u}_\infty \cdot (\nabla \cdot \mathbb{F}_\infty) - \mu_\infty. \end{aligned}$$

As in (5.9) with the functions $(\mathbf{u}_{n_k}, p_{n_k})$ and with $a = 0$, and moreover, taking the limsup when $n_k \to +\infty$ we have

$$
\begin{aligned}
&\limsup_{n_k\to+\infty}\left(\int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,dx+\int_0^t\int |\nabla \mathbf{u}_{n_k}|^2\ \phi_R w_{\gamma,\epsilon} dx\,ds\right)\\
&\le \int \frac{|\mathbf{u}_{0,\infty}(x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,dx\\
&-\sum_{i=1}^3\int_0^t\int \partial_i \mathbf{u}_\infty\cdot \mathbf{u}_\infty\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&+\sum_{i=1}^3\int_0^t\int [(\frac{|\mathbf{u}_\infty|^2}{2})v_{\infty,i}+p_\infty u_{\infty,i}](w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&-\sum_{1\le i,j\le 3}(\int_0^t\int F_{\infty,i,j}u_{\infty,j}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds-\int_0^t\int F_{\infty,i,j}\partial_i u_{\infty,j}\ \phi_R\,dx\,ds).
\end{aligned}
$$

Now, recall that $\mathbf{u}_{n_k}=\mathbf{u}_{0,n_k}+\int_0^t \partial_t \mathbf{u}_{n_k}\,ds$ and then we see that for all $t\in(0,T)$, $\mathbf{u}_{n_k}(t,\cdot)$ converges to $\mathbf{u}_\infty(t,\cdot)$ in $\mathcal{D}'(\mathbb{R}^3)$. Moreover, as $\mathbf{u}_{n_k}(t,\cdot)$ is bounded in $L^2_{w_\gamma}(\mathbb{R}^3)$ we get that $\mathbf{u}_{n_k}(t,\cdot)$ converges to $\mathbf{u}_\infty(t,\cdot)$ in $L^2_{\text{loc}}(\mathbb{R}^3)$. Then, we have obtained that

$$
\int \frac{|\mathbf{u}_\infty(t,x)|^2}{2}\phi_R w_{\gamma,\epsilon}\,dx\le \limsup_{n_k\to+\infty}\int\frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,dx.
$$

On the other hand, as $\nabla\mathbf{u}_{n_k}$ is weakly convergent to $\nabla\mathbf{u}_\infty$ in $L^2_t L^2_{w_\gamma}$, we have

$$
\int_0^t\int\frac{|\nabla\mathbf{u}_\infty(s,x)|^2}{2}\phi_R w_{\gamma,\varepsilon}\,dx\,ds\le\limsup_{n_k\to+\infty}\int_0^t\int|\nabla\mathbf{u}_{n_k}|^2\ \phi_R w_{\gamma,\epsilon}dx\,ds.
$$

Thus, taking the limit when $R\to 0$ and $\varepsilon\to 0$, for every $t\in(0,T)$ we get :

$$
\begin{aligned}
&\|\mathbf{u}_\infty(t)\|^2_{L^2_{w_\gamma}}+2\int_0^t\|\nabla\mathbf{u}_\infty(s)\|^2_{L^2_{w_\gamma}}\,ds\\
\le&\|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}}-\int_0^t\int(\nabla|\mathbf{u}_\infty|^2)\cdot\nabla w_\gamma\,dx\,ds\\
&+\int_0^t\int[(\frac{|\mathbf{u}_\infty|^2}{2})\mathbf{v}]\cdot\nabla w_\gamma\,dx\,ds+2\int_0^t\int p_\infty\mathbf{u}_\infty\cdot\nabla w_\gamma dx\,ds\\
&-\sum_{1\le i,j\le 3}(\int_0^t\int F_{\infty,i,j}(\partial_i u_{\infty,j})w_\gamma\,dx\,ds+\int_0^t\int F_{\infty,i,j}u_{\infty,i}\partial_j(w_\gamma)\cdot\nabla w_\gamma\,dx\,ds).
\end{aligned}
$$

Now, in this estimate we take the limsup when t goes to 0 in order to find

$$
\lim_{t\to 0}\|\mathbf{u}_\infty(t)\|^2_{L^2_{w_\gamma}}=\|\mathbf{u}_{0,\infty}\|^2_{L^2_{w_\gamma}}.
$$

which implies strongly convergence of the solution to the initial data (since we have weak convergence and convergence of the norms in a Hilbert space). The proof is thus finished. ◇

Theorem 10. *Let $\alpha\in[3,\frac{10}{3}]$ and $\gamma\in[0,\frac{6}{\alpha})$. Let $\boldsymbol{u}_0$ be divergence-free vector fields such that $\boldsymbol{u}_0\in L^2_{w_\gamma}(\mathbb{R}^3)$. Let $\mathbb{F}$ be a tensor such that $\mathbb{F}\in L^2((0,+\infty),L^2_{w_\gamma})$. Let $\boldsymbol{v}$ be a time dependent divergence free vector-field which fulfills for every $T>0$, $\boldsymbol{v}\in L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})$.*

Then, the problem

$$(LNS) \begin{cases} \partial_t \boldsymbol{u} = \Delta \boldsymbol{u} - (\boldsymbol{v} \cdot \nabla) \boldsymbol{u} - \nabla p + \nabla \cdot \mathbb{F}, \\ \nabla \cdot \boldsymbol{u} = 0, \ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \end{cases}$$

has a solution $(\boldsymbol{u}, p)$ *which satisfies the assumptions of Lemma 5.1.*

Proof. For the initial data $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0) \in L^2(\mathbb{R}^3)$, for the forcing tensors $\mathbb{F}_R = \phi_R \mathbb{F} \in L^2((0,T), L^2)$, and for $\mathbf{v}_R = \mathbb{P}(\phi_R \mathbf{v})$, we consider the solution $(\mathbf{u}_{R,\epsilon}, p_{R,\epsilon})$ of the approximated system

$$\begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{v}_R * \theta_\epsilon) \cdot \nabla) \mathbf{u}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_R, \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0, \ \mathbf{u}_{R,\epsilon}(0, \cdot) = \mathbf{u}_{0,R}, \end{cases}$$

where $\mathbf{u}_{R,\epsilon} \in \mathcal{C}([0,T), L^2(\mathbb{R}^3)) \cap L^2([0,T), \dot{H}^1(\mathbb{R}^3))$ for every $0 < T < +\infty$, and $(\mathbf{u}_{R,\epsilon}, p_{R,\epsilon})$ verify all the assumptions of Theorem 5.1 (with energy equality). Since $\frac{\alpha\gamma}{2} < 3$, we have

$$\|\mathbf{v}_R * \theta_\epsilon\|_{L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})} < C \|\mathbf{v}_R\|_{L^\alpha_{w_{\alpha\gamma/2}}} < C \|\mathbf{v}\|_{L^\alpha_{w_{\alpha\gamma/2}}}. \tag{5.11}$$

Let R_n be a sequence converging to $+\infty$ and ϵ_n a sequence converging to 0 and let us denote $\mathbf{u}_{0,n} = \mathbf{u}_{0,R_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{v}_n = \mathbf{v}_{R_n} * \epsilon_n$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\epsilon_n}$.

As $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_0$ in $L^2_{w_\gamma}$, $\mathbb{F}_n$ is strongly convergent to $\mathbb{F}$ in $L^2((0,T), L^2_{w_\gamma})$, and moreover, as $\mathbf{v}_n$ is bounded in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$ (since we have (5.11)), we can apply Theorem 9, so there exists $(\mathbf{u}, \mathbf{V}, p)$ and there exists an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in $\mathbb{N}$ such that:

- $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T_0), L^2_{w_\gamma})$, $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}$ in $L^2((0,T_0), L^2_{w_\gamma})$.
- $\mathbf{v}_{n_k}$ converges weakly to $\mathbf{V}$ in $L^\alpha((0,T_0), L^\alpha_{w_{\alpha\gamma/2}})$, the sequence p_{n_k} converges weakly to p in $L^a((0,T_0), L^\beta_{w_{\gamma\beta}}) + L^2((0,T_0), L^2_{w_\gamma})$, for a parameter $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha})$ and a parameter a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$.
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\text{loc}}([0,T_0) \times \mathbb{R}^3)$,

and moreover, $(\mathbf{u}, p)$ is a solution of the advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{V} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F}, \\ \nabla \cdot \mathbf{u} = 0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0. \end{cases}$$

which verifies:

- the map $t \in [0,T_0) \mapsto \mathbf{u}(t)$ is weakly continuous from $[0,T_0)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$.
- there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that we have the local energy balance

$$\partial_t \left(\frac{|\mathbf{u}|^2}{2}\right) = \Delta \left(\frac{|\mathbf{u}|^2}{2}\right) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2} \mathbf{V}\right) - \nabla \cdot (p \mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

If we verify that $\mathbf{V} = \mathbf{v}$ the proof is finished. As $\mathbf{v}_n = \theta_{\epsilon_n} * (\mathbf{v}_n - \mathbf{v}) + \theta_{\epsilon_n} * \mathbf{v}$, then we verify that $\mathbf{v}_{n_k}$ is convergent to $\mathbf{v}$ in $\mathcal{D}'((0,T_0)\times\mathbb{R}^3)$. Thus we have $\mathbf{V} = \mathbf{v}$. ◇

5.3 The mollified linearized problem.

We fix $1 < \lambda < +\infty$.

Let θ be an radially decreasing function in $\mathcal{D}(\mathbb{R}^3)$ with $\int \theta\, dx = 1$, in particular θ is non-negative. We define

$$\theta_{\epsilon,t}(x) = \frac{1}{(\epsilon\sqrt{t})^3}\,\theta(\frac{x}{\epsilon\sqrt{t}}). \tag{5.12}$$

We look for a discretely selfsimilar solution of the mollified problem

$$(NS_\epsilon)\begin{cases} \partial_t \mathbf{u}_\epsilon = \Delta \mathbf{u}_\epsilon - ((\mathbf{u}_\epsilon * \theta_{\epsilon,t})\cdot\nabla)\mathbf{u}_\epsilon - \nabla p_\epsilon + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\mathbf{u}_\epsilon = 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \end{cases}$$

we refocus the problem in the search of a fixed point for the application $\mathbf{v} \mapsto L_\epsilon(\mathbf{v})$ where $L_\epsilon(\mathbf{v})$ is a solution of the mollified linearized problem

$$(LNS_\epsilon)\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - ((\mathbf{v} * \theta_{\epsilon,t})\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0, \end{cases}$$

Lemma 5.2. *Let $\alpha \in [3, \frac{10}{3}]$ and $\gamma \in [1, \frac{6}{\alpha})$. Let $\boldsymbol{u}_0$ be a λ-DSS divergence-free vector field which belong to $L^2_{w_\gamma}(\mathbb{R}^3)$. Let $\mathbb{F}$ be a λ-DSS tensor wich satisfies $\mathbb{F} \in L^2_{loc}((0,+\infty), L^2_{w_\gamma})$. Moreover, let $\boldsymbol{v}$ be a λ-DSS time-dependent divergence free vector-field such that for every $T > 0$, $\boldsymbol{v} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$.*

Then, the linearized mollified problem (LNS_ϵ) has a unique solution $(\boldsymbol{u}, p)$ which satisfies all the conclusions of Theorem 10. Moreover, the function $\boldsymbol{u}$ is a λ-DSS vector field.

Proof. As we have $|\mathbf{v}(t,.) * \theta_{\epsilon,t}| \le \mathcal{M}_{\mathbf{v}(t,.)}$, we find

$$\|\mathbf{v}(t) * \theta_{\epsilon,t}\|_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})} \le C_{\alpha,\gamma}\|\mathbf{v}\|_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})}.$$

Theorem 10 provides a solution $(\mathbf{u}, p)$ on the interval of time $(0,T)$. Moreover, as $\mathbf{v} * \theta_{\epsilon,t}$ belongs the space to $L^2_t L^\infty_x(K)$ for every compact subset K of $(0,T)\times\mathbb{R}^3$, we can use Corollary 5.2 to conclude that this solution $(\mathbf{u}, p)$ is unique.

We will prove that this solution is λ-DSS. Let $\tilde{\mathbf{u}}(t,x) = \frac{1}{\lambda}\mathbf{u}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ and $\tilde{p}(t,x) = \frac{1}{\lambda^2}p(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. Remark that $\mathbf{v} * \theta_{\epsilon,t}$ is λ-DSS. In fact,

$$\begin{aligned} \lambda(\mathbf{v} * \theta_{\epsilon,t})(\lambda^2 t, \lambda x) &= \lambda \int_{\mathbb{R}^3} \mathbf{v}(\lambda^2 t, \lambda x - y)\frac{1}{(\epsilon\sqrt{\lambda^2 t})^3}\theta(\frac{y}{\epsilon\sqrt{\lambda^2 t}})dy \\ &= \int_{\mathbb{R}^3} \lambda\mathbf{v}(\lambda^2 t, \lambda x - \lambda y)\frac{1}{(\epsilon\sqrt{\lambda^2 t})^3}\theta(\frac{\lambda y}{\epsilon\sqrt{\lambda^2 t}})\lambda^3 dy \\ &= \int_{\mathbb{R}^3} \mathbf{v}(t, x-y)\frac{1}{(\epsilon\sqrt{t})^3}\theta(\frac{y}{\epsilon\sqrt{t}}) \\ &= (\mathbf{v} * \theta_{\epsilon,t})(t,x). \end{aligned}$$

Then, we get $(\tilde{\mathbf{u}}, \tilde{p})$, is a solution of (LNS_ϵ) on $(0, T)$. Thus, we have the identities $(\tilde{\mathbf{u}}, \tilde{p}) = (\mathbf{u}, p)$ from which we conclude that $(\mathbf{u}, p)$ is λ-DSS.

$\diamond$

5.4 The mollified Navier–Stokes equations.

For $\alpha \in [3, \frac{10}{3}]$ and for $\mathbf{v} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$ the terms $\mathbf{u}$ of the solution provided by Lemma 5.2 belongs to $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$ by interpolation. Then the map $L_{\epsilon,\alpha} : \mathbf{v} \mapsto \mathbf{u}$ where $L_{\epsilon,\alpha}\mathbf{v} = \mathbf{u}$ is well defined from

$$X_{T,\gamma,\alpha} = \{\mathbf{v} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}}) \ / \ \mathbf{v} \text{ is } \lambda - \text{DSS}\}$$

to $X_{T,\gamma,\alpha}$. At this point, we introduce the following technical lemmas:

Lemma 5.3. *For $\gamma > \frac{10}{\alpha} - 2$, $X_{T,\gamma,\alpha}$ is a Banach space, and the norms $\|v\|_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})}$ and $\|v\|_{L^\alpha((0,T/\lambda^2)\times B(0,\frac{1}{\lambda}))}$ are equivalents.*

Proof. Using a change of variables, we get

$$\int_0^T \int_{B(0,1)} |\mathbf{v}(t,x)|^\alpha \, dx \, dt = \lambda^{5-\alpha} \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{v}(t,x)|^\alpha \, dx \, dt$$

and, for $k \in \mathbb{N}$,

$$\int_0^T \int_{\lambda^{k-1}<|x|<\lambda^k} |\mathbf{v}(t,x)|^\alpha \, dx \, dt = \lambda^{(5-\alpha)(k+1)} \int_0^{\frac{T}{\lambda^{2(k+1)}}} \int_{\frac{1}{\lambda^2}<|x|<\frac{1}{\lambda}} |\mathbf{v}(t,x)|^\alpha \, dx \, dt.$$

then splitting the integral in spatial variable we find

$$\begin{aligned} &\|\mathbf{v}\|^\alpha_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})} \\ &= \int_0^T \int_{B(0,1)} |\mathbf{v}(t,x)|^\alpha w_{\alpha\gamma/2} \, dx \, dt + \sum_{k=1}^{+\infty} \int_0^T \int_{\lambda^{k-1}<|x|<\lambda^k} |\mathbf{v}(t,x)|^\alpha w_{\alpha\gamma/2} \, dx \, dt \\ &\leq \lambda^{5-\alpha-\frac{\alpha\gamma}{2}} \|\mathbf{v}\|^\alpha_{L^\alpha((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))} + C \sum_{k=1}^{+\infty} \lambda^{(5-\alpha-\frac{\alpha\gamma}{2})(k+1)} \|\mathbf{v}\|^\alpha_{L^\alpha((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))} \end{aligned}$$

Since $\gamma > \frac{10}{\alpha} - 2$ we have $5 - \alpha - \frac{\alpha\gamma}{2} < 0$. Thus, $\sum_{k=1}^{+\infty} \lambda^{(5-\alpha-\frac{\alpha\gamma}{2})(k+1)} < +\infty$, and

$$\|\mathbf{v}\|^\alpha_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})} \leq C \|\mathbf{v}\|^\alpha_{L^\alpha((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))}.$$

The other estimate is clear

$$\|\mathbf{v}\|^\alpha_{L^\alpha((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))} \leq \|\mathbf{v}\|^\alpha_{L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})}.$$

Remark 5.1. *In the condition $\gamma > \frac{10}{\alpha} - 2$ in Lemma 5.3, we observe that values of α close to $\frac{10}{3}$ permit to consider values of γ close to 1.*

Lemma 5.4. *Let $\alpha \in [3, \frac{10}{3})$ and $\gamma \in (\frac{10}{\alpha} - 2, \frac{6}{\alpha})$, the mapping $L_{\epsilon,\alpha}$ is continuous and compact on $X_{T,\gamma,\alpha}$.*

Proof. Let $\mathbf{v}_n$ be a bounded sequence in $X_{T,\gamma,\alpha}$ and let $\mathbf{u}_n = L_{\epsilon,\alpha}\mathbf{v}_n$. Remark that, as $\frac{\alpha\gamma}{2} < 3$ the sequence $\mathbf{v}_n(t) * \theta_{\epsilon,t}$ is bounded in $X_{T,\gamma,\alpha}$. Using Theorem 5.1 and

Corollary 5.1 we have that the sequence $\mathbf{u}_n$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and moreover $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

Thus, by Theorem 9 there exists $\mathbf{u}_\infty$, p_∞, $\mathbf{V}_\infty$, and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in $\mathbb{N}$ such that we have:

- $\mathbf{u}_{n_k}$ converges *-weakly to $\mathbf{u}_\infty$ in $L^\infty((0,T), L^2_{w_\gamma})$, $\nabla\mathbf{u}_{n_k}$ converges weakly to $\nabla\mathbf{u}_\infty$ in $L^2((0,T), L^2_{w_\gamma})$.
- $\mathbf{v}_{n_k} * \theta_{\epsilon,t}$ converges weakly to $\mathbf{V}_\infty$ in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$.
- p_{n_k} converges weakly to p in $L^a((0,T_0), L^\beta_{w_{\gamma\beta}}) + L^2((0,T_0), L^2_{w_\gamma})$, for a parameter $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha})$ and a parameter a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$.
- $\mathbf{u}_{n_k}$ converges strongly to $\mathbf{u}_\infty$ in $L^2_{\text{loc}}([0,T)\times\mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every $R > 0$, we have

$$\lim_{k\to+\infty} \int_0^{T_0} \int_{|y|<R} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^2\, ds\, dy = 0,$$

- and

$$\begin{cases} \partial_t \mathbf{u}_\infty = \Delta\mathbf{u}_\infty - (\mathbf{V}_\infty\cdot\nabla)\mathbf{u}_\infty - \nabla p_\infty + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\mathbf{u}_\infty = 0,\ \ \mathbf{u}_{0,\infty} = \mathbf{u}_0. \end{cases}$$

We will prove the compactness of $L_{\epsilon,\alpha}$. As before $\sqrt{w_\gamma}\mathbf{u}_n$ is bounded in $L^{10/3}((0,T)\times\mathbb{R}^3)$ by interpolation hence strong convergence of $\mathbf{u}_{n_k}$ in $L^2_{\text{loc}}([0,T)\times\mathbb{R}^3)$ implies the strong convergence of $\mathbf{u}_{n_k}$ in $L^\alpha_{\text{loc}}((0,T)\times\mathbb{R}^3)$.

Moreover, we have that $\mathbf{u}_\infty$ is still λ-DSS (a stable property under weak limits). With these information we obtain that $\mathbf{u}_\infty \in X_{T,\gamma,\alpha}$ and

$$\lim_{n_k\to+\infty} \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{u}_{n_k}(s,y) - \mathbf{u}_\infty(s,y)|^\alpha\, ds\, dy = 0,$$

which proves that $L_{\epsilon,\alpha}$ is compact.

Now, we prove the continuity of $L_{\epsilon,\alpha}$. Let $\mathbf{v}_n$ be such that $\mathbf{v}_n$ is convergent to $\mathbf{v}_\infty$ in $X_{T,\gamma,\alpha}$. Then if we take a convergent subsequence of $\mathbf{u}_n = L_{\epsilon,\alpha}\mathbf{v}_n$ with limit $\mathbf{u}_\infty$, we necessarily have $\mathbf{u}_\infty = L_{\epsilon,\alpha}(\mathbf{v}_\infty)$, thus the relatively compact sequence $\mathbf{u}_n$ can have only one limit point which is $L_{\epsilon,\alpha}(\mathbf{v}_\infty)$, hence $\mathbf{u}_n$ is converges to $L_{\epsilon,\alpha}(\mathbf{v}_\infty)$. Then $L_{\epsilon,\alpha}$ is continuous. ◇

Lemma 5.5. *Let $\alpha \in [3, \frac{10}{3})$. Let $\gamma \in (\frac{10}{\alpha} - 2, \frac{6}{\alpha})$. If $\mu \in [0,1]$ and $\boldsymbol{u}$ solves $\boldsymbol{u} = \mu L_{\epsilon,\alpha}(\boldsymbol{u})$ then*

$$\|\boldsymbol{u}\|_{X_{T,\gamma,\alpha}} \leq C_{\boldsymbol{u}_0,\mathbb{F},\gamma,\alpha,T,\lambda}$$

where the constant $C_{\boldsymbol{u}_0,\mathbb{F},\gamma,\alpha,T,\lambda}$ depends only on $\boldsymbol{u}_0$, $\mathbb{F}$, γ, α, T and λ (but not on μ nor on ϵ).

Proof. We let $\tilde{\mathbf{u}} = \frac{1}{\mu}\mathbf{u}$, so that

$$\begin{cases} \partial_t\tilde{\mathbf{u}} = \Delta\tilde{\mathbf{u}} - ((\mathbf{u} * \theta_{\epsilon,t})\cdot\nabla)\tilde{\mathbf{u}} - \frac{1}{\mu}\nabla p + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\tilde{\mathbf{u}} = 0,\ \ \tilde{\mathbf{u}}(0,\cdot) = \mathbf{u}_0, \end{cases}$$

where $\nabla p = \nabla(\sum_i \sum_j \mathcal{R}_i\mathcal{R}_j[(\mathbf{u}_i * \theta_{\epsilon,t})\mathbf{u}_j] - \mu F_{i,j})$.

Multiplying by μ, we obtain

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - ((\mathbf{u} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} - \nabla p + \nabla \cdot \mu \mathbb{F}, \\ \nabla \cdot \mathbf{u} = 0, \ \mathbf{u}(0,\cdot) = \mu \mathbf{u}_0. \end{cases}$$

We consider the constant $\mathbf{C}_{\alpha,\gamma}$ given in the Corollary 5.3, and we take $T_0 \in (0,T)$ such that

$$\mathbf{C}_{\alpha,\gamma}\left(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \le 1,$$

which implies

$$\mathbf{C}_{\alpha,\gamma}\left(1 + \|\mu\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mu\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds\right)^2 T_0 \le 1,$$

then we have the controls

$$\begin{aligned} \sup_{0\le t\le T_0} \|\mathbf{u}(t)\|^2_{L^2_{w_\gamma}} &+ \int_0^{T_0} \|\nabla \mathbf{u}\|^2_{L^2_{w_\gamma}}\, ds \\ &\le \mathbf{C}_{\alpha,\gamma}(1 + \mu^2\|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \mu^2 \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds) \\ &\le \mathbf{C}_{\alpha,\gamma}(1 + \|\mathbf{u}_0\|^2_{L^2_{w_\gamma}} + \int_0^{T_0} \|\mathbb{F}\|^2_{L^2_{w_\gamma}}\, ds) \end{aligned}$$

In particular, by interpolation

$$\int_0^{T_0} \|\mathbf{u}\|^\alpha_{L^\alpha_{w_{\alpha\gamma/2}}}\, ds$$

is bounded by a constant $C_{\mathbf{u}_0,\mathbb{F},\gamma,\alpha}$ and we can go back from T_0 to T, using the self-similarity property. ⋄

Lemma 5.6. *Let $\alpha \in [3, \frac{10}{3})$. Let $\gamma \in (\frac{10}{\alpha} - 2, \frac{6}{\alpha})$. There is at least one solution $\boldsymbol{u}_\epsilon$ of the problem $\boldsymbol{u}_\epsilon = L_{\epsilon,\alpha}(\boldsymbol{u}_\epsilon)$.*

Proof. The uniform estimates for the fixed points of the application μL_ϵ for $0 \le \mu \le 1$ given by Lemma 5.5, and Lemma 5.4 permit to apply Leray–Schauder principle and Schaefer theorem to find a solution of the problem $\mathbf{u}_\epsilon = L_{\epsilon,\alpha}(\mathbf{u}_\epsilon)$. ⋄

5.5 Existence of discretely self-similar solutions.

Proof of Theorem 8

As $\gamma \in (1,2)$ we know that $\mathbf{u}_0 \in L^2_{w_\gamma}$. We take $\alpha \in [3, \frac{10}{3})$ such that $\gamma \in (\frac{10}{\alpha} - 2, \frac{6}{\alpha})$.

We consider $\mathbf{u}_\epsilon$ solutions of the problem $\mathbf{u}_\epsilon = L_{\epsilon,\alpha}(\mathbf{u}_\epsilon)$ given by Lemma 5.6.

Lemma 5.5, shows that $\mathbf{u}_\epsilon * \theta_{\epsilon,t}$ is bounded in $L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$. Theorem 5.1 and Corollary 5.1 allow us to conclude that $\mathbf{u}_\epsilon$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \mathbf{u}_\epsilon$ is bounded in $L^2((0,T), L^2_{w_\gamma})$.

Theorem 9 provides $\mathbf{u}$, $\mathbf{v}$, p and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ converging to 0, such that

- $\mathbf{u}_{\epsilon_k}$ converges *-weakly to $\mathbf{u}$ in $L^\infty((0,T),L^2_{w_\gamma})$, $\nabla\mathbf{u}_{\epsilon_k}$ converges weakly to $\nabla\mathbf{u}$ in $L^2((0,T),L^2_{w_\gamma})$
- $\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t}$ converges weakly to $\mathbf{v}$ in $L^\alpha((0,T),L^\alpha_{w_{\alpha\gamma/2}})$
- p_{ϵ_k} converges weakly to p in $L^a((0,T_0),L^\beta_{w_{\gamma\beta}}) + L^2((0,T_0),L^2_{w_\gamma})$, for a parameter $\beta \in \left(\frac{5}{4},\frac{6\alpha}{10+\alpha}\right)$ and a parameter a verifying $\frac{2}{a}+\frac{3}{\beta}=\frac{5}{\alpha}+\frac{3}{2}$.
- $\mathbf{u}_{\epsilon_k}$ converges strongly to $\mathbf{u}$ in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$
- and
$$\begin{cases} \partial_t\mathbf{u} = \Delta\mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{u} - \nabla p + \nabla\cdot\mathbb{F}, \\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}_0 = \mathbf{u}_0. \end{cases}$$

The proof is finished if $\mathbf{v}=\mathbf{u}$. As we have $\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t} = (\mathbf{u}_{\epsilon_k}-\mathbf{u}) * \theta_{\epsilon_k,t} + \mathbf{u} * \theta_{\epsilon_k,t}$. We remark that $\mathbf{u} * \theta_{\epsilon,t}$ converges strongly in $L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$ as ϵ goes to 0 and then $\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t}$ converges to $\mathbf{u}$ in $\mathcal{D}'$. ◇

Chapter 6

The incompressible magneto-hydrodynamics equations

It is natural to look for extend the results obtained for the (NS) equations to the more general setting of the coupled magneto-hydrodynamics system:

$$\text{(MHD)}\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} - \nabla q, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

where the fluid velocity $\mathbf{u} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$, the fluid magnetic field $\mathbf{b} : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$, the fluid pressure $p : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$, and $q : [0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ are the unknowns. On the other hand, the data of the problem are given by the fluid velocity at $t = 0$: $\mathbf{u}_0 : \mathbb{R}^3 \to \mathbb{R}^3$; the magnetic field at $t = 0$, $\mathbf{b}_0 : \mathbb{R}^3 \to \mathbb{R}^3$; and the forcing tensor $\mathbb{F}$.

Observe that Theorem 3 gives sufficient conditions to determine the pressure terms through the formulas

$$p = \sum_{i,j} \mathcal{R}_i\mathcal{R}_j(u_iu_j - b_ib_j - F_{i,j})$$

$$q = \sum_{i,j} \mathcal{R}_i\mathcal{R}_j(u_ib_j - b_iu_j) = 0.$$

Note that the proofs in this chapter also work if one introduces different constants in front of $\Delta\mathbf{u}$ and $\Delta\mathbf{b}$.

6.1 DSS solutions

It is not complicated to verify that the theory developed for discretely self-similar solutions in the Chapter 5 can be generalized to the case of (MHD) equations. We will only give a light about how it works.

First, we look for solutions of the linearized system

$$\text{(MHDL)} \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{c}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{v}\cdot\nabla)\mathbf{b} + (\mathbf{c}\cdot\nabla)\mathbf{u} - \nabla q, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

Proposition 6.1. *Let $\alpha \in [3, \frac{10}{3}]$ and $\gamma \in [0, \frac{6}{\alpha})$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0$ be divergence-free vector fields such that $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$. Let $\mathbb{F} = (F_{i,j})_{1\le i,j\le 3}$ be a tensor belonging to $L^2((0,+\infty), L^2_{w_\gamma})$. Let $\boldsymbol{v}, \boldsymbol{c}$ be time dependent divergence free vector-fields such that, $\boldsymbol{v}, \boldsymbol{c} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$, for every $T > 0$. Then, the advection-diffusion problem (MHDL) has a global solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ which satisfies*

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\boldsymbol{u}$, $\nabla\boldsymbol{b}$ belong to $L^2((0,T), L^2_{w_\gamma})$*
- *The distributions p and q are related with $\boldsymbol{u}$, $\boldsymbol{b}$, $\boldsymbol{v}$, $\boldsymbol{c}$ and $\mathbb{F}$ by*

$$p = \sum_{i,j} \mathcal{R}_i\mathcal{R}_j(v_i u_j - c_i b_j - F_{i,j})$$

and

$$q = \sum_{i,j} \mathcal{R}_i\mathcal{R}_j(v_i b_j - c_i u_j)$$

- *the map $t \in [0,+\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$:*
- *the solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0,+\infty)\times\mathbb{R}^3$ such that*

$$\begin{aligned} \partial_t\Big(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\Big) =& \Delta\Big(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\Big) - |\nabla\boldsymbol{u}|^2 - |\nabla\boldsymbol{b}|^2 - \nabla\cdot\left(\Big(\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2}\Big)\boldsymbol{v}\right) \\ & - \nabla\cdot(p\boldsymbol{u}) - \nabla\cdot(q\boldsymbol{b}) + \nabla\cdot((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c}) \\ & + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu. \end{aligned} \tag{6.1}$$

Sketch of the proof
We consider the mollified problem

$$\begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{v}_R * \theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon} + ((\mathbf{c}_R * \theta_\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla\cdot\mathbb{F}_R, \\ \partial_t \mathbf{b}_{R,\epsilon} = \Delta \mathbf{b}_{R,\epsilon} - ((\mathbf{v}_R * \theta_\epsilon)\cdot\nabla)\mathbf{b}_{R,\epsilon} + ((\mathbf{c}_R * \theta_\epsilon)\cdot\nabla)\mathbf{u}_{R,\epsilon} - \nabla q_{R,\epsilon}, \\ \nabla\cdot\mathbf{u}_{R,\epsilon} = 0,\ \nabla\cdot\mathbf{b}_{R,\epsilon} = 0, \\ \mathbf{u}_{R,\epsilon}(0,\cdot) = \mathbf{u}_{0,R},\ \mathbf{b}_{R,\epsilon}(0,\cdot) = \mathbf{b}_{0,R}, \end{cases}$$

where $(\mathbf{u}_{0,R}, \mathbf{b}_{0,R}) = (\mathbb{P}(\phi_R\mathbf{u}_0), \mathbb{P}(\phi_R\mathbf{b}_0)) \in L^2(\mathbb{R}^3)$, $\mathbb{F}_R = \phi_R\mathbb{F} \in L^2((0,T), L^2)$, and $(\mathbf{v}_R, \mathbf{c}_R) = \mathbb{P}(\phi_R\mathbf{v}, \phi_R\mathbf{c})$.

We find solutions $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon}) \in \mathcal{C}([0,T), L^2(\mathbb{R}^3)) \cap L^2([0,T), \dot{H}^1(\mathbb{R}^3))$ which satisfy the following energy equality, where we omit the subscripts,

$$\begin{aligned}\partial_t(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}) =&\Delta(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}) - |\nabla\mathbf{u}|^2 - |\nabla\mathbf{b}|^2 - \nabla\cdot\left((\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})(\mathbf{v}*\theta_\epsilon)\right)\\ &-\nabla\cdot(p\mathbf{u}) - \nabla\cdot(q\mathbf{b}) + \nabla\cdot((\mathbf{u}\cdot\mathbf{b})(\mathbf{c}*\theta_\epsilon))\\ &+\mathbf{u}\cdot(\nabla\cdot\mathbb{F})\end{aligned}$$

Then, we pass to the limit when R goes to $+\infty$ and ϵ goes to 0, using the a priori controls, described in the following Proposition, and compactness. ⋄

To simplify the notation, for a Banach space $X \subset \mathcal{D}'$ of vector fields endowed with a norm $\|\cdot\|_X$, we denote

$$\|(\mathbf{u},\mathbf{v})\|_X^2 = \|\mathbf{u}\|_X^2 + \|\mathbf{v}\|_X^2, \qquad \text{and} \qquad \|\nabla(\mathbf{u},\mathbf{v})\|_X^2 = \|\nabla\mathbf{u}\|_X^2 + \|\nabla\mathbf{v}\|_X^2.$$

Proposition 6.2. *Let $0 \le \gamma \le 2$ and $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ be a divergence-free vector fields and let $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$ be a tensor. Consider $\alpha \in [3, \frac{10}{3}]$ and $\boldsymbol{v}, \boldsymbol{c} \in L^\alpha((0,T), L^\alpha_{w_{\alpha\gamma/2}})$ two time-dependent divergence free vector-fields.*

Let $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ be a solution of the following advection-diffusion problem (MHDL). which satisfies :

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla\boldsymbol{u}$, $\nabla\boldsymbol{b}$ belong to $L^2((0,T), L^2_{w_\gamma})$*
- *p, q are distributions on $(0,T) \times \mathbb{R}^3$*
- *the map $t \in [0,+\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ is weakly continuous from $[0,+\infty)$ to $L^2_{w_\gamma}$, and is strongly continuous at $t = 0$*
- *the solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$ is suitable : there exists a non-negative locally finite measure μ on $(0,+\infty) \times \mathbb{R}^3$ such that*

$$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}) =&\Delta(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - |\nabla\boldsymbol{b}|^2 - \nabla\cdot\left((\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v}\right)\\ &-\nabla\cdot(p\boldsymbol{u}) - \nabla\cdot(q\boldsymbol{b}) + \nabla\cdot((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c})\\ &+\boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.\end{aligned}$$

Then we have the following controls:

- *If $0 < \gamma \le 2$, then for almost every $\tau \ge 0$, and for $\tau = 0$, we have for all $t \ge \tau$,*

$$\begin{aligned}&\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2_{w_\gamma}} + 2\int_\tau^t (\|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}})ds\\ &\le \|(\boldsymbol{u},\boldsymbol{b})(a)\|^2_{L^2_{w_\gamma}} - \int_\tau^t\int \nabla(|\boldsymbol{u}|^2+|\boldsymbol{b}|^2)\cdot\nabla w_\gamma\,dx\,ds\\ &+\int_\tau^t\int [(\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v}]\cdot\nabla w_\gamma\,dx\,ds + 2\int_\tau^t\int p\boldsymbol{u}\cdot\nabla w_\gamma dx\,ds\\ &+2\int_\tau^t\int q\boldsymbol{b}\cdot\nabla w_\gamma dx\,ds + 2\int_\tau^t\int [(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c}]\cdot\nabla w_\gamma\,dx\,ds\\ &-2\sum_{1\le i,j\le 3}(\int_\tau^t\int F_{i,j}(\partial_i u_j)w_\gamma\,dx\,ds + \int_\tau^t\int F_{i,j}u_j\partial_i w_\gamma\,dx\,ds),\end{aligned} \tag{6.2}$$

the map $t \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$, from $[0, +\infty)$ to $L^2_{w_\gamma}$, is right strongly continuous almost everywhere, and moreover

$$\begin{aligned}\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2_{w_\gamma}} &+ \int_\tau^t \|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}}\,ds \\ &\le \|(\boldsymbol{u},\boldsymbol{b})(\tau)\|^2_{L^2_{w_\gamma}} + C_\gamma \int_\tau^t \|\mathbb{F}(s)\|^2_{L^2_{w_\gamma}}\,ds \\ &\quad + C_{\alpha,\gamma}\int_\tau^t \|(\boldsymbol{v},\boldsymbol{c})(s)\|^{\frac{2\alpha}{2\alpha-3}}_{L^\alpha_{w_{\alpha\gamma/2}}}\|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2_{w_\gamma}}\,ds.\end{aligned} \tag{6.3}$$

- *If $\gamma = 0$, then for almost all $\tau \ge 0$ and for $\tau = 0$, we have for all $t \ge \tau$,*

$$\begin{aligned}\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{L^2} &+ 2\int_\tau^t (\|\nabla(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{L^2})ds \\ &\le \|(\boldsymbol{u},\boldsymbol{b})(\tau)\|^2_{L^2} + 2\sum_{1\le i,j\le 3}\int_\tau^t\int F_{i,j}\partial_i u_j\,dx\,ds,\end{aligned}$$

the map $t \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$, from $[0, +\infty)$ to $L^2_{w_\gamma}$, is right strongly continuous almost everywhere.

Sketch of the proof

As all the terms of order three in the right side of the energy balance are written in a divergence form, we can introduce the weights w_γ as for (NS) to obtain a priori controls.

We will just to illustrate how to pass from (6.2) to (6.3). The rest is very similar to the case of the Navier–Stokes equations.

First, remark that the gradient terms $(\nabla p, \nabla q)$ are *necessarily* related to $(\mathbf{u}, \mathbf{b}, \mathbf{v}, \mathbf{c})$ and $\mathbb{F}$ through the Riesz transforms $\mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formulas

$$\nabla p = \nabla\left(\sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_j(u_iv_j - b_ic_j - F_{i,j})\right),$$

and

$$\nabla q = \nabla\left(\sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_j(v_ib_j - c_iu_j)\right),$$

where :

For $\beta \in (\frac{5}{4}, \frac{6\alpha}{10+\alpha}) \subset (\frac{5}{4}, \frac{3}{2})$ and a verifying $\frac{2}{a} + \frac{3}{\beta} = \frac{5}{\alpha} + \frac{3}{2}$ (so that $a \in (2, \frac{20\alpha}{50-9\alpha})$) we have

$$\begin{aligned}\sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_j(u_iv_j - b_ic_j) &\in L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^d)),\\ \sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_j(v_ib_j - c_iu_j) &\in L^a((0,T), L^\beta_{w_{\beta\gamma}}(\mathbb{R}^d)).\end{aligned}$$

and

$$\sum_{1\le i,j\le 3}\mathcal{R}_i\mathcal{R}_jF_{i,j} \in L^2((0,T), L^2_{w_\gamma}).$$

We write $p = p_1 + p_2$ where

$$p_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_iu_j - c_ib_j), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(F_{i,j}),$$

Since $w_{\frac{2\gamma\alpha}{2+\alpha}} \in \mathcal{A}_{\frac{2\alpha}{2+\alpha}}$ we have the following control

$$\begin{aligned}
&\left|\int_\tau^t \int (|\mathbf{u}|^2\mathbf{v} + |\mathbf{b}|^2\mathbf{v} + 2((\mathbf{u}\cdot\mathbf{b})\mathbf{c}) + 2p_1\mathbf{u} + 2q\mathbf{b})\cdot\nabla w_\gamma\, dx\, ds\right| \\
\leq & C_\gamma \int_\tau^t \int (|\mathbf{u}|^2|\mathbf{v}| + |\mathbf{b}|^2|\mathbf{v}| + 2|\mathbf{u}||\mathbf{b}||\mathbf{c}| + 2|p_1||\mathbf{u}| + 2|q_1||\mathbf{b}|)\, w_\gamma^{3/2}\, dx\, ds \\
\leq & C_{\alpha,\gamma} \int_\tau^t \|w_\gamma^{1/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} (\|w_\gamma|\mathbf{v}||\mathbf{u}|\|_{\frac{2\alpha}{2+\alpha}} + \|w_\gamma|\mathbf{c}|\,|\mathbf{b}|\|_{\frac{2\alpha}{2+\alpha}})\, ds \\
& + C_{\alpha,\gamma} \int_\tau^t \|w_\gamma^{1/2}\mathbf{b}\|_{\frac{2\alpha}{\alpha-2}} (\|w_\gamma|\mathbf{b}||\mathbf{v}|\|_{\frac{2\alpha}{2+\alpha}} + \|w_\gamma|\mathbf{c}|\,|\mathbf{u}|\|_{\frac{2\alpha}{2+\alpha}})\, ds \\
\leq & C_{\alpha,\gamma} \int_\tau^t \|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} (\|w_{\gamma/2}\mathbf{u}\|_2 \|w_{\gamma/2}\mathbf{v}\|_\alpha + \|w_{\gamma/2}\mathbf{b}\|_2\, \|w_\gamma\mathbf{c}\|_\alpha)\, ds \\
& + C_{\alpha,\gamma} \int_\tau^t \|w_{\gamma/2}\mathbf{b}\|_{\frac{2\alpha}{\alpha-2}} (\|w_{\gamma/2}\mathbf{b}\|_2 \|w_{\gamma/2}\mathbf{v}\|_\alpha + \|w_{\gamma/2}\mathbf{u}\|_2 \|w_{\gamma/2}\mathbf{c}\|_\alpha)\, ds.
\end{aligned}$$

At this point we need to estimate each term above but, for the sake of simplicity, we will only treat one term in the right side since the other terms follow the same estimates. We remark that $\frac{2\alpha}{\alpha-2} \in [5,6]$, thus, using the Gagliardo-Nirenberg inequality we find

$$\begin{aligned}
\|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} &\leq C_{\alpha,\gamma} \|\nabla(w_{\gamma/2}\mathbf{u})\|_2^{\frac{3}{\alpha}} \|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}} \\
&\leq C_{\alpha,\gamma} (\|w_{\gamma/2}\mathbf{u}\|_2 + \|w_{\gamma/2}\nabla\mathbf{u}\|_2)^{\frac{3}{\alpha}} \|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}},
\end{aligned}$$

and then, using the Young inequalities with $\frac{3}{2\alpha} + \frac{2\alpha-3}{2\alpha} = 1$ we obtain

$$\begin{aligned}
& C_{\alpha,\gamma} \int_\tau^t \|w_{\gamma/2}\mathbf{u}\|_{\frac{2\alpha}{\alpha-2}} \|w_{\gamma/2}\mathbf{b}\|_2 \|w_{\gamma/2}\mathbf{c}\|_\alpha ds \\
\leq\ & C_{\alpha,\gamma} \int_\tau^t (\|w_{\gamma/2}\mathbf{u}\|_2 + \|w_{\gamma/2}\nabla\mathbf{u}\|_2)^{\frac{3}{\alpha}} \|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}} \|w_{\gamma/2}\mathbf{b}\|_2 \|w_{\gamma/2}\mathbf{c}\|_\alpha ds \\
\leq\ & \frac{1}{16} \int_\tau^t (\|w_{\gamma/2}\nabla\mathbf{u}\|_2^2 + \|w_{\gamma/2}\mathbf{u}\|_2^2)\, ds + C_{\alpha,\gamma} \int_\tau^t (\|w_{\gamma/2}\mathbf{u}\|_2^{\frac{\alpha-3}{\alpha}} \|w_{\gamma/2}\mathbf{b}\|_2 \|w_{\gamma/2}\mathbf{c}\|_\alpha)^{\frac{2\alpha}{2\alpha-3}}\, ds \\
\leq\ & \frac{1}{16} \int_\tau^t (\|w_{\gamma/2}\nabla\mathbf{u}\|_2^2 + \|w_{\gamma/2}\mathbf{u}\|_2^2)\, ds + C_{\alpha,\gamma} \int_\tau^t \|\mathbf{c}\|_{L^\alpha_{w_{\alpha\gamma/2}}}^{\frac{2\alpha}{2\alpha-3}} (\|\mathbf{u}\|_{L^2_{w_\gamma}}^2 + \|\mathbf{b}\|_{L^2_{w_\gamma}}^2)\, ds.
\end{aligned}$$

Treating the other terms in the same way we are able to write

$$\begin{aligned}
&|\int_\tau^t \int (|\mathbf{u}|^2\mathbf{v} + |\mathbf{b}|^2\mathbf{v} + 2((\mathbf{u}\cdot\mathbf{b})\mathbf{c}) + 2p_1\mathbf{u} + 2q_1\mathbf{b})\cdot\nabla(w_\gamma)\, dx\, ds| \\
&\leq \frac{1}{4} \int_\tau^t (\|\nabla(\mathbf{u},\mathbf{b})(s)\|_{L^2_{w_\gamma}}^2 + \|(\mathbf{u},\mathbf{b})(s)\|_{L^2_{w_\gamma}}^2)\, ds \\
&\quad + C_{\alpha,\gamma} \int_\tau^t \|(\mathbf{v},\mathbf{c})(s)\|_{L^\alpha_{w_{\alpha\gamma/2}}}^{\frac{2\alpha}{2\alpha-3}} \|(\mathbf{u},\mathbf{b})(s)\|_{L^2_{w_\gamma}}^2\, ds.
\end{aligned}$$

On the other hand, since $w_\gamma \in \mathcal{A}_2$, we can write

$$\left| \int_\tau^t \int p_2 \mathbf{u} \cdot \nabla w_\gamma dx\, ds \right| \leq C_\gamma \int_\tau^t \int |p_2||\mathbf{u}| w_\gamma \, dx\, ds$$
$$\leq C_\gamma \int_\tau^t \|\mathbf{u}\|^2_{L^2_{w_\gamma}} + \|p_2\|^2_{L^2_{w_\gamma}} \, ds$$
$$\leq C_\gamma \int_\tau^t \|(\mathbf{u}, \mathbf{b})\|^2_{L^2_{w_\gamma}} + \|\mathbb{F}\|^2_{L^2_{w_\gamma}} \, ds.$$

The other terms are easier to treat.

$\diamond$

The rest of procedure to obtain discretely selfsimilar solutions works without significant changes. We apply the Leray–Schauder principle and Schaefer theorem to find a point fixe of the application $L_{\epsilon,\alpha}$ which sends a pair of discretely selfsimilar vector fields $(\mathbf{v}, \mathbf{c})$ into the discretely selfsimilar solution $(\mathbf{u}, \mathbf{b})$ of the mollified linearized problem

$$(LMHD_\epsilon) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} + ((\mathbf{c} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} - \nabla p + \nabla \cdot \mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{b} + ((\mathbf{c} * \theta_{\epsilon,t}) \cdot \nabla)\mathbf{u} - \nabla q \\ \nabla \cdot \mathbf{u} = 0, \ \nabla \cdot \mathbf{b} = 0, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \mathbf{b}(0, \cdot) = \mathbf{b}_0, \end{cases}$$

and we passe to the limit. As in the case of the Navier–Stokes equations, we find

$$p = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j})$$

and

$$q = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (u_i b_j - b_i u_j) = 0$$

Remark that the structure of the function $\theta_{\epsilon,t}$ defined in (5.12) permits to transfer the selfsimilarity property to the solution. Then, we arrive as in the case of the Navier–Stokes equations to the following conclusion.

Theorem 11. *We consider a real number $\lambda > 1$ and we let $\boldsymbol{u}_0, \boldsymbol{b}_0$ be two λ-DSS vector fields, and be locally L^2.*

Let $\gamma \in (1,2)$. We consider a λ-DSS tensor $\mathbb{F}$ which belongs to $L^2_{loc}((0,+\infty), L^2_{w_\gamma})$. Then, the (MHD) equations has a global weak solution $(\boldsymbol{u}, \boldsymbol{b}, p, q)$, with $q = 0$, such that :

- *$\boldsymbol{u}, \boldsymbol{b}$ are λ-DSS vector fields*
- *the distributions p and q is related with $\boldsymbol{u}$, $\boldsymbol{b}$ and $\mathbb{F}$ by*

$$p = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j})$$

$$q = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (u_i b_j - b_i u_j) = 0$$

- *for every $0 < T < +\infty$, $\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T), L^2_{w_\gamma})$ and $\nabla \boldsymbol{u}, \nabla \boldsymbol{b}$ belong to $L^2((0,T), L^2_{w_\gamma})$*

- *the map* $t \in [0,+\infty) \mapsto (\boldsymbol{u}(t), \boldsymbol{b}(t))$ *is weakly continuous from* $[0,+\infty)$ *to* $L^2_{w_\gamma}$*, and is strongly continuous at* $t = 0$
- $(\boldsymbol{u}, \boldsymbol{b}, p)$ *is suitable : it verifies the local energy inequality (6.1).*

In the next section we present a new result which we have not still addressed in the context of the Navier–Stokes equations.

6.2 Weak-strong uniqueness in weighted spaces

The study of uniqueness of Leray weak solutions for the Navier–stokes equations remains an outstanding open problem, so research community has attired the attention to look for supplementary assumptions in order to ensure the uniqueness of solutions. This kind of statements are known as *weak-strong uniqueness results*.

We complement the study of the (MHD) equations in the framework of weighted L^2 spaces with a weak-strong uniqueness theorem. This result, is obtained in the setting of the multiplier space $\mathbb{X}_T$ which we introduce as follows:

For a time $0 < T < +\infty$ fix, let us denote E_T the *energy space* of the time-dependent vector fields $\mathbf{v}$ belonging to $L^\infty((0,T), L^2)$ and such that $\nabla \mathbf{v}$ belongs to $L^2((0,T), L^2)$. E_T is doted by the norm

$$\|\mathbf{v}\|^2_{E_T} = \sup_{0 \leq t \leq T} \|\mathbf{v}(t,.)\|^2_{L^2} + \int_0^T \|\nabla \mathbf{v}(s,.)\|^2_{L^2} ds.$$

Then, we define $\mathbb{X}_T$ the space of pointwise multipliers on $(0,T) \times \mathbb{R}^3$ from E_T to $L^2((0,T), L^2)$, which is a Banach space with the norm:

$$\|\mathbf{u}\|_{\mathbb{X}_T} = \sup_{\|\mathbf{v}\|_{E_T} \leq 1} \|\mathbf{u}\mathbf{v}\|_{L^2((0,T),L^2)}.$$

Moreover, we define $\mathbb{X}^{(0)}_T$ the space of multipliers $\mathbf{u} \in \mathbb{X}_T$ such that for every $t_0 \in [0,T)$ we have

$$\lim_{t_1 \to t_0^+} \|\mathbb{1}_{(t_0,t_1)}(t)\mathbf{u}(t,\cdot)\|_{\mathbb{X}_T} = 0.$$

The multiplier space $\mathbb{X}^{(0)}_T$ gives us a natural and general framework to prove a weak-strong uniqueness criterion. More precisely, based on the classical Prodi-Serrin's type condition (Prodi, 1959; Serrin, 1962) for the (NS) equations, we obtain the following result.

Theorem 12 (Weak-strong uniqueness)**.** *Let* $0 \leq \gamma \leq 2$*. Let* $0 < T < +\infty$*. Let* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ *be divergence-free vector fields, and moreover, consider a forcing tensor* $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$*.*

Let $(\boldsymbol{u}, \boldsymbol{b}, p)$ *and* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}}, \tilde{p})$ *two solutions of the (MHD) system, with initial data* $\boldsymbol{u}_0$*,* $\boldsymbol{b}_0$*, forcing tensor* $\mathbb{F}$*, and such that :*

- $\boldsymbol{u}, \boldsymbol{b}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{b}}$ *belong to the space* $L^\infty((0,T), L^2_{w_\gamma})$ *and* $\nabla \boldsymbol{u}, \nabla \tilde{\boldsymbol{u}}, \nabla \boldsymbol{b}, \nabla \tilde{\boldsymbol{b}} \in L^2((0,T), L^2_{w_\gamma})$

- *the maps* $t \in [0,T) \mapsto (\boldsymbol{u},\boldsymbol{b})(t,.)$ *and* $t \in [0,T) \mapsto (\tilde{\boldsymbol{u}},\tilde{\boldsymbol{b}})(t,.)$ *are weakly continuous from* $[0,T)$ *to* $L^2_{w_\gamma}(\mathbb{R}^3)$*, and are strongly continuous at* $t=0$ *:*

$$\lim_{t\to 0}\|(\boldsymbol{u}(t,.)-\boldsymbol{u}_0,\boldsymbol{b}(t,.)-\boldsymbol{b}_0)\|_{L^2_{w_\gamma}}=0,$$

and

$$\lim_{t\to 0}\|(\tilde{\boldsymbol{u}}(t,.)-\boldsymbol{u}_0,\tilde{\boldsymbol{b}}(t,.)-\boldsymbol{b}_0)\|_{L^2_{w_\gamma}}=0.$$

- *there exist non-negative locally finite measures* μ *and* ν *on* $(0,T)\times\mathbb{R}^3$ *such that*

$$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}) =&\Delta(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2})-|\nabla\boldsymbol{u}|^2-|\nabla\boldsymbol{b}|^2-\nabla\cdot\left([\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2}+p]\boldsymbol{u}\right)\\&+\nabla\cdot[(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{b}]+\boldsymbol{u}\cdot(\nabla\cdot\mathbb{F})-\mu,\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\partial_t(\frac{|\tilde{\boldsymbol{u}}|^2+|\tilde{\boldsymbol{b}}|^2}{2}) =&\Delta(\frac{|\tilde{\boldsymbol{u}}|^2+|\tilde{\boldsymbol{b}}|^2}{2})-|\nabla\tilde{\boldsymbol{u}}|^2-|\nabla\tilde{\boldsymbol{b}}|^2-\nabla\cdot\left([\frac{|\tilde{\boldsymbol{u}}|^2}{2}+\frac{|\tilde{\boldsymbol{b}}|^2}{2}+\tilde{p}]\tilde{\boldsymbol{u}}\right)\\&+\nabla\cdot[(\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{b}})\tilde{\boldsymbol{b}}]+\tilde{\boldsymbol{u}}\cdot(\nabla\cdot\mathbb{F})-\nu.\end{aligned} \tag{6.5}$$

If $\boldsymbol{u},\boldsymbol{b}\in\mathbb{X}^{(0)}_T$ *and if the product* $\boldsymbol{u}\cdot\nabla\tilde{p}$ *is well defined as a distribution, in the sense that* $\nabla\tilde{p}$ *belongs to* L^1_{loc} *and the pointwise product* $\boldsymbol{u}\cdot\nabla\tilde{p}\in L^1_{loc}$*, then we have* $(\boldsymbol{u},\boldsymbol{b},p)=(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{b}},\tilde{p})$*.*

P.G. Lemarié-Rieusset used the space $\mathbb{X}^{(0)}_T$ to prove a weak-strong uniqueness criterion of weak Leray solutions (see the Theorem 12.4, page 359 of the book (Lemarié-Rieusset, 2016)). He proves some examples of embeddings $E\subset\mathbb{X}^{(0)}_T$.

We may observe that we also need the assumption that $\mathbf{u}\cdot\nabla\tilde{p}$ is well-defined in the distributional sense, which essentially is a technical requirement. However, in some particular cases this assumption is no longer required. For example, this is the case of the space $L^p((0,T),L^q)\subset\mathbb{X}^{(0)}_T$ (with $2/p+3/q=1$ and $2<p<+\infty$) where the products above are well defined. Indeed, we remark that $\nabla\tilde{p}\in L^{\hat{p}}((0,T),L^{\hat{q}}_{\text{loc}})$ where $\frac{1}{\hat{p}}=1-\frac{1}{p}$ and $\frac{1}{\hat{q}}=1-\frac{1}{q}$. To verify that, let $\frac{1}{p'}=\frac{1}{\hat{p}}-\frac{1}{2}$ and $\frac{1}{q'}=\frac{1}{\hat{q}}-\frac{1}{2}$, by interpolation we find that $\sqrt{w_\gamma}\tilde{\mathbf{u}},\sqrt{w_\gamma}\tilde{\mathbf{b}}\in L^{p'}((0,T),L^{q'})$. Then, for a test function $\varphi\in\mathcal{D}'(\mathbb{R}^3)$, using the continuity of the Riesz transforms, and moreover, assuming $\mathbb{F}=0$ (only for the sake of simplicity) we have

$$\begin{aligned}\|\varphi\nabla\tilde{p}\|_{L^{\hat{p}}L^{\hat{q}}}&\leq C_\varphi\|w_\gamma\nabla\tilde{p}\|_{L^{\hat{p}}L^{\hat{q}}}\leq C_\varphi\sum_{i,j,k}\|w_\gamma\partial_k(\tilde{u}_i\tilde{u}_j)+\partial_k(\tilde{b}_i\tilde{b}_j)\|_{L^{\hat{p}}L^{\hat{q}}}\\&\leq C(\|\sqrt{w_\gamma}\tilde{\mathbf{u}}\|_{L^{p'}L^{q'}}\|\sqrt{w_\gamma}\nabla\tilde{\mathbf{u}}\|_{L^2L^2}+\|\sqrt{w_\gamma}\tilde{\mathbf{b}}\|_{L^{p'}L^{q'}}\|\sqrt{w_\gamma}\nabla\tilde{\mathbf{b}}\|_{L^2L^2}).\end{aligned}$$

Thus, if $\mathbf{u}\in L^p((0,T),L^q)$ then we have $\mathbf{u}\cdot\nabla\tilde{p}\in L^1_{loc}([0,T]\times\mathbb{R}^3)$.

Proof of Theorem 12

By the characterization of the pressure term given in Chapter 2 (with the dimension $d=3$), we know that p can be taken as follows, let us choose $\varphi\in\mathcal{D}(\mathbb{R}^3)$ such that $\varphi(x) = 1$ on a neighborhood of the origin and let us denote $h_{i,j} = u_i u_j - b_i b_j - F_{i,j}$, and $\Phi_{i,j,\varphi} = (1-\varphi)\partial_i\partial_j G_3$, with $G_3(x) = \frac{1}{|x|}$, then we can take

$$p = \sum_{i,j}(\varphi\partial_i\partial_j G_3) * h_{i,j} + \sum_{i,j}\int(\Phi_{i,j,\varphi}(x-y) - \Phi_{i,j\varphi}(-y))h_{i,j}(y)dy,$$

As $\sqrt{w_\gamma}\mathbf{u}, \sqrt{w_\gamma}\mathbf{b} \in L^\infty((0,T),L^2)$ and $\sqrt{w_\gamma}\nabla\mathbf{u}, \sqrt{w_\gamma}\nabla\mathbf{b} \in L^2((0,T),L^2)$, we obtain by interpolation that $w_\gamma\mathbf{u}$ and $w_\gamma\mathbf{b}$ belong to $L^{\hat{a}}L^{\hat{b}}$ with $\frac{2}{\hat{a}} + \frac{3}{\hat{b}} = \frac{3}{2}$ and $\hat{a} \in [2,+\infty]$. Taking $r \in \left(1, \min\{\frac{3}{2}, \frac{3}{\gamma}\}\right)$ and $\hat{a}$ satisfying $\frac{2}{\hat{a}} + \frac{3}{r} = 3$, we get that

$$\mathcal{R}_i\mathcal{R}_j(u_iu_j), \mathcal{R}_i\mathcal{R}_j(b_ib_j) \in L^{\hat{a}}((0,T), L^r_{w_{r\gamma}}(\mathbb{R}^d)),$$

and by the continuity of the Riesz transforms on $L^2_{w_\gamma}(\mathbb{R}^3)$ we have

$$\mathcal{R}_i\mathcal{R}_jF_{i,j} \in L^2((0,T), L^2_{w_\gamma}(\mathbb{R}^d)).$$

Indeed, the following estimate holds: taking $\hat{b}$ given by $\frac{2}{\hat{a}} + \frac{3}{\hat{b}} = \frac{3}{2}$, we can write

$$\begin{aligned}
&\left\| \mathcal{R}_i\mathcal{R}_j(u_iu_j) \right\|_{L^{\hat{a}}((0,T),L^r_{w_{r\gamma}}(\mathbb{R}^3))} \\
&\leq C_{\gamma,r}\|u_iu_j\|_{L^{\hat{a}}((0,T),L^r_{w_{r\gamma}}(\mathbb{R}^3))} \\
&\leq C_{\gamma,r}\|\sqrt{w_\gamma}\,u_i\|_{L^\infty((0,T),L^2(\mathbb{R}^3))} \cdot \|\sqrt{w_\gamma}\,u_j\|_{L^{\hat{a}}\left((0,T),L^{\hat{b}}(\mathbb{R}^3)\right)} \\
&\leq \gamma^{\frac{1}{\hat{a}}}\,\tilde{C}_{\gamma,r}\,\|\mathbf{u}\|^{1+\frac{\hat{a}-2}{\hat{a}}}_{L^\infty((0,T),L^2_{w_\gamma}(\mathbb{R}^3))} \\
&\quad\times\left(\int_0^T(\|\mathbf{u}(s)\|_{L^2_{w_\gamma}(\mathbb{R}^3)} + \|\nabla\mathbf{u}(s)\|_{L^2_{w_\gamma}(\mathbb{R}^3)})^2ds\right)^{\frac{1}{\hat{a}}}.
\end{aligned}$$

Thus, as the Riesz transforms are well-defined for all the terms composing the pressure terms, we have necessarily the identity

$$p = \sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(u_iu_j - b_ib_j - F_{i,j}).$$

The corresponding identity hold true for the pressure term $\tilde{p}$.

Now, let $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$, $\mathbf{w} = \mathbf{b} - \tilde{\mathbf{b}}$, and $a = p - \tilde{p}$. So, we will prove the identities $\mathbf{v} = 0$, $\mathbf{w} = 0$ and $a = 0$.

Using the identity

$$\frac{|\mathbf{v}|^2 + |\mathbf{w}|^2}{2} = \frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2} + \frac{|\tilde{\mathbf{u}}|^2 + |\tilde{\mathbf{b}}|^2}{2} - \mathbf{u}\cdot\tilde{\mathbf{u}} - \mathbf{b}\cdot\tilde{\mathbf{b}}, \tag{6.6}$$

we write

$$\begin{aligned}
\partial_t(\frac{|\mathbf{v}|^2 + |\mathbf{w}|^2}{2}) =&\partial_t(\frac{|\mathbf{u}|^2 + |\mathbf{b}|^2}{2}) + \partial_t(\frac{|\tilde{\mathbf{u}}|^2 + |\tilde{\mathbf{b}}|^2}{2}) \\
&- \mathbf{u}\cdot\partial_t\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\cdot\partial_t\mathbf{u} - \mathbf{b}\cdot\partial_t\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\cdot\partial_t\mathbf{b}.
\end{aligned} \tag{6.7}$$

Recall that by assumption, the terms $\mathbf{u}\cdot\partial_t\tilde{\mathbf{u}}$ and $\mathbf{b}\cdot\partial_t\tilde{\mathbf{b}}$ are well-defined as distributions so it remains to verify that the terms $\tilde{\mathbf{u}}\cdot\partial_t\mathbf{u}$ and $\tilde{\mathbf{b}}\cdot\partial_t\mathbf{b}$ are also well-defined in the distributional sense. For this we have the following simple lemma.

Lemma 6.1. *Within the framework of Theorem 12, as we have $\boldsymbol{u},\boldsymbol{b}\in\mathbb{X}_T$ then we get $\tilde{\boldsymbol{u}}\cdot\partial_t\boldsymbol{u}\in L^1_{loc}([0,T]\times\mathbb{R}^3)$ and $\tilde{\boldsymbol{b}}\cdot\partial_t\boldsymbol{b}\in L^1_{loc}([0,T]\times\mathbb{R}^3)$.*

Proof. We shall verify that we have $\tilde{\mathbf{u}}\cdot\partial_t\mathbf{u}\in L^1_{loc}([0,T]\times\mathbb{R}^3)$. The treatment for the other term $\tilde{\mathbf{b}}\cdot\partial_t\mathbf{b}$ follows the same lines.

As we have $\partial_t\mathbf{u}=\Delta\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{u}+(\mathbf{b}\cdot\nabla)\mathbf{b}-\nabla p+\nabla\cdot\mathbb{F}$, then we formally write

$$\tilde{\mathbf{u}}\cdot\partial_t\mathbf{u}=\tilde{\mathbf{u}}\cdot\Delta\mathbf{u}-\tilde{\mathbf{u}}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})+\tilde{\mathbf{u}}\cdot((\mathbf{b}\cdot\nabla)\mathbf{b})-\tilde{\mathbf{u}}\cdot\nabla p+\tilde{\mathbf{u}}\cdot(\nabla\cdot\mathbb{F}),$$

and we must prove that each term in the right side belong to $L^1_{loc}([0,T]\times\mathbb{R}^3)$. We detail the computations for the terms $\tilde{\mathbf{u}}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})$ and $\tilde{\mathbf{u}}\cdot\nabla p$. Let $0<t\leq T$ and let $\varphi\in\mathcal{D}(\mathbb{R}^3)$ be an arbitrary test function. We set $\psi\in\mathcal{D}(\mathbb{R}^3)$ such that $0\leq\psi\leq 1$ and $\psi=1$ on $supp(\varphi)$.

For the term $\tilde{\mathbf{u}}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})$, as $div(\mathbf{u})=0$ we have $(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$ and then

$$\begin{aligned}\varphi\tilde{\mathbf{u}}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})\,dx\,ds&=\sum_{i,j}\varphi\tilde{u}_i\partial_j(u_ju_i)\,dx\,ds\\&=\sum_{i,j}\varphi\tilde{u}_i\partial_j(\psi\,u_ju_i)\,dx\,ds,\end{aligned}$$

therefore we can write

$$\begin{aligned}&\left|\int_0^t\int\varphi\tilde{\mathbf{u}}\cdot((\mathbf{u}\cdot\nabla)\mathbf{u})\,dx\,ds\right|\leq C\sum_{i,j}\int_0^t\|\varphi\tilde{u}_i\|_{\dot{H}^1}\|\partial_j(\psi u_ju_i)\|_{\dot{H}^{-1}}\,ds\\ \leq&C\int_0^t\|\nabla(\varphi\tilde{\mathbf{u}})\|_{L^2}\|\psi(\mathbf{u}\otimes\mathbf{u})\|_{L^2}\,ds\leq C\|\nabla(\varphi\tilde{\mathbf{u}})\|_{L^2_tL^2_x}\|\mathbf{u}\otimes(\sqrt{w_\gamma}\mathbf{u})\|_{L^2_tL^2_x}\\ \leq&C_{\gamma,T}(\|\sqrt{w_\gamma}\tilde{\mathbf{u}}\|_{L^\infty L^2_x}+\|\sqrt{w_\gamma}\nabla\tilde{\mathbf{u}}\|_{L^2L^2_x})\|\mathbf{u}\|_{\mathbb{X}_T}\|\sqrt{w_\gamma}\mathbf{u}\|_{E_T}\,ds\\ \leq&C_{\gamma,T}(\|\sqrt{w_\gamma}\tilde{\mathbf{u}}\|_{L^\infty_tL^2_x}+\|\sqrt{w_\gamma}\nabla\tilde{\mathbf{u}}\|_{L^2_tL^2_x})\\&\times\|\mathbf{u}\|_{\mathbb{X}_T}(\|\sqrt{w_\gamma}\mathbf{u}\|_{L^\infty_tL^2_x}+\|\sqrt{w_\gamma}\nabla\mathbf{u}\|_{L^2_tL^2_x})<+\infty.\end{aligned}\tag{6.8}$$

Now, we study the term $\tilde{\mathbf{u}}\cdot\nabla p$. Let us write

$$\begin{aligned}\nabla p=&\nabla\left(\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(u_iu_j-b_ib_j-F_{i,j})\right)\\ =&\nabla\left(\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(u_iu_j-b_ib_j)\right)-\nabla\left(\sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(F_{i,j})\right)\\ =&\nabla p_1+\nabla p_2.\end{aligned}$$

The term $\tilde{\mathbf{u}}\cdot\nabla p_2$ is easily estimated by the hypothesis on the tensor $\mathbb{F}$ and the computations above. Thereafter, for the term ∇p_1, following the same estimates performed in (6.8), and using the fact that $\mathcal{R}_i\mathcal{R}_j$ is a bounded operator in the space L^2, we find

$$\left|\int_0^t \int \varphi \tilde{\mathbf{u}} \cdot \nabla p_1 \, dx \, ds\right| \leq C_{\gamma,T}(\|\sqrt{w_\gamma}\tilde{\mathbf{u}}\|_{L^\infty_t L^2_x} + \|\sqrt{w_\gamma}\nabla\tilde{\mathbf{u}}\|_{L^2_t L^2_x})$$
$$\times \Big(\|\mathbf{u}\|_{\mathbb{X}_T}(\|\sqrt{w_\gamma}\mathbf{u}\|_{L^\infty_t L^2_x} + \|\sqrt{w_\gamma}\nabla\mathbf{u}\|_{L^2_t L^2_x})$$
$$+\|\mathbf{b}\|_{\mathbb{X}_T}(\|\sqrt{w_\gamma}\mathbf{b}\|_{L^\infty_t L^2_x} + \|\sqrt{w_\gamma}\nabla\mathbf{b}\|_{L^2_t L^2_x})\Big) < +\infty.$$

The lemma is proved.

Once all the terms in (6.7) are well-defined as distributions, we use the locally energy balances (6.4) and (6.5) to obtain

$$\begin{aligned}
&\partial_t(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}) + \mu + \nu = \Delta(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}) - |\nabla\mathbf{u}|^2 - |\nabla\mathbf{b}|^2 \\
&-\nabla\cdot\left([\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}+p]\mathbf{u}\right) + \nabla\cdot[(\mathbf{u}\cdot\mathbf{b})\mathbf{b}] + \mathbf{u}\cdot(\nabla\cdot\mathbb{F}) \\
&+\Delta(\frac{|\tilde{\mathbf{u}}|^2+|\tilde{\mathbf{b}}|^2}{2}) - |\nabla\tilde{\mathbf{u}}|^2 - |\nabla\tilde{\mathbf{b}}|^2 - \nabla\cdot\left([\frac{|\tilde{\mathbf{u}}|^2}{2}+\frac{|\tilde{\mathbf{b}}|^2}{2}+\tilde{p}]\tilde{\mathbf{u}}\right) \\
&+\nabla\cdot[(\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}}] + \tilde{\mathbf{u}}\cdot(\nabla\cdot\mathbb{F}) \\
&-\mathbf{u}\cdot\partial_t\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\cdot\partial_t\mathbf{u} - \mathbf{b}\cdot\partial_t\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\cdot\partial_t\mathbf{b},
\end{aligned}$$

which we order as

$$\begin{aligned}
&\partial_t(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}) + \mu + \nu = \underbrace{\Delta(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}) + \Delta(\frac{|\tilde{\mathbf{u}}|^2+|\tilde{\mathbf{b}}|^2}{2})}_{(1)} \\
&\underbrace{-|\nabla\mathbf{u}|^2 - |\nabla\mathbf{b}|^2 - |\nabla\tilde{\mathbf{u}}|^2 - |\nabla\tilde{\mathbf{b}}|^2 - \mathbf{u}\cdot\partial_t\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\cdot\partial_t\mathbf{u} - \mathbf{b}\cdot\partial_t\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\cdot\partial_t\mathbf{b}}_{(2)} \\
&-\nabla\cdot\left([\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}+p]\mathbf{u}\right) - \nabla\cdot\left([\frac{|\tilde{\mathbf{u}}|^2}{2}+\frac{|\tilde{\mathbf{b}}|^2}{2}+\tilde{p}]\tilde{\mathbf{u}}\right) \\
&+\nabla\cdot[(\mathbf{u}\cdot\mathbf{b})\mathbf{b}] + \nabla\cdot[(\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}}] \\
&+\mathbf{u}\cdot(\nabla\cdot\mathbb{F}) + \tilde{\mathbf{u}}\cdot(\nabla\cdot\mathbb{F}).
\end{aligned}$$

Now, in order to rewrite that in a most convenient form, we use (6.6) to treat the terms (1) and (2), so that

$$\begin{aligned}
&\partial_t(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}) + \mu + \nu = \Delta(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}) - |\nabla\mathbf{v}|^2 - |\nabla\mathbf{w}|^2 \\
&\underbrace{-(\partial_t\mathbf{u}-\Delta\mathbf{u})\cdot\tilde{\mathbf{u}} - (\partial_t\tilde{\mathbf{u}}-\Delta\tilde{\mathbf{u}})\cdot\mathbf{u} - (\partial_t\mathbf{b}-\Delta\mathbf{b})\cdot\tilde{\mathbf{b}} - (\partial_t\tilde{\mathbf{b}}-\Delta\tilde{\mathbf{b}})\cdot\mathbf{b}}_{(3)} \\
&-\nabla\cdot\left([\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}+p]\mathbf{u}\right) - \nabla\cdot\left([\frac{|\tilde{\mathbf{u}}|^2}{2}+\frac{|\tilde{\mathbf{b}}|^2}{2}+\tilde{p}]\tilde{\mathbf{u}}\right) \\
&+\nabla\cdot[(\mathbf{u}\cdot\mathbf{b})\mathbf{b}] + \nabla\cdot[(\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}}] \\
&+\mathbf{u}\cdot(\nabla\cdot\mathbb{F}) + \tilde{\mathbf{u}}\cdot(\nabla\cdot\mathbb{F}).
\end{aligned}$$

Thereafter, to study (3) we use the fact that $(\mathbf{u}, \mathbf{b}, p)$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{b}}, \tilde{p})$ are two solutions of the equations (MHD). Thus, we find

$$\begin{aligned}
\partial_t\Big(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\Big)+\mu+\nu &= \Delta\Big(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\Big)-|\nabla\mathbf{v}|^2-|\nabla\mathbf{w}|^2 \\
&\underbrace{-\nabla\cdot\left(\frac{|\mathbf{u}|^2}{2}\mathbf{u}+\frac{|\tilde{\mathbf{u}}|^2}{2}\tilde{\mathbf{u}}\right)+((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{u}}+((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{u}}_{(4)} \\
&\underbrace{-\nabla\cdot\left(\frac{|\mathbf{b}|^2}{2}\mathbf{u}+\frac{|\tilde{\mathbf{b}}|^2}{2}\tilde{\mathbf{u}}\right)+((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{b}+((\mathbf{u}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{b}}}_{(5)} \\
&+\nabla\cdot((\mathbf{u}\cdot\mathbf{v})\mathbf{b})-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{u}-((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{u}} \\
&+\nabla\cdot((\tilde{\mathbf{u}}\cdot\tilde{\mathbf{v}})\tilde{\mathbf{b}})-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{b}-((\mathbf{b}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{b}} \\
&-\nabla\cdot(a\mathbf{v}).
\end{aligned}$$

We look for rewrite the right side as a sum of terms of the form

$$\nabla\cdot((\mathbf{x}\cdot\mathbf{y})\mathbf{z})$$

where at least two elements of $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ belong to $\{\mathbf{v}, \mathbf{w}\}$, or terms of the form

$$((\mathbf{x}\cdot\nabla)\mathbf{y})\cdot\mathbf{z}$$

where $\mathbf{y} \in \{\mathbf{v}, \mathbf{w}\}$ and at least one element of $\{\mathbf{x}, \mathbf{z}\}$ belongs to $\{\mathbf{u}, \mathbf{b}\}$. As we will see, this fact will us permit to use the hypothesis $\mathbf{u}, \mathbf{b} \in \mathbb{X}_T$ to get a good control and apply the Grönwall inequality.

For the term (4), we observe that

$$\begin{aligned}
&((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{u}}+((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{u} \\
&=((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\tilde{\mathbf{u}}+((\mathbf{u}\cdot\nabla)\tilde{\mathbf{u}})\cdot\tilde{\mathbf{u}}-((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}+((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u})\cdot\mathbf{u} \\
&=\nabla\cdot\left(\frac{|\tilde{\mathbf{u}}|^2}{2}\mathbf{u}+\frac{|\mathbf{u}|^2}{2}\tilde{\mathbf{u}}\right)-((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}-((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{v}+((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{u} \\
&=\nabla\cdot\left(\frac{|\tilde{\mathbf{u}}|^2}{2}\mathbf{u}+\frac{|\mathbf{u}|^2}{2}\tilde{\mathbf{u}}-\frac{|\mathbf{v}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot\mathbf{u},
\end{aligned}$$

hence

$$(4)=-\nabla\cdot\left(\frac{\mathbf{v}\cdot(\mathbf{u}+\tilde{\mathbf{u}})}{2}\mathbf{v}+\frac{|\mathbf{v}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}$$

We treat (5) in a similar way. As

$$\begin{aligned}
&((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{b}+((\mathbf{u}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{b}} \\
&=-((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}+((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{b})\cdot\mathbf{b}+((\mathbf{u}\cdot\nabla)\mathbf{w})\cdot\tilde{\mathbf{b}}+((\mathbf{u}\cdot\nabla)\tilde{\mathbf{b}})\cdot\tilde{\mathbf{b}} \\
&=\nabla\cdot\left(\frac{|\mathbf{b}|^2}{2}\tilde{\mathbf{u}}+\frac{|\tilde{\mathbf{b}}|^2}{2}\mathbf{u}\right)-((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}+((\mathbf{u}\cdot\nabla)\mathbf{w})\cdot\tilde{\mathbf{b}}
\end{aligned}$$

we find

$$(5)=-\nabla\cdot\left(\frac{\mathbf{w}\cdot(\mathbf{b}+\tilde{\mathbf{b}})}{2}\mathbf{v}+\frac{|\mathbf{w}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}$$

With these identities on (4) and (5), we get

$$\begin{aligned}
&\partial_t\left(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\right)+\mu+\nu=\Delta\left(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\right)-|\nabla\mathbf{v}|^2-|\nabla\mathbf{w}|^2\\
&-\nabla\cdot\left(\frac{\mathbf{v}\cdot(\mathbf{u}+\tilde{\mathbf{u}})}{2}\mathbf{v}+\frac{|\mathbf{v}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}\\
&-\nabla\cdot\left(\frac{\mathbf{w}\cdot(\mathbf{b}+\tilde{\mathbf{b}})}{2}\mathbf{v}+\frac{|\mathbf{w}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}\\
&\underbrace{+\nabla\cdot((\mathbf{u}\cdot\mathbf{b})\mathbf{b})-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{u}-((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{u}}}_{(6)}\\
&\underbrace{+\nabla\cdot((\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}})-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{b}-((\mathbf{b}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{b}},}_{(7)}\\
&-\nabla\cdot(a\mathbf{v}).
\end{aligned}$$

In order to obtain (6) and (7), remark that

$$\begin{aligned}
&\nabla\cdot((\mathbf{u}\cdot\mathbf{b})\mathbf{b})+\nabla\cdot((\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}})\\
&=\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\tilde{\mathbf{b}})+\nabla\cdot((\mathbf{u}\cdot\tilde{\mathbf{b}})\mathbf{b})\\
&\quad+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})-\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{b})+\nabla\cdot((\mathbf{u}\cdot\tilde{\mathbf{b}})\tilde{\mathbf{b}})\\
&=\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})+\nabla\cdot((\mathbf{u}\cdot\mathbf{b})\tilde{\mathbf{b}})+\nabla\cdot((\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\mathbf{b})
\end{aligned}$$

which gives

$$\begin{aligned}
(6)+(7)=&\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})+\nabla\cdot((\mathbf{u}\cdot\mathbf{b})\tilde{\mathbf{b}})+\nabla\cdot((\tilde{\mathbf{u}}\cdot\tilde{\mathbf{b}})\mathbf{b})\\
&-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{u}-((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{u}}-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{b}-((\mathbf{b}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{b}}\\
=&\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})\\
&+((\tilde{\mathbf{b}}\cdot\nabla\mathbf{u})\cdot\mathbf{b})+((\tilde{\mathbf{b}}\cdot\nabla\mathbf{b})\cdot\mathbf{u})+((\mathbf{b}\cdot\nabla\tilde{\mathbf{u}})\cdot\tilde{\mathbf{b}})+((\mathbf{b}\cdot\nabla\tilde{\mathbf{b}})\cdot\tilde{\mathbf{u}})\\
&-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{b}})\cdot\mathbf{u}-((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\tilde{\mathbf{u}}-((\tilde{\mathbf{b}}\cdot\nabla)\tilde{\mathbf{u}})\cdot\mathbf{b}-((\mathbf{b}\cdot\nabla)\mathbf{u})\cdot\tilde{\mathbf{b}}\\
=&\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\mathbf{w})\mathbf{b})\\
&-((\mathbf{w}\cdot\nabla)\mathbf{w})\cdot\mathbf{u}-((\mathbf{w}\cdot\nabla)\mathbf{v})\cdot\mathbf{b}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\partial_t\left(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\right)+\mu+\nu=\Delta\left(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\right)-|\nabla\mathbf{v}|^2-|\nabla\mathbf{w}|^2\\
&-\nabla\cdot\left(\frac{\mathbf{v}\cdot(\mathbf{u}+\tilde{\mathbf{u}})}{2}\mathbf{v}+\frac{|\mathbf{v}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}\\
&-\nabla\cdot\left(\frac{\mathbf{w}\cdot(\mathbf{b}+\tilde{\mathbf{b}})}{2}\mathbf{v}+\frac{|\mathbf{w}|^2}{2}\mathbf{u}\right)+((\mathbf{v}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}\\
&+\nabla\cdot((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w})+\nabla\cdot((\mathbf{u}\cdot\mathbf{w})\mathbf{w})+\nabla\cdot((\mathbf{v}\cdot\mathbf{w})\mathbf{b})\\
&-((\mathbf{w}\cdot\nabla)\mathbf{w})\cdot\mathbf{u}-((\mathbf{w}\cdot\nabla)\mathbf{v})\cdot\mathbf{b}\\
&-\nabla\cdot(a\mathbf{v}),
\end{aligned}$$

which we rewrite in the following way

$$\begin{aligned}
&\partial_t\Big(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\Big)+|\nabla\mathbf{v}|^2+|\nabla\mathbf{w}|^2+\mu+\nu\\
=&\Delta\Big(\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\Big)-\nabla\cdot\underbrace{\Big(\frac{\mathbf{v}\cdot(\mathbf{u}+\tilde{\mathbf{u}})}{2}\mathbf{v}+\frac{\mathbf{w}\cdot(\mathbf{b}+\tilde{\mathbf{b}})}{2}\mathbf{v}+\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\mathbf{u}\Big)}_{\mathbf{A}_1}\\
&+\nabla\cdot\underbrace{\big((\mathbf{v}\cdot\tilde{\mathbf{b}})\mathbf{w}+(\mathbf{u}\cdot\mathbf{w})\mathbf{w}+(\mathbf{v}\cdot\mathbf{w})\mathbf{b}\big)}_{\mathbf{A}_2}\\
&\underbrace{-((\mathbf{w}\cdot\nabla)\mathbf{w}))\cdot\mathbf{u}-((\mathbf{w}\cdot\nabla)\mathbf{v}))\cdot\mathbf{b}+((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot\mathbf{u}+((\mathbf{v}\cdot\nabla)\mathbf{w})\cdot\mathbf{b}}_{A_3}\\
&-\nabla\cdot(a\mathbf{v}).
\end{aligned}\tag{6.9}$$

We will apply (6.9) to a suitable test function. First, we consider the function α_{η,t_0,t_1} defined in (5.7), which converges almost everywhere to $\mathbb{1}_{[t_0,t_1]}$ when $\eta\to 0$ and such that $\partial_t\alpha_{\eta,t_0,t_1}$ is the difference between two identity approximations, the first one in t_0 and the second one in t_1.

Thereafter, we consider a non-negative function $\phi\in\mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x|\le 1$ and to 0 for $|x|\ge 2$ and we set

$$\phi_R(x)=\phi(\frac{x}{R}).$$

For $\epsilon>0$, we denote $w_{\gamma,\epsilon}=\frac{1}{(1+\sqrt{\epsilon^2+|x|^2})^\delta}$. We may observe that $\alpha_{\eta,a,s}(t)\phi_R(x)w_{\gamma,\epsilon}(x)$ belongs to $\mathcal{D}((0,T)\times\mathbb{R}^3)$ and $\alpha_{\eta,a,s}(t)\phi_R(x)w_{\gamma,\epsilon}(x)\ge 0$. Thus, applying (6.9) to this particular test function, we obtain

$$\begin{aligned}
-&\iint\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds+\iint(|\nabla\mathbf{v}|^2+|\nabla\mathbf{w}|^2)\ \alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}dx\,ds\\
&\le-\sum_i\iint\partial_i(\mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w})\ \alpha_{\eta,t_0,t_1}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&\quad-\sum_i\iint(\mathbf{A}_1+\mathbf{A}_2)_i\,\alpha_{\eta,t_0,t_1}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&\quad+\iint A_3\,\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds\\
&\quad+\sum_i\iint(a\mathbf{v})_i\,\alpha_{\eta,t_0,t_1}(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds.
\end{aligned}$$

In this inequality, we take the limit when $\eta\to 0$. By the dominated convergence theorem we obtain (when the limit in the left side is well-defined)

$$\begin{aligned}
-&\lim_{\eta\to 0}\iint\frac{|\mathbf{v}|^2+|\mathbf{w}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R w_{\gamma,\epsilon}\,dx\,ds+\int_{t_0}^{t_1}\int(|\nabla\mathbf{v}|^2+|\nabla\mathbf{w}|^2)\ \phi_R w_{\gamma,\epsilon}dx\,ds\\
&\le-\sum_i\int_{t_0}^{t_1}\int\partial_i(\mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{w})\ (w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds\\
&\quad-\sum_i\int_{t_0}^{t_1}\int(\mathbf{A}_1+\mathbf{A}_2)_i\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds+\int_{t_0}^{t_1}\int A_3\,\phi_R w_{\gamma,\epsilon}\,dx\,ds\\
&\quad+\sum_i\int_{t_0}^{t_1}\int(a\mathbf{v})_i\,(w_{\gamma,\epsilon}\partial_i\phi_R+\phi_R\partial_i w_{\gamma,\epsilon})\,dx\,ds.
\end{aligned}$$

We take t_0 and t_1 two Lebesgue points of the measurable function

$$A_{R,\epsilon}(t) = \int (|\mathbf{v}(t,x)|^2 + |\mathbf{w}(t,x)|^2)\phi_R(x) w_{\gamma,\epsilon}(x)\, dx,$$

so that we have

$$-\int\int \frac{|\mathbf{v}|^2 + |\mathbf{w}|^2}{2} \partial_t \alpha_{\eta,a,s} \phi_R w_{\gamma,\epsilon}\, dx\, ds = -\frac{1}{2}\int \partial_t \alpha_{\eta,a,s} A_{R,\epsilon}(s)\, ds,$$

and

$$\lim_{\eta\to 0} -\int\int \frac{|\mathbf{v}|^2 + |\mathbf{w}|^2}{2} \partial_t \alpha_{\eta,a,s} \phi_R w_{\gamma,\epsilon}\, dx\, ds = \frac{1}{2}(A_{R,\epsilon}(t_1) - A_{R,\epsilon}(t_0)).$$

Thereafter, the continuity at 0 of $\mathbf{v}$ and $\mathbf{w}$ allows us to let t_0 goes to 0 and thus we replace t_0 by 0 in this inequality. Additionally, if we let t_1 goes to t, where $t \in (0,T)$, then by weak continuity we know that $A_{R,\epsilon}(t) \leq \liminf_{t_1\to t} A_{R,\epsilon}(t_1)$, so that we may replace t_1 by t as well. In conclusion, for every $t \in (0,T)$ we have

$$\begin{aligned}
&\int \frac{|\mathbf{v}(t,\cdot)|^2 + |\mathbf{w}(t,\cdot)|^2}{2} \phi_R w_{\gamma,\epsilon}\, dx + \int_0^t \int (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{w}|^2)\ \phi_R w_{\gamma,\epsilon} dx\, ds \\
&\leq -\sum_i \int_0^t \int \partial_i(\mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w})\ (w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad -\sum_i \int_0^t \int (\mathbf{A}_1 + \mathbf{A}_2)_i\ (w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds \\
&\quad + \int_0^t \int A_3\, w_\gamma\, dx\, ds + \sum_i \int_0^t \int (a\mathbf{v})_i\ (w_{\gamma,\epsilon}\partial_i\phi_R + \phi_R \partial_i w_{\gamma,\epsilon})\, dx\, ds.
\end{aligned}$$

In this inequality, we take now the limit when $R \to +\infty$, and after that, the limit when $\epsilon \to 0$ to get:

$$\begin{aligned}
&\int \frac{|\mathbf{v}(t,\cdot)|^2 + |\mathbf{w}(t,\cdot)|^2}{2} w_\gamma\, dx + \int_0^t \int (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{w}|^2)\, w_\gamma dx\, ds \\
&\leq -\int_0^t \int \nabla(\mathbf{v}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}) \cdot \nabla w_\gamma\, dx\, ds - \underbrace{\int_0^t \int (\mathbf{A}_1 + \mathbf{A}_2)\cdot \nabla w_\gamma\, dx\, ds}_{I_1} \\
&\quad + \underbrace{\int_0^t \int A_3\, w_\gamma\, dx\, ds}_{I_2} + \underbrace{\int_0^t \int a\mathbf{v}\cdot \nabla w_\gamma\, dx\, ds}_{I_3}.
\end{aligned}$$

Now, we shall estimate the terms I_1, I_2 and I_3. Recall we denote

$$\int (|\mathbf{v}|^2 + |\mathbf{w}|^2) w_\gamma dx = \|\sqrt{w_\gamma}(\mathbf{v},\mathbf{w})\|_{L^2}^2, \int (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{w}|^2) w_\gamma dx = \|\sqrt{w_\gamma}\nabla(\mathbf{v},\mathbf{w})\|_{L^2}^2.$$

Lemma 6.2. $|I_1| \leq C_\gamma \int_0^t \|\sqrt{w_\gamma}(\boldsymbol{v},\boldsymbol{w})(s,\cdot)\|_{L^2}^2\, ds + \frac{1}{4}\int_0^t \|\sqrt{w_\gamma}\nabla(\boldsymbol{v},\boldsymbol{w})(s,\cdot)\|_{L^2}^2\, ds.$

Proof. As $|\nabla w_\gamma| \leq C_\gamma w_{\frac{3}{2}\gamma}$ we have

$$|I_1| \leq C_\gamma \int_0^t \int |\mathbf{A}_1 + \mathbf{A}_2| w_{3/2\gamma} dx\, ds.$$

Then, as each term in the expression $\mathbf{A}_1 + \mathbf{A}_2$ writes down as the product of three vectors: $(\mathbf{x}\cdot\mathbf{y})\mathbf{z}$ where at least two vectors belong to $\{\mathbf{v},\mathbf{w}\}$ and the third one belongs to $\{\mathbf{u},\mathbf{b},\tilde{\mathbf{u}},\tilde{\mathbf{b}}\}$. So, we will estimate the generic expression $(\mathbf{x}\cdot\mathbf{y})\mathbf{z}$, where, we assume that $\mathbf{x},\mathbf{z}\in\{\mathbf{v},\mathbf{w}\}$ and $\mathbf{y}\in\{\mathbf{u},\mathbf{b},\tilde{\mathbf{u}},\tilde{\mathbf{b}}\}$. Remark that for $\delta>0$ (which we will fix later), using the Hölder inequalities and the Young inequalities we obtain

$$\begin{aligned}\int|(\mathbf{x}\cdot\mathbf{y})\mathbf{z}|w_{3/2\gamma}dx &\le \int(\sqrt{w_\gamma}|\mathbf{x}|)(\sqrt{w_\gamma}|\mathbf{z}|)(\sqrt{w_\gamma}|\mathbf{y}|)dx\\ \le&\|\sqrt{w_\gamma}\mathbf{x}\|_{L^3}\|\sqrt{w_\gamma}\mathbf{z}\|_{L^6}\|\sqrt{w_\gamma}\mathbf{y}\|_{L^2}\le \underbrace{\delta^{-1}\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^3}+\delta\|\sqrt{w_\gamma}\mathbf{z}\|^2_{L^6}\|\sqrt{w_\gamma}\mathbf{y}\|^2_{L^2}}_{(a)}.\end{aligned}$$

Interpolation inequalities gives

$$\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^3}\le\delta^{-2}\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^2}+\delta^2\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^6},$$

therefore we can write

$$(a)\le\delta^{-3}\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^2}+\delta\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^6}+\delta\|\sqrt{w_\gamma}\mathbf{z}\|^2_{L^6}\|\sqrt{w_\gamma}\mathbf{y}\|^2_{L^2}=(b).$$

At this point, we apply the Sobolev embedding to estimate $\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^6}$ and $\|\sqrt{w_\gamma}\mathbf{z}\|^2_{L^6}$, and we find

$$\begin{aligned}(b)\le&\,\delta^{-3}\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^2}+\delta(\|\sqrt{w_\gamma}\mathbf{x}\|^2_{L^2}+\|\sqrt{w_\gamma}\nabla\mathbf{x}\|^2_{L^2})\\ &+\delta(\|\sqrt{w_\gamma}\mathbf{z}\|^2_{L^2}+\|\sqrt{w_\gamma}\nabla\mathbf{z}\|^2_{L^2})\|\sqrt{w_\gamma}\mathbf{y}\|^2_{L^2}.\end{aligned}$$

Using the previous estimate we get

$$\begin{aligned}(b)\le&\delta^{-3}\|\sqrt{w_\gamma}(\mathbf{v},\mathbf{w})\|^2_{L^2}+C\delta(\|\sqrt{w_\gamma}(\mathbf{v},\mathbf{w})\|^2_{L^2}+\|\sqrt{w_\gamma}\nabla(\mathbf{v},\mathbf{w})\|^2_{L^2})\\ &+C\delta(\|\sqrt{w_\gamma}(\mathbf{v},\mathbf{w})\|^2_{L^2}+\|\sqrt{w_\gamma}\nabla(\mathbf{v},\mathbf{w})\|^2_{L^2})\left(\sup_{0<s<T}\|\sqrt{w_\gamma}\mathbf{y}\|^2_{L^2}\right).\end{aligned}$$

We set the parameter δ small enough to find

$$\max\left[C\delta,C\delta\left(\sup_{0<s<T}\|\sqrt{w_\gamma}\mathbf{y}\|^2_{L^2}\right)\right]\le 1/64,$$

and thus we finally get

$$\int|(\mathbf{x}\cdot\mathbf{y})\mathbf{z}|w_{3/2\gamma}dx\le C_\gamma\|\sqrt{w_\gamma}(\mathbf{v},\mathbf{w})\|^2_{L^2}+\frac{1}{32}\|\sqrt{w_\gamma}\nabla(\mathbf{v},\mathbf{w})\|^2_{L^2}.$$

Integration in temporal variable gives us the desired estimate.

To study I_2, we use the assumption $\mathbf{u},\mathbf{b}\in\mathbb{X}_T$.

Lemma 6.3. *For all* $0<t<T$, *we have*

$$\begin{aligned}|I_2|\le&C\|\boldsymbol{u}\|_{\mathbb{X}_T}\left(\sup_{0\le s\le t}\|\sqrt{w_\gamma}(\boldsymbol{v},\boldsymbol{w})(s)\|^2_{L^2}+\int_0^t\|\sqrt{w_\gamma}\nabla(\boldsymbol{v},\boldsymbol{w})(s,\cdot)\|^2_{L^2}\,ds\right)\\ &+C\|\boldsymbol{b}\|_{\mathbb{X}_T}\left(\sup_{0\le s\le t}\|\sqrt{w_\gamma}(\boldsymbol{v},\boldsymbol{w})(s,\cdot)\|^2_{L^2}+\int_0^t\|\sqrt{w_\gamma}\nabla(\boldsymbol{v},\boldsymbol{w})(s,\cdot)\|^2_{L^2}\,ds\right).\end{aligned}$$

Proof. Each term in the expression A_3 is written in the form $\sum_{i,j=1}^{3} x_j(\partial_j z_i)y_i$, where, $\{\mathbf{x},\mathbf{z}\}$ belong to $\{\mathbf{v},\mathbf{w}\}$ and $\mathbf{y}\in\{\mathbf{u},\mathbf{b}\}$. By the Cauchy-Schwarz inequalities (in the spatial and temporal variables) we find

$$\int_0^t \|w_\gamma|\mathbf{x}||\nabla\mathbf{z}||\mathbf{y}|(s,\cdot)\|_{L^1}\,ds \leq \left(\int_0^t \|\sqrt{w_\gamma}|\mathbf{x}||\mathbf{y}|(s,\cdot)\|_{L^2}^2\,ds\right)^{1/2} \times\left(\int_0^t \|\sqrt{w_\gamma}|\nabla\mathbf{z}|(s,\cdot)\|_{L^2}^2\,ds\right)^{1/2}.$$

Thereafter, by definition of the multiplier space $\mathbb{X}_T$, and by definition of the energy space E_T we have

$$\begin{aligned}\int_0^t \|w_\gamma|\mathbf{x}||\nabla\mathbf{z}||\mathbf{y}|(s,\cdot)\|_{L^1}\,ds &\leq \|\mathbf{y}\|_{\mathbb{X}_T}\|\sqrt{w_\gamma}\mathbf{x}\|_{E_T}\left(\int_0^t \|\sqrt{w_\gamma}|\nabla\mathbf{z}|(s,\cdot)\|_{L^2}^2\,ds\right)^{1/2}\\ &\leq C\,\|\mathbf{y}\|_{\mathbb{X}_T}\left(\|\sqrt{w_\gamma}\mathbf{x}\|_{E_T}^2+\int_0^t \|\sqrt{w_\gamma}|\nabla\mathbf{z}|(s,\cdot)\|_{L^2}^2\,ds\right),\end{aligned}$$

that is the desired estimate.

Finally, we study I_3.

Lemma 6.4. $|I_3|\leq C\int_0^t \|\sqrt{w_\gamma}(v,w)(s,\cdot)\|_{L^2}^2\,ds+\frac{1}{4}\int_0^t \|\sqrt{w_\gamma}\nabla(v,w)(s,\cdot)\|_{L^2}^2\,ds.$

Proof. Remark we have $p-\tilde{p}=\sum_{i,j}\mathcal{R}_i\mathcal{R}_j(u_iu_j-\tilde{u}_i\tilde{u}_j-b_ib_j+\tilde{b}_i\tilde{b}_j)$. As $\mathbf{v}=\mathbf{u}-\tilde{\mathbf{u}}$ and $\mathbf{w}=\mathbf{b}-\tilde{\mathbf{b}}$, we have

$$u_iu_j-\tilde{u}_i\tilde{u}_j-b_ib_j+\tilde{b}_i\tilde{b}_j=v_iu_j+\tilde{u}_iv_j-w_jb_j-\tilde{b}_iw_j.$$

Then, since $|\nabla w_\gamma|\leq C_\gamma w_{\frac{3}{2}\gamma}$ the Hölder inequalities gives

$$\begin{aligned}|I_3| \leq& C_\gamma\left\|w_\gamma\left(\sum_{i,j}\mathcal{R}_i\mathcal{R}_j(u_iu_j-\tilde{u}_i\tilde{u}_j-b_ib_j+\tilde{b}_i\tilde{b}_j)\right)\right\|_{L^{\frac{6}{5}}}\|\sqrt{w_\gamma}\mathbf{v}\|_{L^6}\\ \leq& C_\gamma\left\|w_\gamma\left(\sum_{i,j}\mathcal{R}_i\mathcal{R}_j(v_iu_j+\tilde{u}_iv_j-w_jb_j-\tilde{b}_iw_j)\right)\right\|_{L^{\frac{6}{5}}}\|\sqrt{w_\gamma}\mathbf{v}\|_{L^6}\\ \leq& C_\gamma(\|w_\gamma(|\mathbf{u}|+|\tilde{\mathbf{u}}|)|\mathbf{v}|\,\|_{L^{\frac{6}{5}}}+\|w_\gamma(|\mathbf{b}|+|\tilde{\mathbf{b}}|)|\mathbf{w}|\,\|_{L^{\frac{6}{5}}})\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^6}\\ \leq& C_\gamma\|\sqrt{w_\gamma}(|\mathbf{u}|+|\tilde{\mathbf{u}}|)\|_{L^2}\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^3}\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^6}\\ &+C_\gamma\|\sqrt{w_\gamma}(|\mathbf{b}|+|\tilde{\mathbf{b}}|)\|_{L^2}\|\sqrt{w_\gamma}\mathbf{w}\,\|_{L^3}\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^6}\\ \leq& C_\gamma\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^3}\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^6}\|\sqrt{w_\gamma}(|\mathbf{u}|+|\tilde{\mathbf{u}}|)\|_{L^2}\\ &+C_\gamma\|\sqrt{w_\gamma}\mathbf{w}\,\|_{L^3}\|\sqrt{w_\gamma}\mathbf{v}\,\|_{L^6}\|\sqrt{w_\gamma}(|\mathbf{b}|+|\tilde{\mathbf{b}}|)\|_{L^2}\\ &=(a).\end{aligned}$$

For $\delta > 0$, using the Young inequalities, and thereafter the Sobolev embedding we find

$$\begin{aligned}(a) \leq & C_\gamma \delta^{-1} \| \sqrt{w_\gamma} \mathbf{v} \|_{L^3}^2 + C_\gamma \delta \| \sqrt{w_\gamma} \mathbf{v} \|_{L^6}^2 \| \sqrt{w_\gamma} (|\mathbf{u}| + |\tilde{\mathbf{u}}|) \|_{L^2}^2 \\ & + C_\gamma \delta^{-1} \| \sqrt{w_\gamma} \mathbf{w} \|_{L^3}^2 + C\delta \| \sqrt{w_\gamma} \mathbf{v} \|_{L^6}^2 \| \sqrt{w_\gamma} (|\mathbf{b}| + |\tilde{\mathbf{b}}|) \|_{L^2}^2 \\ \leq & C\delta^{-3} \| \sqrt{w_\gamma} \mathbf{v} \|_{L^2}^2 + C_\gamma \delta (\| \sqrt{w_\gamma} \mathbf{v} \|_{L^2}^2 + \| \sqrt{w_\gamma} |\nabla \mathbf{v}| \|_{L^2}^2) \\ & + C_\gamma \delta (\| \sqrt{w_\gamma} \mathbf{v} \|_{L^2}^2 + \| \sqrt{w_\gamma} |\nabla \mathbf{v}| \|_{L^2}^2) \left(\sup_{0 \leq s \leq T} \| \sqrt{w_\gamma} (|\mathbf{u}| + |\tilde{\mathbf{u}}|) \|_{L^2} \right) \\ & + C_\gamma \delta^{-2} \| \sqrt{w_\gamma} \mathbf{w} \|_{L^2}^2 + C_\gamma \delta (\| \sqrt{w_\gamma} \mathbf{w} \|_{L^2}^2 + \| \sqrt{w_\gamma} |\nabla \mathbf{w}| \|_{L^2}^2) \\ & + C_\gamma \delta (\| \sqrt{w_\gamma} \mathbf{v} \|_{L^2}^2 + \| \sqrt{w_\gamma} |\nabla \mathbf{v}| \|_{L^2}^2) \left(\sup_{0 \leq s \leq T} \| \sqrt{w_\gamma} (|\mathbf{b}| + |\tilde{\mathbf{b}}| \|_{L^2} \right).\end{aligned}$$

We set the parameter δ small enough and the lemma is proved.

Now, we resume the proof of the theorem. We consider $0 \leq t_0 < t_1 < T$ if we suppose that $\mathbf{v} = \mathbf{w} = 0$ on $[0, t_0]$, for all $t \in [t_0, t_1]$. we have obtained

$$\begin{aligned} & \| (\mathbf{v}, \mathbf{w})(t, \cdot) \|_{L^2_{w_\gamma}}^2 + \frac{1}{2} \int_0^t \| \nabla (\mathbf{v}, \mathbf{w}) \|_{L^2_{w_\gamma}}^2 \, ds \\ \leq & C \int_0^t \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \\ & + C \| \mathbb{1}_{(t_0, t_1)} \mathbf{u} \|_{X_T} \left(\sup_{0 \leq s \leq t_1} \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 ds + \int_0^t \| \nabla (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \right) \\ & + C \| \mathbb{1}_{(t_0, t_1)} \mathbf{b} \|_{X_T} \left(\sup_{0 \leq s \leq t_1} \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 + \int_0^t \| \nabla (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \right). \end{aligned}$$

We then take the supremum on $[0, t_1]$, to get

$$\begin{aligned} & \sup_{0 \leq t \leq t_1} \| (\mathbf{v}, \mathbf{w})(t, \cdot) \|_{L^2_{w_\gamma}}^2 + \frac{1}{2} \int_0^{t_1} \| \nabla (\mathbf{v}, \mathbf{w}) \|_{L^2_{w_\gamma}}^2 \, ds \\ & \leq C \int_0^{t_1} \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \\ & \quad + C \| \mathbb{1}_{(t_0, t_1)} \mathbf{u} \|_{X_T} \left(\sup_{0 \leq s \leq t_1} \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 + \int_0^{t_1} \| \nabla (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \right) \\ & \quad + C \| \mathbb{1}_{(t_0, t_1)} \mathbf{b} \|_{X_T} \left(\sup_{0 \leq s \leq t_1} \| (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 + \int_0^{t_1} \| \nabla (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds \right). \end{aligned}$$

Let us denote

$$f(t_1) = \sup_{0 \leq t \leq t_1} \| (\mathbf{v}, \mathbf{w})(t) \|_{L^2_{w_\gamma}}^2 + \frac{1}{2} \int_0^{t_1} \| \nabla (\mathbf{v}, \mathbf{w})(s) \|_{L^2_{w_\gamma}}^2 \, ds.$$

We take $T_0 \in (t_0, T)$ such that for $t_1 \in (t_0, T_0)$ the quantities $\| \mathbb{1}_{(t_0, t_1)} \mathbf{u} \|_{X_T}$ and $\| \mathbb{1}_{(t_0, t_1)} \mathbf{b} \|_{X_T}$ are small enough so that for all $t_0 < t_1 < T_0$:

$$f(t_1) \leq C \int_0^{t_1} f(s) ds.$$

The Grönwall's lemma permits us to conclude $(\mathbf{u}, \mathbf{b}, p) = (\tilde{\mathbf{u}}, \tilde{\mathbf{b}}, \tilde{p})$ on $[t_0, T_0]$. As $t_0 \in [0, T)$ is arbitrary we have $(\mathbf{u}, \mathbf{b}, p) = (\tilde{\mathbf{u}}, \tilde{\mathbf{b}}, \tilde{p})$ on $[0, T)$. ◇

6.3 Local Morrey spaces

There exists a slight generalisation of Theorem 4 in the case of dimension 2 and 3. It is obtained by consider a bigger space than $L^2(w_2)$, and close to that. Instead of dealing with weighted Lebesgue spaces, one may deal with a kind of local Morrey space, the space B^2_γ.

In this section we state some previous results on weighted spaces and local Morrey spaces. We state these results in $\mathbb{R}^d$.

Definition 6.1. *For $\gamma \geq 0$ and $p \in (1, \infty)$. We denote $B^p_\gamma(\mathbb{R}^d)$ the Banach space of all functions $u \in L^p_{\text{loc}}(\mathbb{R}^d)$ such that :*

$$\|u\|_{B^p_\gamma} = \sup_{R \geq 1} \left(\frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx \right)^{1/p} < +\infty.$$

Moreover, for $0 < T \leq +\infty$, we define $B^p_\gamma L^p(0, T)$ as the Banach space of all functions $u \subset (L^p_t L^p_x)_{\text{loc}}([0, T] \times \mathbb{R}^d)$ such that

$$\|u\|_{B^p_\gamma L^p(0,T)} = \sup_{R \geq 1} \left(\frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t, x)|^p \right)^{\frac{1}{p}} dx \, dt < +\infty.$$

To simplify notation in what follows, we will write $B^2_2 = B_2$.

Additionally, $B^p_{\gamma,0}$ is defined as the subspace of all functions $u \in B^p_\gamma$ such that

$$\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_{B(0,R)} |u(x)|^p \, dx = 0.$$

Similarly, $B^p_{\gamma,0} L^p(0, T)$ is the subspace of all functions $u \in B^p_\gamma L^p(0, T)$ such that

$$\lim_{R \to +\infty} \frac{1}{R^\gamma} \int_0^T \int_{B(0,R)} |u(t, x)|^p \, dx \, dt = 0.$$

The following result highlights how B^p_γ is strongly linked with the weighted spaces $L^p_{w_\gamma} = L^p(w_\gamma \, dx)$, with $w_\gamma = (1 + |x|)^{-\gamma}$, which are considered in (Fernández-Dalgo and Jarrín, 2021b; Fernández-Dalgo and Lemarié-Rieusset, 2020b).

Lemma 6.5. *Let $\gamma \geq 0$ and $\gamma < \delta < +\infty$. We have the continuous embedding*

$$L^p_{w_\gamma} \subset B^p_{\gamma,0} \subset B^p_\gamma \subset L^p_{w_\delta},$$

and moreover, for $0 < T \leq +\infty$,

$$L^p((0, T), L^p_{w_\gamma}) \subset B^p_{\gamma,0} L^p(0, T) \subset B^p_\gamma L^p(0, T) \subset L^p((0, T), L^p_{w_\delta}).$$

Proof. First, consider $u \in L^p_{w_\gamma}$. It is easy to see that $\|u\|_{B^p_\gamma} \leq 2^{\gamma/p}\|u\|_{L^p_{w_\gamma}}$ and by dominated convergence we have

$$\frac{1}{R^\gamma}\int_{|x|\leq R}|u|^p\,dx = \int_{|x|\leq R}\frac{|u|^p}{(1+|x|)^\gamma}\frac{(1+|x|)^\gamma}{R^\gamma}\,dx$$

converges to zero when $R \to +\infty$. Then $L^p_{w_\gamma} \hookrightarrow B^p_{\gamma,0}$. To study the other part, we observe that

$$\begin{aligned}\int \frac{|u|^p}{(1+|x|)^\delta}\,dx &= \int_{|x|\leq 1}\frac{|u|^p}{(1+|x|)^\delta}\,dx + \sum_{n\in\mathbb{N}}\int_{2^{n-1}\leq|x|\leq 2^n}\frac{|u|^p}{(1+|x|)^\delta}\,dx\\ &\leq \int_{|x|\leq 1}|u|^p\,dx + \sum_{n\in\mathbb{N}}\frac{1}{(1+2^{n-1})^\delta}\int_{2^{n-1}\leq|x|\leq 2^n}|u|^p\,dx\\ &\leq \int_{|x|\leq 1}|u|^p\,dx + c\sum_{n\in\mathbb{N}}\frac{1}{2^{\delta n}}\int_{2^{n-1}\leq|x|\leq 2^n}|u|^p\,dx\\ &\leq (1+c\sum_{n\in\mathbb{N}}\frac{1}{2^{(\delta-\gamma)n}})\sup_{R\geq 1}\frac{1}{R^\gamma}\int_{|x|\leq R}|u|^p\,dx,\end{aligned}$$

therefore, $B^p_\gamma \subset L^p_{w_\delta}$.

Similarly, for all $\delta > \gamma$, $B^p_\gamma L^p(0,T) \subset L^p((0,T), L^p_{w_\delta})$.

To prove that $L^p((0,T), L^p_{w_\gamma}) \subset B^p_{\gamma,0}L^p(0,T)$. Consider $\lambda > 1$, $n \in \mathbb{N}$ and denote $u_n(t,x) = u(t,\lambda^n x)$. As we have

$$\begin{aligned}&\sup_{R\geq 1}\frac{1}{(\lambda^n R)^\gamma}\int_0^T\int_{|x|\leq\lambda^n R}|u(t,x)|^p\,dx\,dt = \sup_{R\geq 1}\frac{\lambda^{(d-\gamma)n}}{R^\gamma}\int_0^T\int_{|x|\leq R}|u(t,\lambda^n x)|^p\,dx\,dt\\ &=\lambda^{(d-\gamma)n}\|u_n\|^p_{B^p_\gamma L^p(0,T)} \leq C\lambda^{(d-\gamma)n}\|u_n\|^p_{L^pL^p_{w_\gamma}} \leq C\int_0^T\int|u(s,x)|^p\frac{1}{(\lambda^n+|x|)^\gamma}\,dx\,dt,\end{aligned}$$

by dominated convergence we can take the limit when $n \to +\infty$ and conclude. ◇

Thereafter, we have the following result involving the interpolation theory of Banach spaces:

Theorem 13. *B^p_γ can be found by interpolation: for all $0 < \gamma < \delta < \infty$ we have*

$$B^p_\gamma = [L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}$$

and $\|\cdot\|_{B^p_\gamma}$ and $\|\cdot\|_{[L^p,L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}}$ are equivalent norms.

Proof. Consider $f \in B^p_\gamma$. For $A < 1$, we let $f_0 = 0$ and $f_1 = f$. Then, $f = f_0 + f_1$ and $\|f_1\|_{L^p_{w_\delta}} \leq CA^{\frac{\gamma}{\delta}-1}\|f\|_{B^p_\gamma}$.

For $A > 1$, we denote $R = A^{\frac{p}{\delta}} > 1$. We write $f_0 = f\mathbb{1}_{|x|\leq R}$ and $f_1 = f\mathbb{1}_{|x|>R}$. Then, we find

$$\|f_0\|_p \leq C\|f\|_{B^p_\gamma}R^{\frac{\gamma}{p}} = CA^{\frac{\gamma}{\delta}}\|f\|_{B^p_\gamma}$$

and

$$\begin{aligned}
\|f_1\|^p_{L^p_{w_\delta}} &= \sum_{n\in\mathbb{N}} \int_{2^{n-1}R\le|x|\le 2^n R} \frac{|u|^p}{(1+|x|)^\delta}\,dx \\
&\le CR^{\gamma-\delta} \sum_{n\in\mathbb{N}} \frac{1}{2^{(\delta-\gamma)n}} \|f\|^p_{B^p_\gamma} \\
&= CA^{(\frac{\gamma}{\delta}-1)p} \|f\|^p_{B^p_\gamma}.
\end{aligned}$$

Therefore, $B^p_\gamma \hookrightarrow [L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}$.

Now, consider $f \in [L^p, L^p_{w_\delta}]_{\frac{\gamma}{\delta},\infty}$, then there exists $c > 0$ such that for all $A > 0$, there exist $f_0 \in L^p$ and $f_1 \in L^p_{w_\delta}$ such that $f = f_0 + f_1$,

$$\|f_0\|_p \le cA^{\frac{\gamma}{\delta}} \qquad \text{and} \qquad \|f_1\|_{L^p_{w_\delta}} \le cA^{\frac{\gamma}{\delta}-1}.$$

For each $j \in \mathbb{N}$, we take $A = 2^{\frac{j\delta}{p}}$ so that

$$\begin{aligned}
&\frac{1}{2^{j\gamma}} \int_{|x|<2^j} |f|^p\,dx \\
&\le C\left(\frac{1}{2^{j\gamma}} \int_{|x|<2^j} |f_0|^p\,dx + \frac{1}{2^{j\gamma}} \int_{|x|<2^j} |f_1|^p\,dx\right) \\
&\le C\left(\frac{1}{2^{j\gamma}} \|f_0\|^p_p + \frac{C}{2^{j\gamma}} \int_{|x|\le 1} \frac{|f_1|^p}{(1+|x|)^\delta}\,dx + C\sum_{k=1}^{j} \frac{2^{k\delta}}{2^{j\gamma}} \int_{2^{k-1}<|x|<2^k} \frac{|f_1|^p}{(1+|x|)^\delta}\,dx\right) \\
&\le C\left(\frac{1}{2^{j\gamma}} \|f_0\|^p_p + C' 2^{j(\delta-\gamma)} \|f_1\|^p_{L^p_{w_\delta}}\right) \\
&\le C''.
\end{aligned}$$

We have found $\sup_{j\in\mathbb{N}} \frac{1}{2^{j\gamma}} \int_{|x|<2^j} |f|^p\,dx < +\infty$ and hence $\sup_{R\ge 1} \frac{1}{R^\gamma} \int_{|x|<R} |f|^p\,dx < +\infty$. ◇

An important corollary of Theorem 13 reads as follows:

Corollary 6.1. *If $\delta \in (0,d)$ and $p \in (1,+\infty)$, then :*

- *The Riesz transforms $\mathcal{R}_j$ are bounded on the space B^p_δ :*

$$\|\mathcal{R}_j f\|_{B^p_\delta} \le C_{p,\delta} \|f\|_{B^p_\delta}$$

- *The Hardy–Littlewood maximal function operator is bounded on the space B^p_δ :*

$$\|\mathcal{M}_f\|_{B^p_\delta} \le C_{p,\delta} \|f\|_{B^p_\delta}.$$

Proof. By Theorem 13 we know that $B^p_\delta = [L^p, L^p_{w_{\delta_0}}]_{\frac{\delta}{\delta_0},\infty}$, for some $\delta < \delta_0 < d$. Then, we conclude the continuity of these operators on $L^p_{w_{\delta_0}}$. ◇

6.4 Solutions in local Morrey spaces

Local Morrey spaces B^2_d occur naturally in the setting of homogeneous statistical solutions, we refer to (Vishik and Fursikov, 1977) and (Dostoglou, 2002) for exemple.

For the (NS) equations, in the paper (Bradshaw, Kukavica, and Tsai, 2019), the main theorem on global existence given in (Fernández-Dalgo and Lemarié-Rieusset, 2020b) is improved with respect to the initial data. We *relax* the method developed in the second paper to *enlarge* the initial data space and thus we generalize the previous results to the framework of the (MHD) equations.

Our main result in this direction reads as follows:

Theorem 14 (Local and global existence)**.** *Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_2(\mathbb{R}^3)$ be divergence-free vector fields. Let $\mathbb{F}$ be a tensor belonging to $B_2L^2(0,T)$. Then, there exists a time $T_0 \in (0,T)$ such that the system (MHD) admits a solution $(\boldsymbol{u}, \boldsymbol{b}, p)$ with the following properties :*

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T_0), B_2)$ and $\nabla \boldsymbol{u}, \nabla \boldsymbol{b}$ belong to $B_2L^2(0,T_0)$*
- *the pressure p is related to $\boldsymbol{u}, \boldsymbol{b}$ and $\mathbb{F}$ by the formula:*

$$p = \sum_{1 \le i,j \le 3} \mathcal{R}_i \mathcal{R}_j (u_i u_j - b_i b_j - F_{i,j})$$

- *the map $t \in [0,T) \mapsto (\boldsymbol{u}(t,\cdot), \boldsymbol{b}(t,\cdot))$ is $*$-weakly continuous from $[0,T)$ to $B_2(\mathbb{R}^3)$, and for all compact set $K \subset \mathbb{R}^3$,*

$$\lim_{t \to 0} \|(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_0, \boldsymbol{b}(t,\cdot) - \boldsymbol{b}_0)\|_{L^2(K)} = 0,$$

- *the solution $(\boldsymbol{u}, \boldsymbol{b}, p)$ is suitable : there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that*

$$\begin{aligned} \partial_t \Big(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\Big) =& \Delta\Big(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}\Big) - |\nabla \boldsymbol{u}|^2 - |\nabla \boldsymbol{b}|^2 - \nabla \cdot \Big(\big(\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2} + p\big) \boldsymbol{u} \Big) \\ &+ \nabla \cdot [(\boldsymbol{u} \cdot \boldsymbol{b}) \boldsymbol{b}] + \boldsymbol{u} \cdot (\nabla \cdot \mathbb{F}) - \mu. \end{aligned}$$

We also obtain for all $0 \le t \le T_0$,

$$\begin{aligned} &\max\{\|(\boldsymbol{u},\boldsymbol{b})(t)\|^2_{B_2}, \|\nabla(\boldsymbol{u},\boldsymbol{b})\|^2_{B_2L^2(0,T_0)}\} \\ &\le C\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|^2_{B_2} + C\|\mathbb{F}\|^2_{B_2L^2(0,t)} + C\int_0^t \|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{B_2} + \|(\boldsymbol{u},\boldsymbol{b})(s)\|^6_{B_2} ds. \end{aligned} \tag{6.10}$$

Moreover, if the data verify:

$$\lim_{R \to +\infty} R^{-2} \int_{|x| \le R} |\boldsymbol{u}_0(x)|^2 + |\boldsymbol{b}_0(x)|^2 \, dx = 0$$

$$\lim_{R \to +\infty} R^{-2} \int_0^{+\infty} \int_{|x| \le R} |\mathbb{F}(t,x)|^2 \, dx \, ds = 0,$$

then we get a global weak solution $(\boldsymbol{u}, \boldsymbol{b}, p)$.

We prove a global control on the solutions (6.10) which is not exhibited in (Bradshaw, Kukavica, and Tsai, 2019). In the setting of the space $B_2(\mathbb{R}^3)$, the control on the pressure p is a little more technical.

Getting back to the (NS) equations, the global existence and uniqueness of solutions for the 2D case with initial data $\mathbf{u}_0 \in B_2(\mathbb{R}^2)$ was an open problem proposed by A. Basson in (Basson, 2006b). In Section 6.5 we make a discussion on problematic arising on the local and global existence for the 2D case, and moreover, we give a sketch of the proof of a result analogous to Theorem 14 in dimension 2.

Our main theorem bases on results for the equations:

$$(MHD_\epsilon)\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{v}\cdot\nabla)\mathbf{u} + (\mathbf{c}\cdot\nabla)\mathbf{b} - \nabla p + \nabla\cdot\mathbb{F}, \\ \partial_t \mathbf{b} = \Delta \mathbf{b} - (\mathbf{v}\cdot\nabla)\mathbf{b} + (\mathbf{c}\cdot\nabla)\mathbf{u} - \nabla q, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{b} = 0, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \mathbf{b}(0,\cdot) = \mathbf{b}_0. \end{cases}$$

where we will consider $(\mathbf{v},\mathbf{c}) = (\mathbf{u}*\theta_\epsilon, \mathbf{b}*\theta_\epsilon)$, where, for $0<\varepsilon<1$ and for a fixed, non-negative and radially non increasing test function $\theta \in \mathcal{D}(\mathbb{R}^3)$ which is equals to 0 for $|x|\geq 1$ and $\int \theta\, dx = 1$; we define $\theta_\varepsilon(x) = \frac{1}{\varepsilon^3}\theta(x/\varepsilon)$.

6.4.1 A priori estimates

Theorem 15. *Let $0<T<+\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_2$ be a divergence-free vector fields and let $\mathbb{F}$ be a tensor such that $\mathbb{F}\in B_2L^2(0,T)$. Moreover, let $(\boldsymbol{u},\boldsymbol{b},p,q)$ be a solution of the problem (MHD_ϵ) or a solution of the problem (MHD).*

We suppose that:

- *$\boldsymbol{u},\boldsymbol{b}$ belongs to $L^\infty((0,T),B_2)$ and $\nabla\boldsymbol{u}, \nabla\boldsymbol{b}$ belongs to $B_2L^2(0,T)$.*
- *The map $t\in[0,T)\mapsto \boldsymbol{u}(t,\cdot)$ is $*$-weakly continuous from $[0,T)$ to B_2, and for all compact set $K\subset\mathbb{R}^3$ we have:*

$$\lim_{t\to 0} \|(\boldsymbol{u}(t,\cdot)-\boldsymbol{u}_0, \boldsymbol{b}(t,\cdot)-\boldsymbol{b}_0)\|_{L^2(K)} = 0.$$

In particular, we can take p and q related to $\boldsymbol{u}$, $\boldsymbol{b}$ and $\mathbb{F}$ by

$$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i\mathcal{R}_j(v_iu_j - c_ib_j - F_{i,j}) \ \text{and}\ \ q = \sum_{1\leq i,j\leq 3}\mathcal{R}_i\mathcal{R}_j(v_ib_j - c_ju_i).$$

- *The solution $(\boldsymbol{u},\boldsymbol{b},p,q)$ is suitable : there exists a non-negative locally finite measure μ on $(0,T)\times\mathbb{R}^3$ such that*

$$\begin{aligned}\partial_t(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}) =&\Delta(\frac{|\boldsymbol{u}|^2+|\boldsymbol{b}|^2}{2}) - |\nabla\boldsymbol{u}|^2 - |\nabla\boldsymbol{b}|^2 - \nabla\cdot\left((\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2})\boldsymbol{v} + p\boldsymbol{u}\right)\\ &+\nabla\cdot((\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{c} + q\boldsymbol{b}) + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu.\end{aligned} \tag{6.11}$$

Then, there exists a constant $C\geq 1$, which does not depend on T, and not on $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ $\boldsymbol{u},\boldsymbol{b}$, $\mathbb{F}$ nor ϵ, such that:

- *We have on* $[0, T)$:

$$\begin{aligned} &\max\{\|(\boldsymbol{u},\boldsymbol{b})(t)\|_{B_2}^2, \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{B_2L^2(0,t)}^2\} \\ &\leq C\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2 + C\|\mathbb{F}\|_{B_2L^2(0,t)}^2 + C\int_0^t \|(\boldsymbol{u},\boldsymbol{b})(s)\|_{B_2}^2 + \|(\boldsymbol{u},\boldsymbol{b})(s)\|_{B_2}^6 ds. \end{aligned} \tag{6.12}$$

- *Moreover, if* $T_0 < T$ *is small enough:*

$$C\left(1+\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2 + \|\mathbb{F}\|_{B_2L^2(0,T_0)}^2\right)^2 T_0 \leq 1,$$

then the following estimate with respect to the data holds:

$$\begin{aligned} &\sup_{0\leq t\leq T_0} \max\{\|(\boldsymbol{u},\boldsymbol{b})(t,.)\|_{B_2}^2, \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{B_2L^2(0,t)}^2\} \\ &\qquad \leq C\left(1+\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2 + \|\mathbb{F}\|_{B_2L^2(0,T_0)}^2\right). \end{aligned} \tag{6.13}$$

Proof. We focus only in the case $(\mathbf{v},\mathbf{c}) = (\mathbf{u}*\theta_\varepsilon, \mathbf{b}*\theta_\varepsilon)$, as the case $(\mathbf{v},\mathbf{c}) = (\mathbf{u},\mathbf{b})$ can be treated in a similar way.

We start by proving (6.12). As usual, we look to apply the energy balance (6.11) to a suitable test function.

Consider the function α_{η,t_0,t_1} defined in (5.7) and take a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equals to 1 for $|x| \leq 1/2$ and is equals to 0 for $|x| \geq 1$; and for $R \geq 1$ we set

$$\phi_R(x) = \phi(\frac{x}{R}). \tag{6.14}$$

Applying the energy balance (6.11) to the test function $\alpha_{\eta,t_0,t_1}\phi_R$, we obtain

$$\begin{aligned} &-\iint \frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\, dx\, ds + \iint |\nabla\mathbf{u}|^2 + |\nabla\mathbf{b}|^2\, \alpha_{\eta,t_0,t_1}\phi_R dx\, ds \\ &\leq \int\int \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\,\alpha_{\eta,t_0,t_1}\Delta\phi_R\, dx\, ds \\ &\quad + \sum_{i=1}^3 \iint [(\frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{b}|^2}{2})v_i + pu_i]\alpha_{\eta,t_0,t_1}\partial_i\phi_R\, dx\, ds \\ &\quad + \sum_{i=1}^3 \iint [(\mathbf{u}\cdot\mathbf{b})c_i + qb_i]\alpha_{\eta,t_0,t_1}\partial_i\phi_R\, dx\, ds \\ &\quad - \sum_{1\leq i,j\leq 3} (\iint F_{i,j}u_j\alpha_{\eta,t_0,t_1}\partial_i\phi_R\, dx\, ds + \iint F_{i,j}\partial_i u_j\, \alpha_{\eta,t_0,t_1}\phi_R\, dx\, ds). \end{aligned}$$

We let η goes to 0, when the limit in the left side above is well-defined, we obtain by the dominated convergence :

$$\begin{aligned}
-\lim_{\eta\to 0}\iint \frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int |\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\ \phi_R dx\,ds\\
\le \int_{t_0}^{t_1}\int \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\,\Delta\phi_R\,dx\,ds\\
+\sum_{i=1}^{3}\int_{t_0}^{t_1}\int [(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})v_i+pu_i]\partial_i\phi_R\,dx\,ds\\
+\sum_{i=1}^{3}\int_{t_0}^{t_1}\int [(\mathbf{u}\cdot\mathbf{b})c_i+qb_i]\partial_i\phi_R\,dx\,ds\\
-\sum_{1\le i,j\le 3}(\int_{t_0}^{t_1}\int F_{i,j}u_j\partial_i\phi_R\,dx\,ds+\int_{t_0}^{t_1}\int F_{i,j}\partial_i u_j\ \phi_R\,dx\,ds).
\end{aligned}$$

We denote

$$A_R(t)=\int (|\mathbf{u}(t,x)|^2+|\mathbf{b}(t,x)|^2)\phi_R(x)\,dx,$$

so that

$$-\iint(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=-\frac{1}{2}\int \partial_t\alpha_{\eta,t_0,t_1}A_R(s)\,ds,$$

Thus, if t_0 and t_1 are Lebesgue points of $A_R(t)$ we find

$$\lim_{\eta\to 0}-\iint(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=\frac{1}{2}(A_R(t_1)-A_R(t_0)).$$

As ϕ_R is a support compact function we can let t_0 goes to 0 and thus we may replace t_0 by 0 in this inequality. Adittionaly, letting t_1 goes to t, by $*$-weak continuity, $A_R(t)\le \lim_{t_1\to t}A_R(t_1)$, and so we can replace t_1 by $t\in(0,T)$. Therefore, for every $t\in(0,T)$:

$$\begin{aligned}
\int \frac{|\mathbf{u}(t,x)|^2+|\mathbf{b}(t,x)|^2}{2}\phi_R\,dx+\int_0^t\int (|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\ \phi_R\,ds\,dx\\
\le \int \frac{|\mathbf{u}_0(x)|^2+|\mathbf{b}_0(x)|^2}{2}\phi_R\,dx+\int_0^t\int \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\,\Delta\phi_R\,ds\,dx\\
+\sum_{i=1}^{3}\int_0^t\int [(\frac{|\mathbf{u}|^2}{2}+\frac{|\mathbf{b}|^2}{2})v_i+pu_i]\partial_i\phi_R\,dx\,ds\\
+\sum_{i=1}^{3}\int_0^t\int [(\mathbf{u}\cdot\mathbf{b})c_i+qb_i]\partial_i\phi_R\,dx\,ds\\
-\sum_{1\le i,j\le 3}(\int_0^t\int F_{i,j}u_j\partial_i\phi_R\,dx\,ds+\int_0^t\int F_{i,j}\partial_i u_j\ \phi_R\,dx\,ds).
\end{aligned}\tag{6.15}$$

To estimate the second term in the right-hand side, as $R\ge 1$ we write

$$\frac{1}{R^2}\int (|\mathbf{u}|^2+|\mathbf{b}|^2)\Delta\phi_R\,dx\le \frac{C}{R^4}\int_{B(0,R)}(|\mathbf{u}|^2+|\mathbf{b}|^2)\,dx\le C(\|\mathbf{u}\|_{B_2}^2+\|\mathbf{b}\|_{B_2}^2).$$

To treat third and fourth terms, we consider first the expressions where the pressure terms p and q do not appear. By Hölder inequalities and Sobolev embeddings we obtain:

$$\sum_{i=1}^{3}\int\frac{(\mathbf{u}\cdot\mathbf{b})}{2}(b_i*\theta_\epsilon)\partial_i\phi_R\,dx \le \|\mathbf{u}\|_{L^{\frac{12}{5}}(B(0,R))}\|\mathbf{b}\|_{L^{\frac{12}{5}}(B(0,R))}\|\mathbf{b}*\theta_\epsilon\|_{L^6(B(0,R))}\|\nabla\phi_R\|_{L^\infty}$$
$$\le\frac{C}{R}\|\mathbf{u}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{u}\|^{1/4}_{L^6(B(0,R))}\|\mathbf{b}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{b}\|^{5/4}_{L^6(B(0,R+1))}$$
$$\le\frac{C}{R}\|\mathbf{b}\|^{3/4}_{L^2(B(0,R))}\|\mathbf{u}\|^{3/4}_{L^2(B(0,R))}U^{1/4}B^{5/4},$$

where we have denoted

$$U=\left(\int|\phi_{2R}\nabla\mathbf{u}|^2dx\right)^{1/2}+\left(\int_{|x|\le 2R}|\mathbf{u}|^2dx\right)^{1/2}$$

and

$$B=\left(\int|\phi_{2(R+1)}\nabla\mathbf{b}|^2dx\right)^{1/2}+\left(\int_{|x|\le 2(R+1)}|\mathbf{b}|^2dx\right)^{1/2}.$$

Then, by the Young's inequalities for products with $1=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{5}{8}$,

$$\frac{1}{R^2}\sum_{i=1}^{3}\int\frac{(\mathbf{u}\cdot\mathbf{b})}{2}(b_i*\theta_\epsilon)\partial_i\phi_R\,dx$$
$$\le C\Big(\frac{\|\mathbf{u}\|_{L^2(B(0,R))}}{R}\Big)^{3/4}\Big(\frac{\|\mathbf{b}\|_{L^2(B(0,R))}}{R}\Big)^{3/4}\Big(\frac{U}{R}\Big)^{1/4}\Big(\frac{B}{R}\Big)^{5/4}$$
$$\le C\|(\mathbf{u},\mathbf{b})\|^6_{B_2}+C\|(\mathbf{u},\mathbf{b})\|^2_{B_2}+\frac{C_0}{R^2}\int|\phi_{2R}\nabla\mathbf{u}|^2+|\phi_{2(R+1)}\nabla\mathbf{b}|^2\,dx,$$

where $C_0>0$ is an arbitrarily small constant to be fixed later.

Now, we use the following technical lemma which will be proved below, in order to estimate the expressions where the terms p and q appear.

Lemma 6.6. *Under the hypothesis of Theorem 15, p and q belong to $L^{3/2}_{\text{loc}}$ and there exist an arbitrarily small constant $C_0>0$ and a constant $C=C(C_0)>0$, which do not depend on : T, $\boldsymbol{u}$, $\boldsymbol{b}$, $\boldsymbol{u}_0$, $\boldsymbol{b}_0$, $\mathbb{F}$ nor ϵ; such that for all $R\ge 1$ and for all $0\le t\le T$,*

$$\frac{1}{R^2}\sum_{i=1}^{3}\int_0^t\int(pu_i+qb_i)\,\partial_i\phi_R\,ds\,dx$$
$$\le C\|\mathbb{F}\|^2_{B_2L^2(0,t)}+C\int_0^t\|(\boldsymbol{u},\boldsymbol{b})(s)\|^2_{B_2}+\|(\boldsymbol{u},\boldsymbol{b})(s)\|^6_{B_2}$$
$$+\frac{C_0}{R^2}\int\int_0^t|\varphi_{2(5R+1)}\nabla\boldsymbol{u}|^2+|\varphi_{2(5R+1)}\nabla\boldsymbol{b}|^2\,dx.$$

Finally, for the fifth term in the right side of (6.15) we have

$$\left|\frac{1}{R^2}\sum_{1\le i,j\le 3}\int_0^t\int F_{i,j}(\partial_i u_j)\phi_R\,dx\,ds\right|\le C\|\mathbb{F}\|^2_{B_2L^2(0,t)}+\frac{C_0}{R^2}\int_0^t\int_{|x|<R}|\nabla\mathbf{u}|^2\,dx\,ds,$$

and

$$\left|\frac{1}{R^2}\sum_{1\le i,j\le 3}\int_0^t\int F_{i,j}u_i\partial_j(\phi_R)\,dx\,ds\right|\le C\|\mathbb{F}\|^2_{B_2L^2(0,t)}+C\int_0^t\|\mathbf{u}(s)\|^2_{B_2}\,ds,$$

where $C_0 > 0$ always denote a small enough constant.

Once we dispose of all these estimates, we conclude that

$$\begin{aligned}
&\int \Big(\frac{|\mathbf{u}(t,x)|^2}{2} + \frac{|\mathbf{b}(t,x)|^2}{2}\Big)\phi_R \, dx + \int_0^t \int (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{b}|^2) \ \phi_R \, ds \, dx \\
&\leq \int \Big(\frac{|\mathbf{u}(0,x)|^2}{2} + \frac{|\mathbf{b}(0,x)|^2}{2}\Big)\phi_R \, dx + C\|\mathbb{F}\|^2_{B_2 L^2(0,t)} \, ds \\
&\quad + C \int_0^t \|(\mathbf{u},\mathbf{b})(s,\cdot)\|^2_{B_2} + \|(\mathbf{u},\mathbf{b})(s,\cdot)\|^6_{B_2} \, ds \\
&\quad + \frac{C_0}{R^2} \int \int_0^t |\varphi_{2(5R+1)} \nabla \mathbf{u}|^2 + |\varphi_{2(5R+1)} \nabla \mathbf{b}|^2 \, dx,
\end{aligned}$$

and the desired control (6.12) follows. To finish this proof, (6.13) follows directly from (6.12) and the Lemma at the end of this section. $\diamond$

Proof of Lemma 6.6. As in the proof of the theorem above, we consider the case $(\mathbf{v},\mathbf{c}) = (\mathbf{u} * \theta_\varepsilon, \mathbf{b} * \theta_\varepsilon)$. We focus only on the expression which involves the pressure p, because the computations for the other expression, where q appears, are very similar.

We have

$$\frac{1}{R^2} \sum_{k=1}^{3} \int_0^t \int_{|x|\leq R} |p u_k| |\partial_k \phi_R| \, dx \, ds \leq \frac{c}{R^3} \sum_{k=1}^{3} \int_0^t \int_{|x|\leq R} |p u_k| \, dx \, ds,$$

then, recalling the formula

$$p = \sum_{1\leq i,j\leq 3} \mathcal{R}_i \mathcal{R}_j ((u_i * \theta_\varepsilon) u_j - (b_i * \theta_\varepsilon) b_j - F_{i,j}),$$

we find

$$\begin{aligned}
&\frac{1}{R^2} \sum_{k=1}^{3} \int_0^t \int_{|x|\leq R} |p u_k| |\partial_k \phi_R| \, dx \, ds \\
&\leq \frac{c}{R^3} \sum_{k=1}^{3} \int_0^t \int_{|x|\leq R} |u_k \sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j ((u_i * \theta_\varepsilon) u_j)| \, dx \, ds \\
&\quad + \frac{c}{R^3} \sum_{k=1}^{3} \int_0^t \int_{|x|\leq R} |u_k \sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j ((b_i * \theta_\varepsilon) b_j - F_{i,j})| \, dx \, ds.
\end{aligned}$$

In view that we have the same information on $\mathbf{u}$ and $\mathbf{b}$, it is enough to analyse the last term above. For $R \geq 1$, we define :

$$p_1 = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y|<5R} (\theta_\epsilon * b_i) b_j), \ \ p_2 = -\sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y|\geq 5R} (\theta_\epsilon * b_i) b_j),$$

and

$$p_3 = -\sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y|<5R} F_{i,j}), \ \ p_4 = \sum_{i,j} \mathcal{R}_i \mathcal{R}_j (\mathbb{1}_{|y|\geq 5R} F_{i,j}).$$

By the Young's inequalities we get,

$$\frac{c}{R^3}\sum_{k=1}^{3}\int_0^t\int_{|x|\le R}|u_k\sum_{i,j=1}^{3}\mathcal{R}_i\mathcal{R}_j((b_i*\theta_\varepsilon)b_j-F_{i,j})|\,dx\,ds$$
$$\le \frac{C}{R^3}\int_0^t\int_{|x|\le R}(|p_1|^{3/2}+|p_2|^{3/2}+|\mathbf{u}|^3+|p_3|^2+|p_4|^2+|\mathbf{u}|^2)dx\,ds,$$

and next we will study each term.

To study p_1, we use the continuity of $\mathcal{R}_i$ on $L^{\frac{3}{2}}(\mathbb{R}^3)$ and the fact that $supp(\theta_\epsilon)\subset \overline{B(0,1)}$ to obtain

$$\begin{aligned}\int_{|x|\le R}|p_1|^{3/2}dx &\le C\int|p_1|^{3/2}dx\le C\int|(\mathbb{1}_{|x|<5R}(\theta_\epsilon*\mathbf{b})\otimes\mathbf{b})|^{3/2}dx\\
&\le C\left(\int|\mathbb{1}_{|x|<5R}(\theta_\epsilon*\mathbf{b})|^3dx\right)^{1/2}\left(\int|\mathbb{1}_{|y|<5R}\mathbf{b}|^3\,dx\right)^{1/2}\\
&\le C\left(\int_{|x|\le 5R}\int_{|x-z|\le 1}\theta_\epsilon(x-z)|\mathbf{b}(z)|^3dzdx\right)^{1/2}\left(\int|(\mathbb{1}_{|y|<5R}\mathbf{b})|^3\,dx\right)^{1/2}\\
&\le C\left(\int_{|x|\le 5R}\int_{|z|\le 5R+1}\theta_\epsilon(x-z)|\mathbf{b}(z)|^3dzdx\right)^{1/2}\left(\int|(\mathbb{1}_{|y|<5R}\mathbf{b})|^3\,dx\right)^{1/2}\\
&\le C\int_{|z|\le 5R+1}|\mathbf{b}|^3\,dz,\end{aligned}$$

so we see that

$$\int_{|x|\le R}|\mathbf{u}|^3+|p_1|^{3/2}\,dx\le C\int_{|x|\le 5R+1}|\mathbf{u}|^3+|\mathbf{b}|^3\,dx.$$

Using a Sobolev embedding, we find

$$\begin{aligned}&\frac{C}{R^3}\int_{|x|\le 5R+1}|\mathbf{u}|^3\,dx\le\frac{C}{R^3}\|\mathbf{u}\|^{3/2}_{L^2(B(0,5R+1))}\|\mathbf{u}\|^{3/2}_{L^6(B(0,5R+1))}\\
&\le\frac{C}{R^{3/2}}\|\mathbf{u}\|^{3/2}_{L^2(B(0,5R+1))}\left(\left(\frac{1}{R^2}\int|\phi_{2(5R+1)}\nabla\mathbf{u}|^2dx\right)^{\frac{1}{2}}+\left(\frac{1}{R^2}\int_{|x|\le 2(5R+1)}|\mathbf{u}|^2dx\right)^{\frac{1}{2}}\right)^{\frac{3}{2}}\\
&\le C\|\mathbf{u}\|^6_{B_2}+C_0\|\mathbf{u}\|^2_{B_2}+\frac{C_0}{R^2}\int|\phi_{2(5R+1)}\nabla\mathbf{u}|^2dx,\end{aligned}$$

where $C_0>0$ is a arbitrarily small constant. Similar estimates works for $\mathbf{b}$.

To study p_2. Remark first that there exist a constant $C>0$, which does not depend on $R\ge 1$, such that for all $|x|\le R$ and all $|y|\ge 5R$, the kernel of the operator $\mathcal{R}_i\mathcal{R}_j$ named $\mathbb{K}_{i,j}$ verifies $|\mathbb{K}_{i,j}(x-y)|\le\frac{C}{|y|^3}$ (we refer to (Grafakos, 2009) for a proof), and then we may write:

$$
\begin{aligned}
&\left(\int_{|x|\leq R}|p_2|^{3/2}dx\right)^{2/3}\\
&\leq C\sum_{i,j}\left(\int_{|x|\leq R}\left(\int|\mathbb{K}_{i,j}(x-y)|\,|(\theta_\epsilon*b_i)(y)b_j(y)|\,\mathbb{1}_{|y|\geq 5R}\,dy\right)^{3/2}dx\right)^{2/3}\\
&\leq C\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|(\theta_\epsilon*\mathbf{b})\otimes\mathbf{b}|\,dy\right)^{3/2}dx\right)^{2/3}\\
&\leq CR^2\int_{|y|\geq 5R}\frac{1}{|y|^3}|(\theta_\epsilon*\mathbf{b})\otimes\mathbf{b}|dy\\
&\leq CR^2\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\theta_\epsilon*\mathbf{b}|^2dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}\int_{|y-z|<1}\theta_\epsilon(y-z)|\mathbf{b}(z)|^2dz\,dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\left(\int_{|y|\geq 5R}\int_{|z|\geq 5R-1}\frac{1}{|z|^3}\theta_\epsilon(y-z)|\mathbf{b}(z)|^2dz\,dy\right)^{1/2}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbf{b}|^2dy\right)^{1/2}\\
&\leq CR^2\int_{|z|\geq 5R-1}\frac{1}{|z|^3}|\mathbf{b}|^2dz.
\end{aligned}
$$

As $B_2(\mathbb{R}^3)\subset L^2_{w_3}(\mathbb{R}^3)$, we finally get

$$\frac{C}{R^3}\int_{|y|\leq R}|p_2|^{3/2}dx\leq C\left(\int\frac{1}{(1+|z|)^3}|\mathbf{b}|^2\right)^{3/2}\leq C\|\mathbf{b}\|^3_{B_2}.$$

Now, we treat the terms p_3 and p_4 which involve the tensor $\mathbb{F}$. For p_3, continuity of the Riesz transform $\mathcal{R}_i$ on L^2 gives

$$\frac{c}{R^3}\int_0^t\int_{|x|\leq R}|p_3|^2dx\,ds\leq\frac{C}{R^3}\sum_{i,j}\int_0^t\int_{|x|<5R}|\mathbb{F}_{i,j}|^2dx\,ds\leq C\|\mathbb{F}\|^2_{B_2L^2(0,t)}.$$

For the term p_4, observe that

$$
\begin{aligned}
\left(\int_{|x|\leq R}|p_4|^2dx\right)^{1/2}C&\leq\sum_{i,j}\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}|\mathbb{K}_{i,j}(x-y)\mathbb{F}_{i,j}|\,dy\right)^2dx\right)^{1/2}\\
&\leq C\sum_{i,j}\left(\int_{|x|\leq R}\left(\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbb{F}_{i,j}|\,dy\right)^2dx\right)^{1/2}\\
&\leq C\sum_{i,j}R^{3/2}\int_{|y|\geq 5R}\frac{1}{|y|^3}|\mathbb{F}_{i,j}|\,dy,
\end{aligned}
$$

then, taking $0 < \delta < 1$, by the Hölder inequalities we find

$$\begin{aligned}\frac{C}{R^3}\int_0^t\int_{|x|\leq R}|p_4|^2\,dx\,ds &\leq C\sum_{i,j}\int_0^t\left(\int\frac{1}{(1+|x|)^3}|\mathbb{F}_{i,j}|\,dx\right)^2 ds\\ &\leq C\sum_{i,j}\int_0^t\int\frac{1}{(1+|x|)^{2+\delta}}|\mathbb{F}_{i,j}|^2\,dx\,ds\\ &\leq C\sum_{i,j}\int\frac{1}{(1+|x|)^{2+\delta}}\int_0^t|\mathbb{F}_{i,j}|^2\,ds\,dx\\ &\leq C\|\mathbb{F}\|^2_{B_2L^2(0,t)}\end{aligned}$$

and the proof is finished. ◇

Before to finish this section we present the following lemma, used in the proof of the theorem above, which is a variant of Lemma 3.5.

Lemma 6.7. *Let α be a non-negative bounded measurable function on $[0,T)$ which satisfies, for three constants $A, B > 0$ and $b > 1$,*

$$\alpha(t) \leq A + B\int_0^t 1+\alpha(s)^b\,ds.$$

If $T_0 > 0$ and $T_1 = \min(T, T_0, \frac{1}{4(b-1)B(A+BT_0)^{b-1}})$, we have, for every $t \in [0,T_1]$, $\alpha(t) \leq \sqrt{2}(A+BT_0)$.

Proof. We define

$$\Phi(t) = A + BT_0 + B\int_0^t \alpha^b\,ds \text{ and } \Psi(t) = A + BT_0 + B\int_0^t \Phi(s)^b\,ds,$$

so that for all $t \in [0,T_1]$, $\alpha \leq \Phi \leq \Psi$, and then

$$\Psi'(t) = B\Phi(t)^b \leq B\Psi(t)^b$$

so

$$\frac{1}{\Psi(0)^{b-1}} - \frac{1}{\Psi(t)^{b-1}} \leq (b-1)Bt,$$

and we conclude

$$\Psi(t)^{b-1} \leq \frac{\Psi(0)^{b-1}}{1-(b-1)B\Psi(0)^{b-1}t} \leq 2\Psi(0)^{b-1}.$$

◇

6.4.2 A stability result

Theorem 16. *Let $0 < T < +\infty$. Let $\boldsymbol{u}_{0,n}, \boldsymbol{b}_{0,n}$ be divergence-free vector fields belonging to B_2. Let $\mathbb{F}_n$ be tensors belonging to $B_2L^2(0,T)$. Suppose that $(\boldsymbol{u}_n, \boldsymbol{b}_n, p_n, q_n)$ is a solution of the (MHD_{ϵ_n}) problem, with $\epsilon_n \to 0^+$, or a solution of the (MHD) problem (in which* *case* $q_n = 0$) *:*

$$\begin{cases} \partial_t \boldsymbol{u}_n = \Delta \boldsymbol{u}_n - (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n + (\boldsymbol{c}_n \cdot \nabla)\boldsymbol{b}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n, \\ \partial_t \boldsymbol{b}_n = \Delta \boldsymbol{b}_n - (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{b}_n + (\boldsymbol{c}_n \cdot \nabla)\boldsymbol{u}_n - \nabla q_n, \\ \nabla \cdot \boldsymbol{u}_n = 0, \ \nabla \cdot \boldsymbol{b}_n = 0, \\ \boldsymbol{u}_n(0,\cdot) = \boldsymbol{u}_{0,n}, \ \boldsymbol{b}_n(0,\cdot) = \boldsymbol{b}_{0,n}. \end{cases}$$

which verifies the same hypothesis of Theorem 15.

We assume that $(\boldsymbol{u}_{0,n}, \boldsymbol{b}_{0,n})$ *is strongly convergent to* $(\boldsymbol{u}_{0,\infty}, \boldsymbol{b}_{0,\infty})$ *in* B_2*, and the sequence* $\mathbb{F}_n$ *is strongly convergent to* $\mathbb{F}_\infty$ *in* $B_2L^2(0,T)$*. Then, there exist* $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty, p_\infty)$ *and an increasing sequence* $(n_k)_{k\in\mathbb{N}}$ *with values in* $\mathbb{N}$ *such that:*

- $(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ *converges *-weakly to* $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ *in* $L^\infty((0,T), B_2)$*,* $(\nabla\boldsymbol{u}_{n_k}, \nabla\boldsymbol{b}_{n_k})$ *converges weakly to* $(\nabla\boldsymbol{u}_\infty, \nabla\boldsymbol{b}_\infty)$ *in* $B_2L^2(0,T)$*.*
- $(\boldsymbol{u}_{n_k}, \boldsymbol{b}_{n_k})$ *converges strongly to* $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty)$ *in* $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$*.*
- *For* $2 < \gamma < 5/2$*, the sequence* (p_{n_k}, q_{n_k})*converges weakly to* $(p_\infty, 0)$ *in* $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_\gamma})$*.*

Moreover, $(\boldsymbol{u}_\infty, \boldsymbol{b}_\infty, p_\infty)$ *is a solution of the problem* (MHD)*:*

$$\begin{cases} \partial_t \boldsymbol{u}_\infty = \Delta \boldsymbol{u}_\infty - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty + (\boldsymbol{b}_\infty \cdot \nabla)\boldsymbol{b}_\infty - \nabla p_\infty + \nabla \cdot \mathbb{F}_\infty, \\ \partial_t \boldsymbol{b}_\infty = \Delta \boldsymbol{b}_\infty - (\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{b}_\infty + (\boldsymbol{b}_\infty \cdot \nabla)\boldsymbol{u}_\infty \\ \nabla \cdot \boldsymbol{u}_\infty = 0, \ \nabla \cdot \boldsymbol{b}_\infty = 0, \\ \boldsymbol{u}_\infty(0,\cdot) = \boldsymbol{u}_{0,\infty}, \ \boldsymbol{b}_\infty(0,\cdot) = \boldsymbol{b}_{0,\infty}, \end{cases}$$

and verifies all the hypothesis of Theorem 15.

Proof. We will verify that the sequence $(\mathbf{u}_n, \mathbf{b}_n)$ satisfy the hypothesis of the Rellich lemma. Remark first that: since for $2 < \gamma$ we have that $\mathbf{u}_n, \mathbf{b}_n$ is bounded in $L^\infty((0,T), B_2) \subset L^\infty((0,T), L^2_{w_\gamma})$ and moreover, since we have that $\nabla\mathbf{u}_n, \nabla\mathbf{b}_n$ is bounded in $B_2L^2(0,T) \subset L^2((0,T), L^2_{w_\gamma})$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have that $(\varphi\mathbf{u}_n, \varphi\mathbf{b}_n)$ are bounded in $L^2((0,T), H^1)$. On the other hand, for the pressure p_n and the term q_n we write $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{n,i}u_{n,j} - c_{n,i}b_{n,j}), \quad p_{n,2} = -\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(F_{n,i,j}),$$

and

$$q_n = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{n,i}b_{n,j} - c_{n,i}u_{n,j}).$$

From now on, we fix $\gamma \in (2, \frac{5}{2})$. Interpolation inequalities and the continuity of the Riesz transforms in the Lebesgue weighted spaces permit to conclude that the sequence $(p_{n,1}, q_{n,1})$ is bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$. Indeed, to treat the term $p_{n,1}$ recall that for $0 < \gamma < 5/2$ the weight $w_{6\gamma/5}$ belongs to the Muckenhoupt class

$\mathcal{A}_p(\mathbb{R}^3)$, with $p \in (1, +\infty)$, then

$$\|\sum_{i,j} \mathcal{R}_i\mathcal{R}_j(\mathbf{u}_{n,i}\mathbf{u}_{n,j})w_\gamma\|_{L^{6/5}} \leq \|(\mathbf{u}_n \otimes \mathbf{u}_n)w_\gamma\|_{L^{6/5}} \leq \|\sqrt{w_\gamma}\mathbf{u}_n\|_{L^2}^{\frac{3}{2}}\|\sqrt{w_\gamma}\mathbf{u}_n\|_{L^6}^{\frac{1}{2}}$$

$$\leq \|\sqrt{w_\gamma}\mathbf{u}\|_{L^2}^{\frac{3}{2}}(\|\sqrt{w_\gamma}\mathbf{u}\|_{L^2} + \|\sqrt{w_\gamma}\nabla\mathbf{u}\|_{L^2})^{\frac{1}{2}}.$$

Similar estimates holds for the term $q_{n,1}$. Of course, the sequences $q_{n,2}$ and $p_{n,2}$ are bounded in $L^2((0,T), L^2_{w_\gamma})$. Then, we obtain that the sequence $(\varphi\partial_t\mathbf{u}_n, \varphi\partial_t\mathbf{b}_n)$ is bounded in $L^2L^2 + L^2W^{-1,6/5} + L^2H^{-1} \subset L^2((0,T), H^{-2})$. Thus, we can apply the Rellich lemma, there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$, and there exists a couple of functions $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ such that $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$. We also know that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{u}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{b}_{n_k} * \theta_{\epsilon_{n_k}})$ converges strongly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$.

As the sequence $(\mathbf{u}_n, \mathbf{b}_n)$ is bounded in $L^\infty((0,T), L^2_{w_\gamma})$ and $(\nabla\mathbf{u}_n, \nabla\mathbf{b}_n)$ is bounded in $L^2((0,T), L^2_{w_\gamma})$, we get $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges *-weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^\infty((0,T), L^2_{w_\gamma})$, and $(\nabla\mathbf{u}_{n_k}, \nabla\mathbf{b}_{n_k})$ converges weakly to $(\nabla\mathbf{u}_\infty, \nabla\mathbf{b}_\infty)$ in $L^2((0,T), L^2_{w_\gamma})$. Additionally, by the Sobolev embeddings and the interpolation inequalities we have $(\mathbf{u}_{n_k}, \mathbf{b}_{n_k})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Moreover, we find that $(\mathbf{v}_{n_k}, \mathbf{c}_{n_k}) = (\mathbf{v}_{n_k} * \theta_{\epsilon_{n_k}}, \mathbf{c}_{n_k} * \theta_{\epsilon_{n_k}})$ converges weakly to $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ in $L^3((0,T), L^3_{w_{3\gamma/2}})$ as well, since it is bounded in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Thus, the terms $v_{n_k,i}u_{n_k,j}$, $c_{n_k,i}b_{n_k,j}$, $v_{n_k,i}b_{n_k,j}$ and $c_{n_k,i}u_{n_k,j}$ are weakly convergent in $(L^{6/5}L^{6/5})_{\text{loc}}$ and hence in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.

Those terms are bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$, then they are weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$. Therefore, defining $p_\infty = p_{\infty,1} + p_{\infty,2}$ with

$$p_{\infty,1} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{\infty,i}u_{\infty,j} - c_{\infty,i}b_{\infty,j}), \quad p_2 = -\sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(F_{\infty,i,j}),$$

and

$$q_\infty = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathcal{R}_i\mathcal{R}_j(v_{\infty,i}b_{\infty,j} - c_{\infty,i}u_{\infty,j}),$$

we conclude that $(p_{n_k,1}, q_{n_k,1})$ is weakly convergent in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $(p_{\infty,1}, q_{\infty,1})$, and $p_{n_k,2}$ is strongly convergent in $L^2((0,T), L^2_{w_\gamma})$ to $p_{\infty,2}$.

As $\mathbf{v}_{n_k} = \theta_{\epsilon_{n_k}} * (\mathbf{u}_{n_k} - \mathbf{u}) + \theta_{\epsilon_{n_k}} * \mathbf{u}$, then we verify that $\mathbf{v}_{n_k}$ is convergent to $\mathbf{u}$ in $L^\alpha_{\text{loc}}([0,T_0) \times \mathbb{R}^3)$. Thus, we find $(\mathbf{v}_\infty, \mathbf{c}_\infty) = (\mathbf{u}_\infty, \mathbf{b}_\infty)$ and then $q_\infty = 0$ and $(\mathbf{u}_\infty, \mathbf{b}_\infty, p_\infty)$ verify the three first equations in the system (MHD) in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$.

It remains to verify the initial conditions. Since $(\partial_t\mathbf{u}_\infty, \partial_t\mathbf{b}_\infty)$ are locally in L^2H^{-2} the distribution $(\mathbf{u}_\infty, \mathbf{b}_\infty)$ has a representative such that $t \mapsto (\mathbf{u}_\infty(t,.), \mathbf{b}_\infty(t,.))$ is continuous from $[0,T)$ to $\mathcal{D}'(\mathbb{R}^3)$ (hence *-weakly continuous from $[0,T)$ to B_2) and additionally, they coincide with $\mathbf{u}_\infty(0,.) + \int_0^t \partial_t\mathbf{u}_\infty\, ds$ and $\mathbf{b}_\infty(0,.) + \int_0^t \partial_t\mathbf{b}_\infty\, ds$. Then, we get in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$,

$$\mathbf{u}_\infty(0,.)+\int_0^t \partial_t\mathbf{u}_\infty\,ds=\mathbf{u}_\infty=\lim_{n_k\to+\infty}\mathbf{u}_{n_k}=\lim_{n_k\to+\infty}\mathbf{u}_{n_k,0}+\int_0^t\partial_t\mathbf{u}_{n_k}\,ds$$
$$=\mathbf{u}_{\infty,0}+\int_0^t\partial_t\mathbf{u}_\infty\,ds,$$

hence $\mathbf{u}_\infty(0,.)=\mathbf{u}_{\infty,0}$. Similarly $\mathbf{b}_\infty(0,.)=\mathbf{b}_{\infty,0}$. Thus $(\mathbf{u}_\infty,\mathbf{b}_\infty,p_\infty)$ is a solution of the (MHD^*) system.

Now, we study the local energy balance. We define

$$\begin{aligned}A_{n_k}=&-\partial_t(\frac{|\mathbf{u}_{n_k}|^2+|\mathbf{b}_{n_k}|^2}{2})+\Delta(\frac{|\mathbf{u}_{n_k}|^2+|\mathbf{b}_{n_k}|^2}{2})-\nabla\cdot\left((\frac{|\mathbf{u}_{n_k}|^2}{2}+\frac{|\mathbf{b}_{n_k}|^2}{2})\mathbf{v}_{n_k}\right)\\&-\nabla\cdot(p_{n_k}\mathbf{u}_{n_k})-\nabla\cdot(q_{n_k}\mathbf{b}_{n_k})+\nabla\cdot((\mathbf{u}_{n_k}\cdot\mathbf{b}_{n_k})\mathbf{c}_{n_k})\\&+\mathbf{u}_{n_k}\cdot(\nabla\cdot\mathbb{F}_{n_k}).\end{aligned}$$

By interpolation $(\mathbf{u}_n,\mathbf{b}_n)$ is bounded in $L^{10/3}((0,T),L^{10/3}_{w_{5\gamma/3}})$, then $(\mathbf{u}_{n_k},\mathbf{b}_{n_k})$ are locally bounded in $L^{10/3}_tL^{10/3}_x$ and locally strongly convergent in $L^2_tL^2_x$. Therefore, $(\mathbf{u}_{n_k},\mathbf{b}_{n_k})$ converges strongly in $(L^3_tL^3_x)_{loc}$. By Lemma 6.6 we know that (p_{n_k},q_{n_k}) are locally bounded in $L^{3/2}_tL^{3/2}_x$. Thus the quantity A_{n_k} converges in the distributional sense to

$$\begin{aligned}A_\infty=&-\partial_t(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2})+\Delta(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2})-\nabla\cdot\left((\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})\mathbf{u}_\infty\right)\\&-\nabla\cdot(p_\infty\mathbf{u}_\infty)+\nabla\cdot((\mathbf{u}_\infty\cdot\mathbf{b}_\infty)\mathbf{b}_\infty)\\&+\mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty).\end{aligned}$$

By hypothesis, there exist μ_{n_k} a non-negative locally finite measure on $(0,T)\times\mathbb{R}^3$ such that

$$\begin{aligned}\partial_t(\frac{|\mathbf{u}_{n_k}|^2+|\mathbf{b}_{n_k}|^2}{2})=&\Delta(\frac{|\mathbf{u}_{n_k}|^2+|\mathbf{b}_{n_k}|^2}{2})-|\nabla\mathbf{u}_{n_k}|^2-|\nabla\mathbf{b}_{n_k}|^2\\&-\nabla\cdot\left((\frac{|\mathbf{u}_{n_k}|^2}{2}+\frac{|\mathbf{b}_{n_k}|^2}{2})\mathbf{v}_{n_k}\right)-\nabla\cdot(p_{n_k}\mathbf{u}_{n_k})-\nabla\cdot(q_{n_k}\mathbf{b}_{n_k})\\&+\nabla\cdot((\mathbf{u}_{n_k}\cdot\mathbf{b}_{n_k})\mathbf{c}_{n_k})+\mathbf{u}_{n_k}\cdot(\nabla\cdot\mathbb{F}_{n_k})-\mu_{n_k},\end{aligned}$$

or equivalently, $A_{n_k}=|\nabla\mathbf{u}_{n_k}|^2+|\nabla\mathbf{b}_{n_k}|^2+\mu_{n_k}$. Hence, we have

$$A_\infty=\lim_{n_k\to+\infty}|\nabla\mathbf{u}_{n_k}|^2+|\nabla\mathbf{b}_{n_k}|^2+\mu_{n_k}.$$

Let us take $\Phi\in\mathcal{D}((0,T)\times\mathbb{R}^3)$ be a non-negative function. As $\sqrt{\Phi}(\nabla\mathbf{u}_{n_k},\nabla\mathbf{b}_{n_k})$ is weakly convergent to $\sqrt{\Phi}(\nabla\mathbf{u}_\infty,\nabla\mathbf{b}_\infty)$ in $L^2_tL^2_x$, we find

$$\begin{aligned}\iint A_\infty\Phi\,dx\,ds=&\lim_{n_k\to+\infty}\iint A_{n_k}\Phi\,dx\,ds\geq\limsup_{n_k\to+\infty}\iint(|\nabla\mathbf{u}_{n_k}|^2+|\nabla\mathbf{b}_{n_k}|^2)\Phi\,dx\,ds\\&\geq\iint(|\nabla\mathbf{u}_\infty|^2+|\nabla\mathbf{b}_\infty|^2)\Phi\,dx\,ds.\end{aligned}$$

Thus, there exists a non-negative locally finite measure μ_∞ on $(0,T)\times\mathbb{R}^3$ such that $A_\infty=(|\nabla\mathbf{u}_\infty|^2+|\nabla\mathbf{b}_\infty|^2)+\mu_\infty$.

We have obtained the desired local energy balance:

$$\partial_t(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}) = \Delta(\frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}) - |\nabla \mathbf{u}_\infty|^2 - |\nabla \mathbf{b}_\infty|^2$$
$$- \nabla\cdot\left((\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})\mathbf{u}_\infty\right) - \nabla\cdot(p_\infty \mathbf{u}_\infty)$$
$$+ \nabla\cdot((\mathbf{u}_\infty\cdot\mathbf{b}_\infty)\mathbf{b}_\infty) + \mathbf{u}_\infty\cdot(\nabla\cdot\mathbb{F}_\infty) - \mu_\infty.$$

To finish this proof, we must to prove the convergence to the initial data $(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})$. Using the local energy balance, we obtain:

$$\int \frac{|\mathbf{u}_n(t,x)|^2+|\mathbf{b}_n(t,x)|^2}{2}\phi_R\,dx + \int_0^t\int (|\nabla\mathbf{u}_n|^2+|\nabla\mathbf{b}_n|^2)\;\phi_R\,dx\,ds$$
$$\leq \int \frac{|\mathbf{u}_{0,n}(x)|^2+|\mathbf{b}_{0,n}(x)|^2}{2}\phi_R\,dx + \int_0^t\int \frac{|\mathbf{u}_n|^2+|\mathbf{b}_n|^2}{2}\;\Delta\phi_R\,dx\,ds$$
$$+ \sum_{i=1}^3\int_0^t\int [(\frac{|\mathbf{u}_n|^2}{2}+\frac{|\mathbf{b}_n|^2}{2})v_{n,i} + p_n u_{n,i}]\partial_i\phi_R\,dx\,ds$$
$$+ \sum_{i=1}^3\int_0^t\int [(\mathbf{u}_n\cdot\mathbf{b}_n)c_{n,i} + q_n b_{n,i}]\partial_i\phi_R\,dx\,ds$$
$$- \sum_{1\leq i,j\leq 3}(\int_0^t\int F_{n,i,j}u_{n,j}\partial_i\phi_R\,dx\,ds + \int_0^t\int F_{n,i,j}\partial_i u_{n,j}\;\phi_R\,dx\,ds).$$

Therefore,

$$\limsup_{n_k\to+\infty}\int \frac{|\mathbf{u}_{n_k}(t,x)|^2+|\mathbf{b}_{n_k}(t,x)|^2}{2}\phi_R\,dx + \int_0^t\int (|\nabla\mathbf{u}_{n_k}|^2+|\nabla\mathbf{b}_{n_k}|^2)\;\phi_R\,dx\,ds$$
$$\leq \int \frac{|\mathbf{u}_0(x)|^2+|\mathbf{b}_0(x)|^2}{2}\phi_R\,dx + \int_0^t\int \frac{|\mathbf{u}_\infty|^2+|\mathbf{b}_\infty|^2}{2}\;\Delta\phi_R\,dx\,ds$$
$$+ \sum_{i=1}^3\int_0^t\int [(\frac{|\mathbf{u}_\infty|^2}{2}+\frac{|\mathbf{b}_\infty|^2}{2})u_{\infty,i} + p_\infty u_{\infty,i}]\partial_i\phi_R\,dx\,ds$$
$$+ \sum_{i=1}^3\int_0^t\int [(\mathbf{u}_\infty\cdot\mathbf{b}_\infty)b_{\infty,i}]\partial_i\phi_R\,dx\,ds$$
$$- \sum_{1\leq i,j\leq 3}(\int_0^t\int F_{\infty,i,j}u_{\infty,j}\partial_i\phi_R\,dx\,ds + \int_0^t\int F_{\infty,i,j}\partial_i u_{\infty,j}\;\phi_R\,dx\,ds).$$

As $\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t\mathbf{u}_{n_k}\,ds$, we observe that $\mathbf{u}_{n_k}(t,.)$ converges to $\mathbf{u}_\infty(t,.)$ in $\mathcal{D}'(\mathbb{R}^3)$, thus, it converges weakly in $L^2_{\text{loc}}(\mathbb{R}^3)$ and we find:

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2}{2}\phi_R\,dx \leq \limsup_{n_k\to+\infty}\int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2}\phi_R\,dx.$$

Moreover, by weakly convergence we know that

$$\int_0^t\int \frac{|\nabla\mathbf{u}_\infty(s,x)|^2}{2}\phi_R\,dx\,ds \leq \limsup_{n_k\to+\infty}\int_0^t\int \frac{|\nabla\mathbf{u}_{n_k}(s,x)|^2}{2}\phi_R\,dx\,ds,$$

of course, a similar estimates hold for $\mathbf{b}_\infty$. Then, we get

$$\int \frac{|\mathbf{u}_\infty(t,x)|^2 + |\mathbf{b}_\infty(t,x)|^2}{2} \phi_R\, dx + \int_0^t \int (|\nabla \mathbf{u}_\infty|^2 + |\nabla \mathbf{b}_\infty|^2)\, \phi_R\, dx\, ds$$
$$\leq \int \frac{|\mathbf{u}_0(x)|^2 + |\mathbf{b}_0(x)|^2}{2} \phi_R\, dx + \int_0^t \int \frac{|\mathbf{u}_\infty|^2 + |\mathbf{b}_\infty|^2}{2} \Delta\phi_R\, dx\, ds$$
$$+ \sum_{i=1}^{3} \int_0^t \int [(\frac{|\mathbf{u}_\infty|^2}{2} + \frac{|\mathbf{b}_\infty|^2}{2}) u_{\infty,i} + p_\infty u_{\infty,i}] \partial_i \phi_R\, dx\, ds$$
$$+ \sum_{i=1}^{3} \int_0^t \int [(\mathbf{u}_\infty \cdot \mathbf{b}_\infty) b_{\infty,i}] \partial_i \phi_R\, dx\, ds$$
$$- \sum_{1 \leq i,j \leq 3} (\int_0^t \int F_{\infty,i,j} u_{\infty,j} \partial_i \phi_R\, dx\, ds + \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j}\, \phi_R\, dx\, ds).$$

Letting t goes to 0,

$$\limsup_{t \to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t,.)\|^2_{L^2(\phi_R(x)dx)} \leq \|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|^2_{L^2(\phi_R(x)dx)}.$$

Reciprocally, by weak convergence

$$\|(\mathbf{u}_{0,\infty}, \mathbf{b}_{0,\infty})\|^2_{L^2(\phi_R(x)dx)} \leq \liminf_{t \to 0} \|(\mathbf{u}_\infty, \mathbf{b}_\infty)(t,.)\|^2_{L^2(\phi_R(x)dx)}.$$

Thus, in the Hilbert space $L^2(\phi_R(x)dx)$, we obtain strong convergence to the initial data.

Proof of Theorem 14

6.4.3 Local in time existence

Consider $\phi_R(x) = \phi(\frac{x}{R})$ given in (6.14), we define $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$, $\mathbf{b}_{0,R} = \mathbb{P}(\phi_R \mathbf{b}_0)$, $\mathbb{F}_R = \phi_R \mathbb{F}$. We denote $(MHD_{R,\epsilon})$ the following approximated problem

$$\begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla)\mathbf{u}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla)\mathbf{b}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_R, \\ \partial_t \mathbf{b}_{R,\epsilon} = \Delta \mathbf{b}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla)\mathbf{b}_{R,\epsilon} + ((\mathbf{b}_{R,\epsilon} * \theta_\epsilon) \cdot \nabla)\mathbf{u}_{R,\epsilon} - \nabla q_{R,\epsilon}, \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0,\ \nabla \cdot \mathbf{b}_{R,\epsilon} = 0, \\ \mathbf{u}_{R,\epsilon}(0,\cdot) = \mathbf{u}_{0,R},\ \mathbf{b}_{R,\epsilon}(0,\cdot) = \mathbf{b}_{0,R}. \end{cases}$$

($\text{MHD}_{R,\epsilon}$) has a unique solution $(\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})$ in $L^\infty((0,+\infty), L^2) \cap L^2((0,+\infty), \dot{H}^1)$. Additionally, this solution belongs to $\mathcal{C}([0,+\infty), L^2)$ and fulfills the hypothesis of Theorem 15. We apply this theorem (for the case $(\mathbf{v}, \mathbf{c}) = (\mathbf{u} * \theta_\epsilon, \mathbf{b} * \theta_\epsilon)$) to obtain uniform controls. More precisely, there exists a constant $C > 0$ such that for T_0 satisfying

$$C \left(1 + \|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|^2_{B_2} + \|\mathbb{F}_R\|^2_{B_2 L^2(0,T_0)}\right)^2 T_0 \leq 1,$$

we have

$$\sup_{0 \leq t \leq T_0} \|\, (\mathbf{u}_{R,\epsilon}, \mathbf{b}_{R,\epsilon})(t)\|^2_{B_2} \leq C \left(1 + \|(\mathbf{u}_{0,R}, \mathbf{b}_{0,R})\|^2_{B_2} + \|\mathbb{F}_R\|^2_{B_2 L^2(0,T_0)}\right)$$

and

$$\|\nabla(\mathbf{u}_{R,\epsilon},\mathbf{b}_{R,\epsilon})\|^2_{B_2L^2(0,T_0)} \le C\left(1+\|(\mathbf{u}_{0,R},\mathbf{b}_{0,R})\|^2_{B_2}+\|(\mathbb{F}_R\|^2_{B_2L^2(0,T_0)}\right).$$

Now, in the setting of Theorem 9, we denote $(\mathbf{u}_{0,n},\mathbf{b}_{0,n})=(\mathbf{u}_{0,R_n},\mathbf{b}_{0,R_n})$, $\mathbb{F}_n=\mathbb{F}_{R_n}$ and $(\mathbf{u}_n,\mathbf{b}_n)=(\mathbf{u}_{R_n,\epsilon_n},\mathbf{b}_{R_n,\epsilon_n})$. Letting $R_n\to+\infty$ and $\epsilon_n\to 0$ we thus obtain a local solution of the (MHD) system satisfying the properties stated in Theorem 14.

6.4.4 Global in time existence

Let $\lambda>1$. For each $n\in\mathbb{N}$ we consider the (MHD) system with initial value

$$(\mathbf{u}_{0,n},\mathbf{b}_{0,n})=(\lambda^n\mathbf{u}_0(\lambda^n\cdot),\lambda^n\mathbf{b}_0(\lambda^n\cdot)),$$

and forcing tensor

$$\mathbb{F}_n=\lambda^{2n}\mathbb{F}(\lambda^{2n}\cdot,\lambda^n\cdot).$$

By the local in time existence, there exists a solution $(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)$ on $(0,T_n)$, where

$$C\left(1+\|(\tilde{\mathbf{u}}_{0,n},\tilde{\mathbf{b}}_{0,n})\|^2_{B_2}+\|\mathbb{F}_n\|^2_{B_2L^2(0,T_n)}\right)^2 T_n=1.$$

Now, we use the scaling of the (MHD) system :

$$(\tilde{\mathbf{u}}_n(t,x),\tilde{\mathbf{b}}_n(t,x))=(\lambda^n\mathbf{u}_n(\lambda^{2n}t,\lambda^n x),\lambda^n\mathbf{b}_n(\lambda^{2n}t,\lambda^n x)),$$

where $(\mathbf{u}_n,\mathbf{b}_n)$ is a solution of the (MHD) on $(0,\lambda^{2n}T_n)$ associated with the initial value $(\mathbf{u}_0,\mathbf{b}_0)$ and forcing tensor $\mathbb{F}$.

Now, we use the following simple remark, which will be proved at the end of this section.

Remark 6.1. *If $\boldsymbol{u}_0,\boldsymbol{b}_0\in B_{2,0}$ and $\mathbb{F}\in B_{2,0}L^2(0,+\infty)$, then*

$$\lim_{n\to+\infty}\frac{\lambda^n}{1+\|(\tilde{\boldsymbol{u}}_{0,n},\tilde{\boldsymbol{b}}_{0,n})\|^2_{B_2}+\|\mathbb{F}_n\|^2_{B_2L^2}}=+\infty.$$

Therefore, we have $\lim_{n\to+\infty}\lambda^{2n}T_n=+\infty$.

Consider $T>0$ arbitrary. Let n_T such that for all $n\ge n_T$, $\lambda^{2n}T_n>T$. Thus, $(\mathbf{u}_n,\mathbf{b}_n)$ is a solution of the (MHD) equations on $(0,T)$.

We denote $(\tilde{\mathbf{u}}_n(t,x),\tilde{\mathbf{b}}_n(t,x))=(\lambda^{n_T}\mathbf{u}_n(\lambda^{2n_T}t,\lambda^{n_T}x),\lambda^{n_T}\mathbf{b}_n(\lambda^{2n_T}t,\lambda^{n_T}x))$. Observe that for $n\ge n_T$, $(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)$ is a solution of the (MHD) system on $(0,\lambda^{-2n_T}T)$ with initial value $(\tilde{\mathbf{u}}_{0,n_T},\tilde{\mathbf{b}}_{0,n_T})$ and forcing tensor $\mathbb{F}_{n_T}$. Then, since we have $\lambda^{-2n_T}T\le T_{n_T}$, we find

$$C\left(1+\|(\tilde{\mathbf{u}}_{0,n_T},\tilde{\mathbf{b}}_{0,n_T})\|^2_{B_2}+\|\mathbb{F}_{n_T}\|^2_{B_2L^2(0,\lambda^{-2n_T}T)}\right)^2\lambda^{-2n_T}T\le 1,$$

and by Theorem 15 we obtain:

$$\sup_{0\le t\le\lambda^{-2n_T}T}\|(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)(t,.)\|^2_{L^2_{w_\gamma}}\le C(1+\|(\tilde{\mathbf{u}}_{0,n_T},\tilde{\mathbf{b}}_{0,n_T})\|^2_{B_2}+\|\mathbb{F}_{n_T}\|^2_{B_2L^2(0,\lambda^{-2n_T}T)})$$

and

$$\|\nabla(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)\|^2_{B_2L^2(0,\lambda^{-2n_T}T)} \leq C(1+\|(\tilde{\mathbf{u}}_{0,n_T},\tilde{\mathbf{b}}_{0,n_T})\|^2_{B_2}+\|\mathbb{F}_{n_T}\|^2_{B_2L^2(0,\lambda^{-2n_T}T)}).$$

These estimates gives uniforms controls for $\mathbf{u}_n$ and $\mathbf{b}_n$, as

$$\|(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)(t)\|^2_{B_2} \geq \lambda^{-n_T}\|(\mathbf{u}_n,\mathbf{b}_n)(\lambda^{2n_T}t,.)\|^2_{B_2},$$

and

$$\|\nabla(\tilde{\mathbf{u}}_n,\tilde{\mathbf{b}}_n)\|^2_{B_2L^2(0,\lambda^{-2n_T}T)} \geq \lambda^{-n_T}\|\nabla(\mathbf{u}_n,\mathbf{b}_n)\|^2_{B_2L^2(0,T)}.$$

To finish, we observe that we have controlled uniformly $\mathbf{u}_n,\mathbf{b}_n$ on $(0,T)$ for $n\geq n_T$. Thus, we are able to apply Theorem 9 and we get a solution on $(0,T)$. As $T>0$ is an arbitrary time, a diagonal permits to obtain a solution $(\mathbf{u},\mathbf{b},p)$ on $(0,+\infty)$. The control in the statement of the theorem for the solution $(\mathbf{u},\mathbf{b},p)$ is given by Theorem 15, and thus the proof is finished. ◇

Proof of Remark 6.1. We detail the computations for $\mathbf{u}_{0,n}$ and $\mathbb{F}_n$. First, we observe that

$$\frac{\|\mathbf{u}_{0,n}\|^2_{B_2}}{\lambda^n} = \sup_{R\geq 1}\frac{1}{\lambda^n R^2}\int_{|x|\leq R}|\lambda^n\mathbf{u}_0(\lambda^n x)|^2\,dx = \sup_{R\geq 1}\frac{1}{(\lambda^n R)^2}\int_{|x|\leq\lambda^n R}|\mathbf{u}_0(x)|^2\,dx,$$

and

$$\lim_{n\to+\infty}\sup_{R\geq 1}\frac{1}{(\lambda^n R)^2}\int_{|x|\leq\lambda^n R}|\mathbf{u}_0(x)|^2\,dx = \lim_{R\to+\infty}\frac{1}{R^2}\int_{|x|\leq R}|\mathbf{u}_0(x)|^2\,dx = 0.$$

Similarly, we find

$$\begin{aligned}\frac{\|\mathbb{F}_n\|^2_{B_2L^2(0,+\infty)}}{\lambda^n} &= \sup_{R\geq 1}\frac{1}{\lambda^n R^2}\int_0^{+\infty}\int_{|x|\leq R}|\lambda^{2n}\mathbb{F}(\lambda^{2n}t,\lambda^n x)|^2\,dx\,ds\\ &= \sup_{R\geq 1}\frac{1}{(\lambda^n R)^2}\int_0^{+\infty}\int_{|x|\leq\lambda^n R}|\mathbb{F}(t,x)|^2\,dx,\end{aligned}$$

and

$$\lim_{n\to+\infty}\sup_{R\geq 1}\frac{1}{(\lambda^n R)^2}\int_0^{+\infty}\int_{|x|\leq\lambda^n R}|\mathbb{F}(t,x)|^2\,dx\,ds = \lim_{R\to+\infty}\frac{1}{R^2}\int_0^{+\infty}\int_{|x|\leq R}|\mathbb{F}(t,x)|^2\,dx\,ds = 0.$$

◇

Navier–Stokes equations in B^2_2 have been recently discussed by Bradshaw, Kukavica and Tsai (Bradshaw, Kukavica, and Tsai, 2019).

The case $d=2$ is more intricate. Indeed, while the Leray projection operator is bounded on $B^2_2(\mathbb{R}^3)$, this is no longer the fact on $B^2_2(\mathbb{R}^2)$, in which case one must be careful in the handling of the pressure. Basson treat this case to find local solutions of Navier–Stokes equations in his Ph. D. thesis in 2006 (Basson, 2006b).

6.5 Solutions in local Morrey spaces in dimension 2

We indicate here how to treat the pressure and moreover, we give a sketch of the proof for the local and global existence of weak suitable solutions of the (MHD) system. It is worth remark that for the (NS) equations, A. Basson obtained in his Ph. D. thesis (Basson, 2006b) the local existence of weak solutions with initial data in $B_2(\mathbb{R}^2)$. Thus, our main contribution is the study of global weak solutions in the generalized setting of the (MHD) system.

The main idea in (Basson, 2006b) consists in giving a useful decomposition for the pressure. First, we fix some notation. We denote $\mathcal{R}$ the vector field of the Riesz transforms and we write $H_{i,j}$ the kernel of the operator $\mathcal{R}_i\mathcal{R}_j$ and $\mathbb{H} = (H_{i,j})$. Consider $\varphi \in \mathcal{D}(\mathbb{R}^2)$ a non negative function supported on $B(0,2)$ such that $\varphi = 1$ on $B(0,1)$. For each $k \in \mathbb{N}$, we define the functions $\psi_k(x) = \varphi(2^{-k-1}x) - \varphi(2^{-k}x)$ and $\chi_k = \varphi(2^{-k-3}x) - \varphi(2^{-k+2}x)$. Then, $\psi_k(x) = 1$ for all $2^{k-1} \leq |x| \leq 2^{k+3}$,

$$\sup(\chi_k) \subset \{x \in \mathbb{R}^2 : 2^{k-2} \leq |x| \leq 2^{k+4}\},$$

and

$$\sup(\psi_k) \subset \{x \in \mathbb{R}^2 : 2^{k} \leq |x| \leq 2^{k+2}\}.$$

For an index-family $\mathcal{A}$, consider $(\mathbf{u}_\alpha)_{\alpha\in\mathcal{A}}$ and $(\mathbf{b}_\alpha)_{\alpha\in\mathcal{A}}$ two families of time dependent vector fields defined on $[0,T)\times\mathbb{R}^2$, and consider a family $(\mathbb{F}_\alpha)_{\alpha\in A}$ of tensors defined on $[0,T)\times\mathbb{R}^2$. We denote for each $\alpha\in\mathcal{A}$,

$$A_\alpha = \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha - \mathbf{b}_\alpha \otimes \mathbf{b}_\alpha - \mathbb{F}_\alpha, \tag{6.16}$$

and we define the terms $p_{\alpha,1}$ and $\nabla p_{\alpha,2}$ by the formulas

$$p_{\alpha,1} = \varphi(x/8)\mathcal{R}\otimes\mathcal{R}(\varphi(A_\alpha)) + \sum_{k=1}^{+\infty} \chi_k \mathcal{R}\otimes\mathcal{R}(\psi_k(A_\alpha)), \tag{6.17}$$

$$\begin{aligned} \nabla p_{\alpha,2} =& \nabla[(1-\varphi(x/8))\mathcal{R}\otimes\mathcal{R}(\varphi(A_\alpha))] \\ &+ \sum_{k=1}^{+\infty} \nabla[(1-\chi_k)\mathcal{R}\otimes\mathcal{R}(\psi_k(A_\alpha))]. \end{aligned} \tag{6.18}$$

As usual, we will consider an approximated solutions $(\mathbf{u}_n, \mathbf{b}_n, p_n)$. After, using the local energy balance where we split the term p_n as in the expressions above, we will find an uniform bound on the approximated solutions. Passing to the limit we will able to exhibit a solution $(\mathbf{u}, \mathbf{b}, p)$ of the (MHD) system.

Theorem 17 (Local and global solutions)**.** *Let $0 < T < +\infty$. Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in B_2(\mathbb{R}^2)$ be divergence-free vector fields. Let $\mathbb{F}$ be a tensor belonging to $B_2L^2(0,T)$. Then, there exists a time $0 < T_0 < T$ such that the system (MHD) has a solution $(\boldsymbol{u}, \boldsymbol{b}, p)$ which satisfies :*

- *$\boldsymbol{u}, \boldsymbol{b}$ belong to $L^\infty((0,T_0), B_2)$ and $\nabla\boldsymbol{u}, \nabla\boldsymbol{b}$ belong to $B_2L^2(0,T_0)$.*
- *The pressure p is related to $\boldsymbol{u}$, $\boldsymbol{b}$ and $\mathbb{F}$ as follows. Let $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^2)$ be a test function such that $\tilde{\varphi}(x) = 1$ on a neighborhood of the origin. We define*

$$\Phi_{i,j,\tilde{\varphi}} = (1-\tilde{\varphi})\partial_i\partial_j G_2.$$

where $G_2 = \frac{1}{2\pi}\ln(\frac{1}{|x|})$ *is a fundamental solution of the operator* $-\Delta$ *(we have* $-\Delta G_2 = \delta_0$*). Then* p *and can be defined by :*

$$\begin{aligned} p_{\tilde{\varphi}}(t,x) =& \sum_{i,j}(\tilde{\varphi}\partial_i\partial_j G_2) * (u_iu_j - b_ib_j - F_{i,j})(t,x) \\ &+ \sum_{i,j}\int (\Phi_{i,j,\tilde{\varphi}}(x-y) - \Phi_{i,j,\tilde{\varphi}}(-y))\,(u_i(t,y)u_j(t,y) \\ &-b_i(t,y)b_j(t,y) - F_{i,j}(t,y))\,dy \end{aligned} \tag{6.19}$$

- *The map* $t \in [0,T) \mapsto (\boldsymbol{u}(t,\cdot), \boldsymbol{b}(t,\cdot))$ *is* $*$*-weakly continuous from* $[0,T)$ *to* $B_2(\mathbb{R}^2)$*, and for all compact set* $K \subset \mathbb{R}^2$ *we have:*

$$\lim_{t\to 0} \|(\boldsymbol{u}(t,\cdot) - \boldsymbol{u}_0, \boldsymbol{b}(t,\cdot) - \boldsymbol{b}_0)\|_{L^2(K)} = 0.$$

- *The solution* $(\boldsymbol{u}, \boldsymbol{b}, p)$ *is suitable : there exists a non-negative locally finite measure* μ *on* $(0,T)\times\mathbb{R}^2$ *such that:*

$$\begin{aligned} \partial_t(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) =& \Delta(\frac{|\boldsymbol{u}|^2 + |\boldsymbol{b}|^2}{2}) - |\nabla \boldsymbol{u}|^2 - |\nabla \boldsymbol{b}|^2 - \nabla\cdot\left([\frac{|\boldsymbol{u}|^2}{2} + \frac{|\boldsymbol{b}|^2}{2} + p]\boldsymbol{u}\right) \\ &+ \nabla\cdot[(\boldsymbol{u}\cdot\boldsymbol{b})\boldsymbol{b}] + \boldsymbol{u}\cdot(\nabla\cdot\mathbb{F}) - \mu. \end{aligned}$$

In particular we have the following global control on the solution: for all $0 \le t \le T_0$*,*

$$\begin{aligned} &\max\{\|(\boldsymbol{u},\boldsymbol{b})(t)\|_{B_2}^2, \|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{B_2L^2(0,T_0)}^2\} \\ &\le C\|(\boldsymbol{u}_0,\boldsymbol{b}_0)\|_{B_2}^2 + C\|\mathbb{F}\|_{B_2L^2(0,t)}^2 + C\int_0^t 1 + \|(\boldsymbol{u},\boldsymbol{b})(s)\|_{B_2}^4 ds. \end{aligned} \tag{6.20}$$

- *Finally, if the data verify:*

$$\lim_{R\to+\infty} R^{-2}\int_{|x|\le R} |\boldsymbol{u}_0(x)|^2 + |\boldsymbol{b}_0(x)|^2\,dx = 0,$$

and

$$\lim_{R\to+\infty} R^{-2}\int_0^{+\infty}\int_{|x|\le R} |\mathbb{F}(t,x)|^2\,dx\,dt = 0,$$

then we get a global solution $(\boldsymbol{u}, \boldsymbol{b}, p)$*.*

Sketch of the proof.

The key point is to provide *a priori* controls for the following approximated solutions. Let ϕ_R as in (6.14). Basson proves in (Basson, 2006b) that one may take a sequence $R_n \to +\infty$ such that $\mathbb{P}(\phi_{R_n}\mathbf{u}_0)$ converges *-weakly to $\mathbf{u}_0$ and $\mathbb{P}(\phi_{R_n}\mathbf{b}_0)$ converges *-weakly to $\mathbf{b}_0$ in B_2. We define $\mathbf{u}_{0,n} = \mathbb{P}(\phi_{R_n}\mathbf{u}_0)$, $\mathbf{b}_{0,n} = \mathbb{P}(\phi_{R_n}\mathbf{b}_0)$, $\mathbb{F}_n = \phi_{R_n}\mathbb{F}$. Then, there exists a unique solution $(\mathbf{u}_n, \mathbf{b}_n)$ of the approximated system (MHD_n):

$$(MHD_n)\begin{cases} \partial_t\mathbf{u}_n = \Delta\mathbf{u}_n - (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n + (\mathbf{b}_n\cdot\nabla)\mathbf{b}_n - \nabla p_n + \nabla\cdot\mathbb{F}_n, \\ \partial_t\mathbf{b}_n = \Delta\mathbf{b}_n - (\mathbf{u}_n\cdot\nabla)\mathbf{b}_n + (\mathbf{b}_n\cdot\nabla)\mathbf{u}_n, \\ \nabla\cdot\mathbf{u}_n = 0,\ \nabla\cdot\mathbf{b}_n = 0, \\ \mathbf{u}_n(0,\cdot) = \mathbf{u}_{0,n},\ \mathbf{b}_n(0,\cdot) = \mathbf{b}_{0,n}, \end{cases}$$

which belongs to $L^\infty((0,+\infty),L^2(\mathbb{R}^2))\cap L^2((0,+\infty),\dot H^1(\mathbb{R}^2))$, and even, belongs to $\mathcal{C}([0,+\infty),L^2(\mathbb{R}^2))$.

Next, we denote $\nabla p_n=\nabla p_{n,1}+\nabla p_{n,2}$, where $p_{n,1}$ and $\nabla p_{n,2}$ are given by (6.17) and (6.18).

We make use of a technical lemma which we will prove later.

Lemma 6.8. *Let $\mathbb{F}=(F_{i,j})_{1\le i,j\le 2}\in L^1_{loc}$ be a tensor. Then we have*

$$\|\nabla[(1-\chi_k)\mathcal{R}_i\mathcal{R}_j(\psi_k F_{i,j})]\|_{L^\infty(\mathbb{R}^2)}\le C\int\frac{\psi_k(y)|\mathbb{F}(y)|}{(1+|y|)^3}\,dy$$
$$\approx C2^{-3k}\int\psi_k(y)|\mathbb{F}(y)|\,dy.$$

and

$$\|\nabla[(1-\varphi(x/8))\mathcal{R}_i\mathcal{R}_j(\varphi F_{i,j})]\|_{L^\infty(\mathbb{R}^2)}\le C\int\varphi(y)|\mathbb{F}(y)|\,dy$$

We get back to (6.16) (and we set $\alpha=(n,i,j)\in\mathbb{N}\times\{1,2\}\times\{1,2\}$), using this lemma we find

$$\|\nabla[(1-\chi_k)\mathcal{R}_i\mathcal{R}_j(\psi_k A_{n,i,j})]\|_{L^\infty(\mathbb{R}^2)}$$
$$\le C\int\frac{\psi_k(|\mathbf{u}_n|^2+|\mathbf{b}_n|^2)}{(1+|y|)^3}\,dy+C\int\frac{\psi_k|\mathbb{F}_n|}{(1+|y|)^3}\,dy$$
$$\approx 2^{-3k}\int\psi_k(|\mathbf{u}_n|^2+|\mathbf{b}_n|^2)dy+2^{-3k}\int\psi_k|\mathbb{F}_n|dy.$$

After summation over k and using the Hölder inequality in the term with the forcing tensor, we get for $2<\gamma_0<4$,

$$\|\nabla p_{n,2}\|_{L^\infty}<C\int\frac{|\mathbf{u}_n|^2+|\mathbf{b}_n|^2}{(1+|x|)^3}\,dy+C\left(\int\frac{|\mathbb{F}_n|^2}{(1+|x|)^{\gamma_0}}\right)^{1/2},\tag{6.21}$$

which shows that $\|\nabla p_{n,2}\|_{L^\infty}$ is uniformly bounded.

Now, we study the term $p_{n,1}$. Let $R\ge 1$ fix, and we take $k_0\in\mathbb{N}$ such that $2^{k_0-1}\le 2R\le 2^{k_0}$. Then, by the localization of the function χ_k,

$$\begin{aligned}\int_{|x|\le 2R}|p_{n,1}|^{3/2}\,dx\le& C\int_{|x|\le 2R}|\varphi(x/8)\mathcal{R}\otimes\mathcal{R}(\varphi(A_n))|^{3/2}\,dx\\&+C\sum_{k=1}^{k_0+1}\int_{|x|\le 2R}|\chi_k\mathcal{R}\otimes\mathcal{R}(\psi_k(A_n))|^{3/2}\,dx\\ \le& C\int_{|x|\le 2R}|\varphi(\mathbf{u}_n\otimes\mathbf{u}_n+\mathbf{b}_n\otimes\mathbf{b}_n)|^{3/2}dx\\&+C\sum_{k=1}^{k_0+1}\int_{|x|\le 2R}|\psi_k(\mathbf{u}_n\otimes\mathbf{u}_n+\mathbf{b}_n\otimes\mathbf{b}_n)|^{3/2}\,dx\\&+C\int|\varphi\mathbb{F}_n|^{3/2}+C\sum_{k=1}^{k_0+1}\int|\psi_k\mathbb{F}_n|^{3/2}\\ \le& C\int_{|x|\le 2^5R}|\mathbf{u}_n|^3+|\mathbf{b}_n|^3dx+C\int_{|x|\le 2^5R}|\mathbb{F}_n|^{3/2}dx.\end{aligned}\tag{6.22}$$

We consider the solutions $(\mathbf{u}_n, \mathbf{b}_n, p_n)$ of the approximated system (MHD_n), but, for simplicity, we will get rid of the index n and we shall just write $(\mathbf{u}, \mathbf{b}, p)$.

This part of the proof seems to the case of dimension 3, for this reason we only detail the main computations.

We define $\alpha_{\eta,t_0,t_1}(t)$ as in (5.7), and ϕ_R as in (6.14). The local energy balance gives

$$\begin{aligned}
&\iint \partial_t(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2})\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds + \iint |\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\ \alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds\\
&\quad\le \iint \frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\ \alpha_{\eta,t_0,t_1}\Delta\phi_R\,dx\,ds\\
&\quad+\sum_{i=1}^{2}\iint(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2})u_i\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds+\iint \mathbf{u}\cdot\nabla p\phi_R\,dx\,ds\\
&\quad+\sum_{i=1}^{2}\iint(\mathbf{u}\cdot\mathbf{b})b_i\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds\\
&\quad-\sum_{1\le i,j\le 2}\iint F_{i,j}u_j\alpha_{\eta,t_0,t_1}\partial_i\phi_R\,dx\,ds-\sum_{1\le i,j\le 2}\iint F_{i,j}\partial_i u_j\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds.
\end{aligned}$$

We divide this expression by R^2, we use $\|\nabla\phi_R\|_{L^\infty}\le c/R$ and $\|\Delta\phi_R\|_{L^\infty}\le c/R^2$, and for the term involving $\mathbb{F}$, we apply the Cauchy-Schwarz inequalities and the Young inequalities. Then, there exist two constants $C_0>0$ and $C=C(C_0)>0$, where C_0 is arbitrarily small, such that

$$\begin{aligned}
&\frac{1}{R^2}\iint \partial_t(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2})\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\frac{1}{R^2}\iint(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\ \alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds\\
\le&\frac{C}{R^2}\iint_{|x|\le 2R}(|\mathbf{u}|^2+|\mathbf{b}|^2)\alpha_{\eta,t_0,t_1}\,dx\,ds+\frac{1}{R^3}\iint_{|x|\le 2R}(\frac{|\mathbf{u}|^3+|\mathbf{b}|^2|\mathbf{u}|}{2})\alpha_{\eta,t_0,t_1}\,dx\,ds\\
&+\underbrace{\frac{2}{R^2}\iint \phi_R(\mathbf{u}\cdot\nabla p)\alpha_{\eta,t_0,t_1}\,dx\,ds}_{(a)}\\
&+\frac{C}{R^2}\iint_{|x|\le 2R}|\mathbb{F}|^2\alpha_{\eta,t_0,t_1}\,dx\,ds\\
&+\frac{C_0}{R^2}\iint_{|x|\le 2R}(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\ \alpha_{\eta,t_0,t_1}\,dx\,ds.
\end{aligned}\tag{6.23}$$

To control the term (a) we use the decomposition $\nabla p=\nabla p_1+\nabla p_2$ where p_1 and ∇p_2 are always given by the equations (6.17) and (6.18). Recalling that $div(\mathbf{u})=0$, we may write

$$\int \phi_R\mathbf{u}\cdot\nabla p\,dx=-\int p_1\mathbf{u}\cdot\nabla\phi_R\,dx+\int\phi_R\mathbf{u}\cdot\nabla p_2\,dx.$$

The following fact is now useful : by the interpolation inequalities we have

$$\begin{aligned}
\int_{|x|\le 2^5R}|\mathbf{u}|^3dx&\le\|\varphi_{2^5R}\mathbf{u}\|_{L^3}^3\le C\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^2\|\nabla(\varphi_{2^5R}\mathbf{u})\|_{L^2}\\
&\le C\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^3+C'\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^2\|\varphi_{2^5R}\nabla\mathbf{u}\|_{L^2}.
\end{aligned}\tag{6.24}$$

Using (6.21) and (6.22), and (6.24), we obtain for $2 < \gamma_0 < 4$ and $2 < \gamma_1 < 10/3$, we obtain

$$
\begin{aligned}
\frac{1}{R^2}\left|\int \phi_R \mathbf{u}\cdot\nabla p\,dx\right| &\le \frac{C}{R^3}\int_{|x|\le 2R}(|\mathbf{u}|^3+|p_1|^{3/2})dx + C\frac{\|\nabla p_2\|_\infty}{R}\left(\int \phi_R|\mathbf{u}|^2\,dx\right)^{1/2}\\
&\le \frac{C}{R^3}\int_{|x|\le 2^5R}(|\mathbf{u}|^3+|\mathbf{b}|^3)dx + C\int\frac{|\mathbb{F}|^{3/2}}{(1+|x|)^3}dx\\
&\quad+\frac{C}{R}(\|\mathbf{u}\|_{B_2}^2+\|\mathbf{b}\|_{B_2}^2)\|\phi_R\mathbf{u}\|_{L^2}+\frac{C}{R^2}\|\phi_R\mathbf{u}\|_{L^2}^2 + C\int\frac{|\mathbb{F}|^2}{(1+|x|)^{\gamma_0}}dx\\
&\le \frac{C}{R^3}\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^3+\frac{C'}{R^3}\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^2\|\varphi_{2^5R}\nabla\mathbf{u}\|_{L^2}+C\left(\int\frac{|\mathbb{F}|^2}{(1+|x|)^{\gamma_1}}\,dx\right)^{3/4}\\
&\quad+\frac{C}{R^3}\|\varphi_{2^5R}\mathbf{b}\|_{L^2}^3+\frac{C'}{R^3}\|\varphi_{2^5R}\mathbf{b}\|_{L^2}^2\|\varphi_{2^5R}\nabla\mathbf{b}\|_{L^2}\\
&\quad+\frac{C}{R}(\|\mathbf{u}\|_{B_2}^2+\|\mathbf{b}\|_{B_2}^2)\left(\int_{|x|\le 2R}|\mathbf{u}|^2dx\right)^{1/2}+\frac{C}{R^2}\|\phi_R\mathbf{u}\|_{L^2}^2dx + C\int\frac{|\mathbb{F}|^2}{(1+|x|)^{\gamma_0}}dx.
\end{aligned}
$$

We can inject this estimate on term (a), to (6.23). Then, we find

$$
\begin{aligned}
&\frac{1}{R^2}\iint \partial_t\left(\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\right)\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds+\frac{1}{R^2}\iint(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\,\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds\\
&\le\frac{C}{R^2}\iint_{|x|\le 2R}(|\mathbf{u}|^2+|\mathbf{b}|^2)\alpha_{\eta,t_0,t_1}\,dx\,ds\\
&\quad+\frac{C}{R^3}\int(\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^3+\|\varphi_{2^5R}\mathbf{b}\|_{L^2}^3)\alpha_{\eta,t_0,t_1}\,ds + C\int(\|\mathbf{u}\|_{B_2}^3+\|\mathbf{b}\|_{B_2}^3)\alpha_{\eta,t_0,t_1}\,ds\\
&\quad+\frac{C}{R^4}\int\|\varphi_{2^5R}\mathbf{u}\|_{L^2}^4\alpha_{\eta,t_0,t_1}\,ds+\frac{C_0}{R^2}\int\|\varphi_{2^5R}\nabla\mathbf{u}\|_{L^2}^2\alpha_{\eta,t_0,t_1}\,ds\\
&\quad+\frac{C}{R^4}\int\|\varphi_{2^5R}\mathbf{b}\|_{L^2}^4\alpha_{\eta,t_0,t_1}\,ds+\frac{C_0}{R^2}\int\|\varphi_{2^5R}\nabla\mathbf{b}\|_{L^2}^2\alpha_{\eta,t_0,t_1}\,ds\\
&\quad+C\int\left(1+\int\frac{|\mathbb{F}|^2}{(1+|x|)^{\gamma_1}}\alpha_{\eta,t_0,t_1}dx\right)ds + C\int\int\frac{|\mathbb{F}|^2}{(1+|x|)^{\gamma_0}}\alpha_{\eta,t_0,t_1}\,dx\,ds\\
&\quad+\frac{C}{R^2}\iint_{|x|\le 2R}|\mathbb{F}|^2\,\alpha_{\eta,t_0,t_1}\,ds\,ds+\frac{C_0}{R^2}\iint_{|x|\le 2R}(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\,\alpha_{\eta,t_0,t_1}\,dx\,ds.
\end{aligned}
$$

At this moment, we let η goes to 0. If we take t_0 and t_1 two Lebesgue points of the function $A_R(s)=\int(|\mathbf{u}(s,x)|^2+|\mathbf{b}(s,x)|^2)\phi_R(x)\,dx$, we get

$$
-\iint\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=-\frac{1}{2}\int\partial_t\alpha_{\eta,t_0,t_1}A_R(s)\,ds,
$$

and

$$
\lim_{\eta\to 0}-\iint\frac{|\mathbf{u}|^2+|\mathbf{b}|^2}{2}\partial_t\alpha_{\eta,t_0,t_1}\phi_R\,dx\,ds=\frac{1}{2}(A_R(t_1)-A_R(t_0)),
$$

so that

$$\begin{aligned}
&\frac{1}{R^2}\int(\frac{|\mathbf{u}(t_1)|^2+|\mathbf{b}(t_1)|^2}{2}-\frac{|\mathbf{u}(t_0)|^2+|\mathbf{b}(t_0)|^2}{2})\phi_R\,dx\\
&\quad+\frac{1}{R^2}\int_{t_0}^{t_1}\int(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\,\phi_R)\,dx\,ds\\
&\leq C\int_{t_0}^{t_1}1+\|\mathbf{u}\|_{B_2}^4+\|\mathbf{b}\|_{B_2}^4\,ds\\
&\quad+C\|\mathbb{F}\|^2_{B_2L^2(t_0,t_1)}\\
&\quad+\frac{C_0}{R^2}\int_{t_0}^{t_1}\int_{|x|\leq 2^5R}|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2\,dx\,ds.
\end{aligned}$$

In this inequality, the continuity at 0 of the map $t\in[0,T)\mapsto(\mathbf{u},\mathbf{b})(t)\in L^2_{loc}(\mathbb{R}^2)$ permit to let t_0 goes to zero. Additionally, by the *-weak continuity of this map we can let t_1 goes to $t\in(0,T)$. We thus conclude that for all $t\in(0,T)$,

$$\begin{aligned}
&\frac{1}{R^2}\int(\frac{|\mathbf{u}(t)|^2+|\mathbf{b}(t)|^2}{2})\phi_R\,dx+\frac{1}{R^2}\int_0^t\int(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\,\phi_R\,dx\\
&\leq C(\|\mathbf{u}_0\|^2_{B_2}+\|\mathbf{b}_0\|^2_{B_2})+C\|\mathbb{F}\|^2_{B_2L^2(0,t)}+C\|\mathbb{G}\|^2_{B_2L^2(0,t)}\\
&\quad+C\int_{t_0}^{t_1}1+\|\mathbf{u}\|^4_{B_2}+\|\mathbf{b}\|^4_{B_2}\,ds\\
&\quad+\frac{C_0}{R^2}\int_0^t\int_{|x|\leq 2^5R}(|\nabla\mathbf{u}|^2+|\nabla\mathbf{b}|^2)\,dx\,ds.
\end{aligned}$$

From this estimate and using the Young inequalities, we obtain the global energy control (6.20) for the approximated solutions.

With these uniform controls, we may follow the ideas in (Basson, 2006b) to obtain a subsequence $(\mathbf{u}_{n_k},\mathbf{b}_{n_k},p_{n_k})$ which converges in the sense of distributions to a local solution $(\mathbf{u},\mathbf{b},p)$ of the (MHD) system on $[0,T_0]$, where

$$T_0\approx\frac{1}{1+\|(\mathbf{u}_0,\mathbf{b}_0)\|^2_{B_2}+\|\mathbb{F}\|^2_{B_2L^2(0,+\infty)}}.$$

We observe that the pressure term is given by the formula $\nabla p=\nabla p_1+\nabla p_2$, with $p_1=\lim_{k\to\infty}p_{n_k,1}$ and $\nabla p_2=\lim_{k\to\infty}\nabla p_{n_k,2}$. We obtain p_1 and ∇p_2 satisfy (6.17) and (6.18) (with $\mathbf{u}_\alpha=\mathbf{u}$). Additionally, by Theorem 3 (with $d=2$) we get that p can be written as in (6.19).

It remains to study the existence of global solutions. This part is obtained in the same way as for dimension 3, see Section 6.4.4. The only difference in the case of dimension 2 is that, we use the Lemma 6.7, with $b=2$. ◇

Proof of Lemma 6.8

We proceed as in (Basson, 2006b). In dimension 2, for the kernel $\mathbb{H}$ we have $|\mathbb{H}(x)|\leq\frac{C}{|x|^2}$ and $|\nabla\mathbb{H}(x)|\leq\frac{C}{|x|^3}$. By the localization properties of the functions ψ_k and χ_k, we obtain for all $k\in\mathbb{N}$, for $|x-y|<\frac{1+|y|}{16}$,

$$(1-\chi_k)(x)\psi_k(y)=0$$

and

$$\nabla\chi_k(x)\psi_k(y)=0.$$

Then we find

$$\mathcal{R}_i\mathcal{R}_j(\psi_k F_{i,j})(x) = \int_{|x-y|>\frac{1+|y|}{16}} H_{i,j}(x-y)\psi_k(y)F_{i,j}dy,$$

and

$$\begin{aligned}
&|\nabla[(1-\chi_k)\mathcal{R}_i\mathcal{R}_j(\psi_k F_{i,j})](x)| \\
\leq&|\nabla(1-\chi_k)(x)\int_{|x-y|>\frac{1+|y|}{16}} H_{i,j}(x-y)\psi_k(y)F_{i,j}dy\,| \\
&+|(1-\chi_k)(x)\int_{|x-y|>\frac{1+|y|}{16}} \nabla H_{i,j}(x-y)\psi_k(y)F_{i,j}dy\,| \\
\leq&C2^{-k}\int \frac{\psi_k(y)|\mathbb{F}(y)|}{(1+|y|)^2}\,dy + C\int \frac{\psi_k(y)|\mathbb{F}(y)|}{(1+|y|)^3}\,dy \\
\approx&C\int \frac{\psi_k(y)|\mathbb{F}(y)|}{(1+|y|)^3}\,dy \approx C2^{-3k}\int \psi_k(y)|\mathbb{F}(y)|\,dy.
\end{aligned}$$

By the localization properties of φ we have that $(1-\varphi(x/8))\varphi(y) = 0$ and $\nabla\varphi(x/8)\varphi(y) = 0$, so the second estimate in the statement of this lemma follows in the same way. ◇

Bibliography

Bahouri, H., J. Y. Chemin, and R. Danchin (2011). *Fourier Analysis and Nonlinear Partial Differential Equations*. Volume 343. Springer.

Basson, A. (2006a). "Homogeneous Statistical Solutions and Local Energy Inequality for 3D Navier–Stokes Equations". In: *Commun. Math. Phys.* 266, pp. 17–35.

Basson, Arnaud (2006b). *Solutions spatialement homogènes adaptées des équations de Navier–Stokes*. Thèse, Université d'Évry.

Boyer, Franck and Pierre Fabrie (2012). *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*. Applied Mathematical Sciences book series, 183. Springer New York.

Bradshaw, Z., I. Kukavica, and T.P. Tsai (2019). "Existence of global weak solutions to the Navier-Stokes equations in weighted spaces". In: *arXiv:1910.06929v1*.

Bradshaw, Z. and T.P. Tsai (2017). "Forward Discretely Self-Similar Solutions of the Navier-Stokes Equations II". In: *Ann. Henri Poincare* 18, no. 3, 1095–1119.

— (2019a). "Discretely self-similar solutions to the Navier-Stokes equations with data in L^2_{loc}". In: *Analysis and PDE* 12, pp. 1943–1962.

— (2019b). "Self-similar solutions to the Navier-Stokes equations: a survey of recent results". In: *preprint arXiv:1802.00038v1*.

— (2020a). "Global existence, Regularity and Uniqueness of infinite energy solutions to the Navier-Stokes equations". In: *Communications in Partial Differential Equations* 9, 1168–1201.

— (2020b). "On the local pressure expansion for the Navier-Stokes equations". In: *preprint arXiv:2001.11526v1*.

Buckmaster, T. and V. Vicol (2019a). "Convex integration and phenomenologies in turbulence". In: *preprint arXiv:1901.09023v2*.

— (2019b). "Nonuniqueness of weak solutions to the Navier-Stokes equation". In: *Annals of Mathematics* 189, pp. 101–144.

Chae, D. and J. Wolf (2018). "Existence of discretely self-similar solutions to the Navier-Stokes equations for initial value in $L^2_{\text{loc}}(\mathbb{R}^3)$". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 35, pp. 1019–1039.

Chamorro, D., P.G. Lemarié-Rieusset, and K. Mayoufi (2018). "The role of the pressure in the partial regularity theory for weak solutions of the Navier–Stokes equations". In: *Arch. Rat. Mech. Anal.* 228.1, pp. 237–277.

Chemin, Jean-Yves (1995). *Fluides parfaits incompressibles*. tome 230. Société mathématique de France, Astérisque.

Dostoglou, S. (2002). "Homogeneous measures and spatial ergodicity of the Navier–Stokes equations". In: *preprint*.

Dubois, S. (2002). "What is a solution to the Navier–Stokes equations?" In: *C. R. Acad. Sci. Paris, Ser. I* 335, pp. 27–32.

Fabes, E., B. F. Jones, and N. Riviere (1972). "The initial value problem for the Navier–Stokes equations with data in L^p". In: *Arch. Ration. Mech. Anal.* 45, pp. 222–240.

Farwig, R. and H. Sohr (1995). "Weighted energy inequalities for the Navier-Stokes equations in exterior domains". In: *Applicable Analysis* 58, pp. 157–173.

Feichtinger, H. et al. (2020). "Navier-Stokes Equation in Super-Critical Spaces $E^s_{p,q}$". In: *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*.
Fernández-Dalgo, P.G and O. Jarrín (2021a). "Discretely self-similar solutions for 3D MHD equations and global weak solutions in weighted L^2 spaces". In: *Journal of Mathematical Fluid Mechanics* 23.
— (2021b). "Weak-strong uniqueness in weighted L^2 spaces and weak suitable solutions in local Morrey spaces for the MHD equations". In: *Journal of Differential Equations* 271, pp. 864–915.
Fernández-Dalgo, P.G and P.G. Lemarié-Rieusset (2020a). "Characterisation of the pressure term in the incompressible Navier–Stokes equations on the whole space". In: *Discrete & Continuous Dynamical Systems*.
— (2020b). "Weak solutions for Navier–Stokes equations with initial data in weighted L^2 spaces". In: *Arch Rational Mech Anal* 237, pp. 347–382.
— (2021). "Weighted energy estimates for the incompressible Navier-Stokes equations and applications to axisymmetric solutions without swirl". In: *Preprint*.
Fujita, H. and T. Kato (1962). "The initial value problem for the Navier–Stokes equations with data in L^p". In: *Rendiconti Seminario Math. Univ. Padova* 32, pp. 243–260.
Furioli, G., P.G. Lemarié-Rieusset, and E. Terraneo (2000). "Unicité dans $\mathrm{L}^3(\mathbb{R}^3)$ et d'autres espaces limites pour Navier–Stokes". In: *Revista Mat. Iberoamericana* 16, pp. 605–667.
Gallay, T. (2017). "Infinite energy solutions of the two-dimensional Navier-Stokes equations". In: *Annales de la Faculté des Sciences de Toulouse* 26, pp. 979–1027.
Grafakos, Loukas (2008). *Classical harmonic analysis (2nd ed.)* Graduate Texts in Mathematics. Springer.
— (2009). *Modern Fourier analysis (2nd ed.)* Graduate Texts in Mathematics. Springer.
Jia, H. and V. Šverák (2014). "Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions". In: *Invent. Math.* 196, pp. 233–265.
Kang, K., H. Miura, and T. P. Tsai (2020). "An ϵ-regularity criterion and estimates of the regular set for Navier-Stokes flows in terms of initial data". In: *preprint arXiv:2006.13145v2*.
Kikuchi, N. and G. Seregin (2007). "Weak solutions to the Cauchy problem for the Navier–Stokes equations satisfying the local energy inequality". In: *Nonlinear equations and spectral theory Amer. Math. Soc. Transl. Ser. 2, Providence,* 220, pp. 141–164.
Kukavica, I. (2003). "On local uniqueness of solutions of the Navier–Stokes equations with bounded initial data". In: *J. Diff. Eq.* 194, pp. 39–50.
Kukavica, I. and V. Vicol (2008). "On local uniqueness of weak solutions to the Navier–Stokes system with BMO^{-1} initial datum". In: *J. Dynam. Differential Equation* 20, pp. 719–732.
L. Caffarelli, R. Kohn and L. Nirenberg (1982). "Partial regularity of suitable weak solutions of the Navier–Stokes equations". In: *Comm. Pure Appl. Math.* 35, pp. 771–831.
Ladyzhenskaya, O. (1968). "On the unique global solvability of the Cauchy problem for the Navier-Stokes equations in the presence of the axial symmetry". In: *Zap. Nauch. Sem. LOMI 7* (in Russian), pp. 155–177.
Lellis, C. De and L. Székelyhidi-Jr (2019). "On turbulence and geometry: from Nash to Onsager". In: *Notices Amer. Math. Soc.* 66, pp. 677–685.
Lemarié-Rieusset, P. G. (1999). "Solutions faibles d'énergie infinie pour les équations de Navier–Stokes dans $\mathbb{R}^3$". In: *C. R. Acad. Sci. Paris, Serie I.* 328, pp. 1133–1138.

Lemarié-Rieusset, Pierre Gilles (2002). *Recent developments in the Navier-Stokes problem*. Chapman & Hall/CRC Research Notes in Mathematics.

— (2016). *The Navier-Stokes problem in the 21st century*. Chapman & Hall/CRC Press.

Leray, J. (1934). "Sur le mouvement d'un liquide visqueux emplissant l'espace". In: *Acta Math.* 63, pp. 193–248.

Prodi, G. (1959). "Un teorema di unicitá per le equazioni de Navier-Stokes". In: *Ann. Mat. Pura Appl.* 48, pp. 173–182.

Serrin, J. (1962). "The initial value problem for the Navier-Stokes equations". In: *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pp. 69–98.

Stein, Elias M. (1993). *Harmonic Analysis, Real Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press, New Jersey.

Vishik, M. I. and A. V. Fursikov (1977). "Solutions statistiques homogènes des systèmes différentiels paraboliques et du système de Navier-Stokes". In: *Ann. Scuola Norm. Sup. Pisa série IV* 63, pp. 531–576.

— (1988). *Mathematical Problems of Statistical Hydromechanics*. Dordrecht: Kluwer Academic Publishers.

Wolf, J. (2017). "On the local pressure of the Navier–Stokes equations and related systems". In: *Adv. Differential Equations* 22, pp. 305–338.

ÉCOLE DOCTORALE
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Hadamard (EDMH)

Titre : Étude de l'existence de solutions faibles d'énergie infinie pour les équations de Navier-Stokes incompressibles

Mots clés : Équations de Navier–Stokes, équations de la magnétohydrodynamique, contrôles d'énergie, solutions discrètement autosimilaires, solutions axialement symétriques

Résumé : Dans cette thèse, on étudie des estimations d'énergie pour les équations de Navier–Stokes, avec des poids en variable d'espace. L'analyse de ces estimées se fait dans un contexte suffisamment robuste pour être appliquées à la construction de solutions discrètement auto-similaires pour des données initiales satisfaisant la faible condition d'être localement de carré intégrable. Ces idées sont également appliquées pour construire des solutions régulières axialement symétriques sans tourbillon pour des données initiales qui appartiennent ainsi que leur gradient à un espace L^2 à poids. Un exemple concret est donné en considérant l'espace $L^2((1+r^2)^{-\gamma/2}dx$, où $r=\sqrt{x_1^2+x_2^2}$ et $\gamma \in (0,2)$.

Title : Study of the existence of infinite energy weak solutions to the incompressible Navier-Stokes equations

Keywords : Navier–Stokes equations, magneto-hydrodynamics equations, energy controls, discretely selfsimilar solutions, axially symmetric solutions

Abstract : In this thesis, energy estimates for the Navier–Stokes equations are studied, in a sufficiently robust context to be applied to the construction of discretely self-similar solutions for initial data satisfying the weak condition to be locally square integrable. These ideas also are applied to construct regular axially symmetrical solutions without swirl, for initial data which together with his gradient belong to a weighted L^2 space, where the weight allows to consider functions which tend to infinity in a piece of the space $\mathbb{R}^3$. More specifically an example is given by consider an axisymmetric initial velocity without swirl such that both the initial velocity and its vorticity belong to $L^2((1+r^2)^{-\gamma/2}dx)$, with $r=\sqrt{x_1^2+x_2^2}$ and $\gamma \in (0,2)$.

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