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Modélisation de carnet d'ordres et gestion de risque de liquidité

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Thèse de doctorat

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Table des matières

Notations

$\mathcal{N}$	Set of all Decreasing integer-valued sequence stationary at 0
n	Element of $\mathcal{N}$
$\mathsf{n}^{\|i}$	i-th coordinates of n
$\mathfrak{b}$	Bid price operator $\mathfrak{b} : \mathcal{N} \to \mathbb{N}$
$\mathfrak{b}^S$	Bid sell price operator $\mathfrak{b}^S : \mathcal{N} \times \mathbb{N} \to \mathbb{N}$
T^S	Sell operator $\mathsf{T}^S : \mathcal{N} \times \mathbb{N} \to \mathcal{N}$
T^C	Cancel operator $\mathsf{T}^C : \mathcal{N} \times \mathbb{N} \times \mathbb{N}^* \to \mathcal{N}$
T^B	Buy operator $\mathsf{T}^B : \mathcal{N} \times \mathbb{N} \times \mathbb{N}^* \to \mathcal{N}$
$\varDelta$	Difference operator $\varDelta : \mathcal{N} \times \mathbb{N}^* \to \mathbb{N}$
$\mathfrak{s}$	Sum operator $\mathfrak{s} : \mathcal{N} \to \mathbb{N}$
$\hat{\mathfrak{s}}$	Partial sum operator $\hat{\mathfrak{s}} : \mathcal{N} \times \mathbb{N} \to \mathbb{N}$
$\mathfrak{g}$	Gain operator $\mathfrak{g} : \mathcal{N} \times \mathbb{N} \to \mathbb{N}$
δ	Notation for random variables for event type
α	Notation for random variables for event size
β	Notation for random variables for event price placement
P	Transition probability on $\mathcal{N}$, $\mathsf{P} : \mathcal{N} \times \mathcal{N} \to [0,1]$
$\mathbf{N}$	Markov chain defined by P on $\mathcal{N}$
$\mathfrak{L}$	Discrete generator
$\varTheta^{condition}$	Set of parameters satisfying the condition

Chapitre 0

Introduction

Motivations

Les progrès technologiques ont profondément changé la structure des marchés financiers qui deviennent de plus en plus électroniques et compétitifs. Les transactions deviennent plus liquides permettant ainsi une meilleure gestion de risque. La majorité des marchés financiers s'éloignent des marchés avec des spécialistes ou market makers et optent pour le mécanisme avec carnet d'ordres. Cela facilite les échanges et permet une réduction importante des risques de liquidité. D'une part, le carnet d'ordres est l'ensemble des ordres d'achat ou de vente disponibles et non exécutés à un instant donné pour un actif financier. D'autre part, le risque de liquidité reste l'un des problèmes les plus étudiés dans les risques financiers mais l'un des moins compris. Nous pouvons cependant noter que le risque de liquidité est étroitement associé au risque dû à l'impact de prix créé par de larges transactions. De plus, la liquidité peut être caractérisée par la résilience, qui correspond à la vitesse à laquelle le prix revient à la normale après un choc aléatoire et sans information [44]. Le fait de "revenir à la normale" implique l'existence d'un prix de référence que l'on peut définir comme un prix d'équilibre déterminé par l'agrégation de l'opinion des différents investisseurs à un instant donné.

La modélisation du risque de liquidité, et plus précisément du carnet d'ordres est d'une importance capitale pour les intervenants des marchés financiers. Les faits stylisés, c'est à dire des observations communes parmi des propriétés observées dans des études statistiques sur différents marchés et différents actifs financiers, imposent des restrictions très contraignantes à la modélisation de carnet d'ordres. Il est en effet nécessaire de comparer les différents faits stylisés et les critères de liquidité tels que l'impact de prix ou le profil moyen de la densité du carnet d'ordres entre les modèles simulés et les données du marché afin de valider leur consistance. L'estimation de paramètres est donc un problème fondamental dans la calibration des modèles à partir des données du marché.

La modélisation du carnet d'ordres permet naturellement aux investisseurs de définir des stratégies d'exécution optimale afin de minimiser l'impact de prix des stratégies et optimiser leurs coûts. Les stratégies d'exécution optimale sont au cœur des problématiques concrètes des acteurs des marchés financiers, en particulier les stratégies optimales d'achat ou de liquidation. Lorsqu'un investisseur possède un nombre conséquent de volume d'un actif financier, le problème de liquidation optimale correspond à la recherche d'un algorithme de vente optimisant son critère de gain sous un certain nombre de contraintes.

Malgré une riche littérature, les deux problématiques des mathématiques financières, c'est à dire, la modélisation de carnet d'ordres et les problèmes d'exécution optimale restent actuellement parmi les sujets plus étudiés.

Cette thèse étudie précisément ces deux sujets d'actualité. Dans la première partie de cette thèse, nous proposons et étudions un modèle de carnet d'ordres à travers l'étude d'une chaîne de Markov dans un espace d'états dénombrables dont les états sont les densités cumulées de carnet d'ordres, autrement dit, la profondeur du carnet, voir Chapitre 2. Le recours à la profondeur du carnet dans la modélisation est à notre connaissance une approche nouvelle. Cette nouvelle approche facilite la résolution des problèmes de liquidation optimale. De plus, cette approche permet surtout de relâcher l'hypothèse imposant que chaque transaction soit de taille constante. Ce relâchement rend notre modèle de carnet d'ordres plus réaliste et plus fidèle à la structure même des marchés financiers. Une fois le modèle fixé, nous spécifions et étudions les comportements de notre carnet d'ordres selon les lois des arrivées des différents types d'ordre. Chaque ordre limite est muni d'un triplet (prix, volume et temps d'arrivée) alors qu'un ordre de marché est caractérisé uniquement par le volume et le temps d'arrivée. Nous étudions le comportement de notre chaîne de Markov, c'est à dire l'évolution du carnet d'ordres, selon les hypothèses fixées sur les lois des arrivés d'ordre (prix, volume, temps). Il est en particulier important d'étudier les cas ou ces lois dépendent de l'état courant de la chaîne de Markov. Dans un premier temps, nous avons prouvé que l'ensemble des paramètres rendant la chaîne de Markov transiente ou récurrente est non vide, voir Chapitre 3. Les résultats théoriques les plus importants de cette partie sont toutefois la caractérisation de cet ensemble, et plus précisément sa description. Pour cela, nous utilisons une approche semi-martingale pour les différents termes du générateur qui dépendent fortement de l'état courant. Nous exhibons une condition sur les paramètres dépendant de faits empiriques. Pour compléter notre étude, nous étudions le problème d'estimation de paramètres en confrontant notre modèle aux données de marché via les faits stylisés et des critères de liquidité. A l'aide des données de marché, nous donnons de manière concrète une paramétrisation pour les différents problèmes de transience et de récurrence, voir Chapitre 4. Enfin, nous revenons au problème initial de liquidation optimale dans le cadre de notre modèle. Le critère d'optimisation consiste à maximiser le gain total sous la contrainte de liquidation, c'est à dire, de vendre l'ensemble du volume initial détenu par l'investisseur.

La partie 2 de cette thèse approfondit l'étude des problèmes de liquidation sous contrainte d'impact de prix. Contrairement à la première partie où l'évolution du carnet d'ordres est représentée par une chaîne de Markov, nous supposons dans cette étude que la forme du carnet est fixé. Dans ce modèle, l'impact de prix est non-linéaire et modélisé par une forme de carnet d'ordres fixe et une résilience sur le volume manquant par rapport à la forme du carnet d'ordre fixe. Afin de tenir compte de l'impact des autres investisseurs, la dynamique du volume manquant à la forme de carnet d'ordres fixe est supposée gouvernée par une EDS avec sauts. Nous avons recours au Principe de la Programmation Dynamique pour obtenir un système d'inéquations variationnelles d'Hamilton-Jacobi-Bellman. Nous caractérisons et montrons que la fonction valeur est l'unique solution de viscosité au système d'inéquations variationnelles d'Hamilton-Jacobi-Bellman associé. Nous utilisons une approche numérique basée sur une méthode itérative de Huang et al. [42] dont le schéma converge vers la fonction valeur grâce aux critères de monotonie, de consistance et de stabilité. Nous illustrons notre étude avec des résultats numériques et donnons des interprétations aux stratégies selon différents types de forme de carnet d'ordres.

Principaux résultats et contributions

Dans la littérature, la liquidité peut être modélisée selon différentes approches. La première approche consiste à étudier une fonction d'impact de prix. Cette approche ne rend pas compte de la structure et du caractère dynamique du carnet d'ordres. Une seconde approche considère une densité fixe de carnet d'ordres. Une dernière approche, plus difficile, est de modéliser la dynamique du carnet d'ordres. Cependant, à notre connaissance, aucun problème de liquidation optimale a utilisé cette dernière approche.

0.0.1 Modélisation de carnet d'ordres

La modélisation de carnet d'ordres consiste à définir les interactions entre les différents types d'ordres envoyés sur le marché.

Les premiers modèles [9], [49], [19] ont cherché à comprendre le mécanisme du carnet d'ordres et ont retrouvé certains faits stylisés pour valider leur modèle. Cependant, leurs modèles ne tiennent pas compte simultanément des 3 principaux types d'ordres. Dans [26], les auteurs proposent un modèle basé sur des processus de Poisson représentant les 3 types d'ordres. Le placement des ordres limites est représenté par une suite de variable aléatoire de loi uniforme. Cont et al. [24] propose une famille de processus de Poisson indépendant à chaque niveau de prix pour les ordres d'annulation et les ordres limites avec. Abergel et Jedidi [2] propose une famille de processus de Poisson indépendant à chaque niveau de prix pour les ordres d'annulation et les ordres limites dans un cadre mobile.

Dans les modèles présentés, la taille des volumes des ordres est supposée constante et égale à une unité, ce qui ne reflète pas le mécanisme réel du carnet d'ordres. De plus, les modèles de carnet d'ordres dans la littérature ne sont pas orientés pour les problèmes de liquidation optimale.

Dans la partie 1 de la thèse, inspiré par le papier de Predoiu et al. [56] dans le cadre d'un problème de liquidation optimale, nous proposons un modèle de carnet d'ordres dont les états sont représentés par la profondeur.

0.0.1.1 Description du modèle

Nous notons $\mathcal{N}$, l'espace dénombrable des profondeurs et n un élément de $\mathcal{N}$.

$$\mathcal{N} = \{(\mathsf{n}^{|1}, \mathsf{n}^{|2}, \mathsf{n}^{|3}, \ldots) : \mathsf{n}^{|i} \in \mathbb{N}, \mathsf{n}^{|i} \geq \mathsf{n}^{|i+1} \text{ et à partir d'un certain entier } k, \mathsf{n}^{|k} = 0\}.$$

$\mathsf{n}^{|1}$ correspond au volume totale d'un état du carnet d'ordres. L'indice $i \in \mathbb{N}^*$ représente les différents niveaux de prix et $\mathsf{n}^{|i} - \mathsf{n}^{|i+1}$ représente le volume disponible au prix i. Nous pouvons définir des fonctions de $\mathcal{N}$ dans $\mathbb{R}$. Le prix au bid peut être définit :

$$\mathfrak{b}(\mathsf{n}) := \max\{k \in \mathbb{N}^* : \mathsf{n}^{|k} > 0\} = \sum_{k=1}^{\infty} \mathbb{1}_{\{\mathsf{n}^{|k} > 0\}},$$

Nous définissons les opérateurs modifiant l'état du carnet d'ordres.

T^B l'opérateur d'achat, représentant la modification du carnet d'ordres après un ordre limite d'achat de taille a et au niveau de prix b :

$$\mathsf{T}^B(\mathsf{n}, a, b)^{|k} := \begin{cases} \mathsf{n}^{|k} + a, & k \leq b, \\ \\ \mathsf{n}^{|k}, & k > b. \end{cases}$$

et T^S : l'opérateur de vente, représentant la modification du carnet d'ordres après un ordre marché de vente de taille a :

$$\mathsf{T}^S(\mathsf{n}, a) := ((\mathsf{n}^{|1} - a)^+, (\mathsf{n}^{|2} - a)^+, (\mathsf{n}^{|3} - a)^+, \ldots),$$

et l'opérateur d'annulation, représentant la modification du carnet d'ordres après une annulation de taille a et au niveau de prix b :

$$\mathsf{T}^C(\mathsf{n}, a, b)^{|k} := \begin{cases} \mathsf{n}^{|k} - a \wedge (\mathsf{n}^{|b} - \mathsf{n}^{|b+1}), & k \leq b, \\ \\ \mathsf{n}^{|k}, & k > b, \end{cases}$$

Nous nous plaçons dans un cadre temporel discret, les instants sont représentés

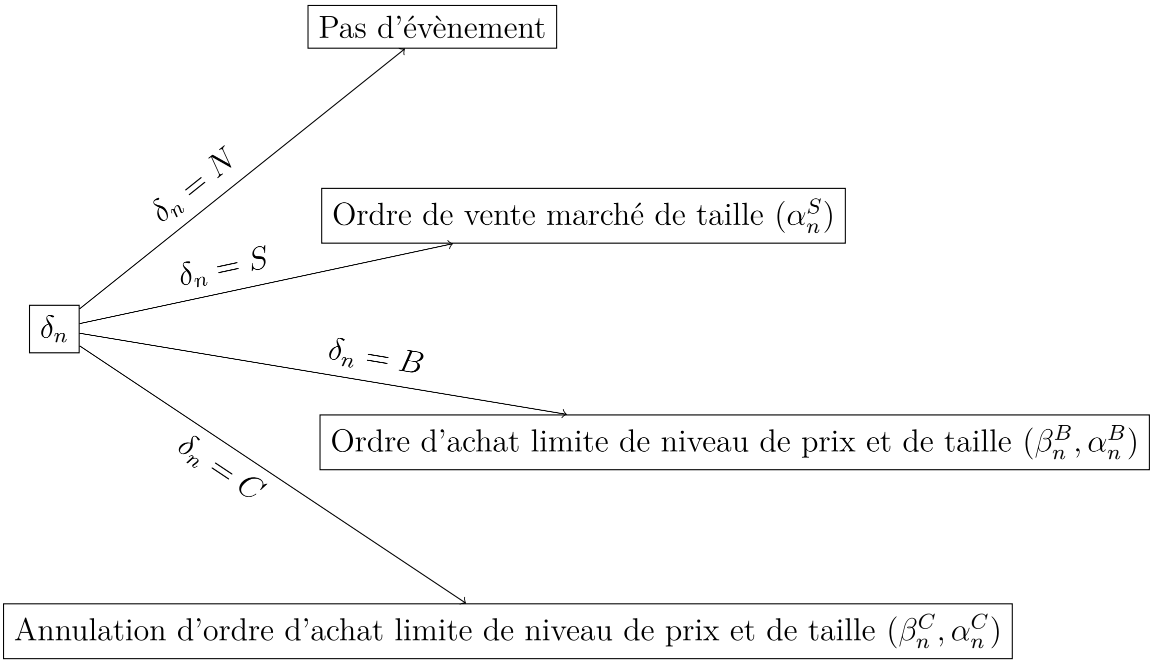


FIGURE 1 – A chaque instant n, nous simulons les variables aléatoires $(\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C)$

par les entiers $\mathbb{N}$. À chaque instant $n \in \mathbb{N}$, nous choisissons :

— δ_n l'indicateur d'évènements prenant ses valeurs dans l'ensemble $\{N, S, B, C\}$ représentant les 4 évènements,

— le niveau de prix d'un ordre d'achat ou d'une annulation d'ordre d'achat par $\beta_n^B, \beta_n^C \in \mathbb{N}^*$,

— la taille de l'ordre par $\alpha_n^S, \alpha_n^B, \alpha_n^C \in \mathbb{N}^*$ correspondant respectivement à un ordre de vente, à un ordre d'achat et une annulation d'ordre d'achat.

Nous avons une représentation des choix des variables aléatoires sur la figure 1.

Soit $(\Omega, \mathcal{A}, \mathbb{P})$ un espace probabilisé, où $\mathcal{A}$ est une σ-algèbre sur Ω et $\mathbb{P}$ est une mesure de probabilité sur $\mathcal{A}$. δ_n ,α_n^S ,α_n^B ,α_n^C ,β_n^B , β_n^C sont des variables aléatoires sur $(\Omega, \mathcal{A}, \mathbb{P})$.

Soit $\mathbf{N}_0 \in \mathcal{N}$ soit un état initial de carnet d'ordres. Nous définissons de manière récursive :

$$\mathbf{N}_{n+1} = \begin{cases} \mathbf{N}_n, & \text{if } \delta_{n+1} = N, \\ \mathsf{T}^S(\mathsf{n}, \alpha_{n+1}^S), & \text{if } \delta_{n+1} = S, \\ \mathsf{T}^B(\mathbf{N}_n, \alpha_{n+1}^B, \beta_{n+1}^B), & \text{if } \delta_{n+1} = B, \\ \mathsf{T}^C(\mathbf{N}_n, \alpha_{n+1}^C, \beta_{n+1}^C), & \text{if } \delta_{n+1} = C, \end{cases} \quad (1)$$

pour $n \in \mathbb{N}$.

Le processus $\mathbf{N} = (\mathbf{N}_n)_{n \in \mathbb{N}}$ représente l'évolution du carnet d'ordres.

Soit $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ la filtration naturelle générée par la suite de variables aléatoires $(\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C)$ pour $n \in \mathbb{N}$.

Après avoir construit le formalisme du carnet d'ordres, la modélisation consiste à choisir les distributions de probabilité des variables aléatoires $\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C$ for $n \in \mathbb{N}$ pour $n \in \mathbb{N}$.

Dans cette thèse, l'évolution du carnet d'ordres est modélisée par une chaîne de Markov. Plus précisément, nous supposons qu'il existe une probabilité $\mathbb{Q}^{\mathsf{n}}[A]$ de $\mathcal{N}$ vers $\{N, S, B, C\} \times (\mathbb{N}^*)^5$ tel que :

$$\begin{aligned} & \mathbb{P}[(\delta_{n+1}, \alpha_{n+1}^s, \alpha_{n+1}^b, \alpha_{n+1}^c, \beta_{n+1}^b, \beta_{n+1}^c) \in A \,|\mathcal{F}_n] \\ = \; & \mathbb{P}[(\delta_{n+1}, \alpha_{n+1}^s, \alpha_{n+1}^b, \alpha_{n+1}^c, \beta_{n+1}^b, \beta_{n+1}^c) \in A \,|\mathbf{N}_n] \\ = \; & \mathbb{Q}^{\mathbf{N}_n}[(\delta, \alpha^s, \alpha^b, \alpha^c, \beta^b, \beta^c) \in A], \end{aligned}$$

Cette hypothèse d'existence donne le caractère markovien de notre modèle. Nous travaillons dans ce cadre là. Caractériser une probabilité $\mathbb{Q}^{\mathsf{x}}$ est équivalent à caractériser les variables aléatoires $\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C$.

Assumption 0.0.1. *Pour tout $\mathsf{x} \in \mathcal{N}$, sous la probabilité $\mathbb{Q}^{\mathsf{x}}$, la variable aléatoire δ est indépendante de $\alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$.*

Nous allons travailler avec l'hypothèse 0.0.1 par la suite. Nous pouvons alors définir

$$p_n(\mathsf{x}) := \mathbb{Q}^{\mathsf{x}}[\delta = N],\; p_s(\mathsf{x}) := \mathbb{Q}^{\mathsf{x}}[\delta = S],\; p_b(\mathsf{x}) := \mathbb{Q}^{\mathsf{x}}[\delta = B],\; p_c(\mathsf{x}) := \mathbb{Q}^{\mathsf{x}}[\delta = C].$$

Les 4 termes définissent les probabilités d'arrivée d'un évènement.

Nous définissons la probabilité de transition P sous l'hypothèse d'indépendance0.0.1 : pour x et y dans $\mathcal{N}$,

$$\begin{aligned} \mathsf{P}(\mathsf{x},\mathsf{y}) \;:=\; & p_n(\mathsf{x})1\!\!1_{\{\mathsf{x}=\mathsf{y}\}} + p_s(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{T}^S(\mathsf{x}, \alpha^s) = \mathsf{y}] \\ & + p_b(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{T}^B(\mathsf{x}, \alpha^B, \beta^B) = \mathsf{y}] + p_c(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{T}^C(\mathsf{x}, \alpha^C\beta^C) = \mathsf{y}], \end{aligned} \tag{2}$$

et nous définissons ainsi le drift pour une fonction F de $\mathcal{N}$ dans $\mathbb{R}^+$

$$\begin{aligned} \mathfrak{L}F(\mathsf{x}) \;:=\; & p_s(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{T}^S(\mathsf{x}, \alpha^s)] - F(\mathsf{x})) \\ & + p_b(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{T}^B(\mathsf{x}, \alpha^B, \beta^B)] - F(\mathsf{x})) + p_c(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{T}^C(\mathsf{x}, \alpha^C\beta^C)] - F(\mathsf{x})). \end{aligned} \tag{3}$$

0.0.1.2 Résultats théoriques

Homogénéité Dans les modèles existants de carnet d'ordres [24], [2], [9], [49], [19], les auteurs supposent que la taille des ordres est constante. L'intérêt de cette hypothèse réside notamment dans la simplification des calculs pour le drift sur le problème de récurrence positive. De plus, les auteurs [24], [2] n'étudient pas la notion de transience de leur chaîne de Markov. Dans notre modèle, nous n'avons pas besoin de l'hypothèse de taille constant pour le problème de récurrence. De plus, nous avons étudié la transience. Nous utilisons les conditions de Foster-Lyapunov pour étudier la récurrence et la transience faisant intervenir le drift. Lorsque nous enlevons l'hypothèse de taille constant, le drift dépend fortement de l'état courant du carnet d'ordres. La notion d'homogénéité consiste à réduire la dépendance du drift par rapport à l'état courant.

Irréductibilité

Nous cherchons l'irréductibilité afin de simplifier l'étude de la récurrence et de la transience. Nous rappelons que l'irréductibilité d'une chaîne de Markov exprime l'existence d'un chemin d'état vers un autre, de probabilité non nulle. Soit $\mathsf{x} \in \mathcal{N}$, nous définissons l'ensemble $\mathcal{S}(\mathsf{x})$ correspond aux niveaux de prix possédant des volumes

$$\mathcal{S}(\mathsf{x}) := \{k \in \mathbb{N}^* : \mathsf{x}^{|k} > \mathsf{x}^{|k+1}\}$$

Theorem 0.0.1. *Supposons*

i. $\mathbb{Q}^{\mathsf{x}}[\beta^b = \mathfrak{b}(\mathsf{x}) - 1] > 0$ *si* $\mathfrak{b}(\mathsf{x}) > 1$, *et* $\mathbb{Q}^{\mathsf{x}}[\beta^b = \mathfrak{b}(\mathsf{x})] > 0$, *si* $\mathfrak{b}(\mathsf{x}) > 0$, *et* $\mathbb{Q}^{\mathsf{x}}[\beta^b = \mathfrak{b}(\mathsf{x}) + 1] > 0$, *pour tout* x.

ii. $\mathbb{Q}^{\mathsf{x}}[\beta^c = k] > 0$ *pour tout* $k \in \mathcal{S}(\mathsf{x})$.

iii. $\mathbb{Q}^{\mathsf{x}}[\alpha^b = 1] > 0$ *et* $\mathbb{Q}^{\mathsf{x}}[\alpha^c = 1] > 0$, *pour tout* x.

iv. $p_n(\mathsf{x}) > 0, p_b(\mathsf{x}) > 0, p_c(\mathsf{x}) > 0$ *pour tout* x.

Alors, la chaîne de Markov $\mathbf{N}$ *est irréductible et apériodique*

Nous définissons Θ^{irr}, l'ensemble des paramètres rendant la chaîne de Markov $\mathbf{N}$ irréductible et apériodique. Nous nous plaçons dans cette ensemble par la suite.

Récurrence

Nous montrons que la chaîne de Markov $\mathbf{N}$ peut être récurrente. Soit $V(\mathsf{x}) := \mathfrak{b}(\mathsf{x}) + H(\mathsf{x})$

Lemma 0.0.1. *Supposons que les variables aléatoires* α^b, β^b *sont intégrables sous* $\mathbb{Q}^{\mathsf{x}}$ *pour tout* $\mathsf{x} \in \mathcal{N}$. *Le drift* $\mathfrak{L}V(\mathsf{x})$ *est bien définit. Si l'ensemble* $\{\mathsf{x} \in \mathcal{N} : \mathfrak{L}V(\mathsf{x}) > 0\}$ *est un ensemble fini, la chaîne de Markov* $\mathbf{N}$ *est récurrent.*

Nous définissons Θ^{rec} l'ensemble des paramètres rendant la chaîne de Markov récurrente.

Nous avons prouvé qu'il existe un ensemble non nul de paramètres pour lequel la chaîne est récurrente. La véritable question est de produire un ensemble concret de paramètres. En effet, nous prouvons que

Proposition 0.0.1. *Soit* $\mathsf{x} \in \mathcal{N}$ *et si* α^C *à support fini dépendant de* $\Delta_{\beta^C}\mathsf{x}$ *Soit* $\Theta^{rec*} \subset \Theta^{rec}$ *l'ensemble des paramètres vérifiant :*

Pour $\mathsf{x}^{|1}$ *assez grand,*

$$p_b(\mathsf{x})D \leq p_c(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\alpha^C],$$

avec $p_c(\mathsf{x})$ *est une constante positive par hypothèse,* $D := \mathbb{E}^{\mathsf{x}}[\alpha^B] + \mathbb{E}^{\mathsf{x}}[\beta^B]$

Alors la chaîne de Markov $\mathbf{N}$ *est récurrente.*

La condition veut dire que pour un niveau de volume $\mathsf{x}^{|1}$ grand, les annulations d'ordre d'achat doivent être supérieurs aux arrivées d'ordre d'achat. L'ensemble Θ^{rec*} est plus explicite, nous utilisons cette ensemble dans la calibration du modèle.

Transience

Dans la recherche de modèles transients, nous allons considérer naturellement les marchés haussiers. Nous montrons que la chaîne peut être transiente.

Theorem 0.0.2. *Supposons que les espérances* $\mathbb{E}^{\mathsf{x}}[(\alpha^s)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^b)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^c)^2]$ *sont uniformément bornées. Supposons que, pour une constante* $\theta > 0$, $\mathfrak{L}H(\mathsf{x}) \geq \theta$ *uniformément. Alors,*

$$\liminf_{n\to\infty} \frac{\mathbf{N}_n^{|1}}{n} > 0.$$

En particulier, la chaîne de Markov $\mathbf{N}$ *est transiente.*

Nous définissons Θ^{tra} l'ensemble des paramètres rendant la chaîne de Markov $\mathbf{N}$ transiente.

Proposition 0.0.2. *Soit* $h, g \in \mathbb{N}^*$. *Supposons que les espérances* $\mathbb{E}^{\mathsf{x}}[(\alpha^s)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^b)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^c)^2]$ *sont uniformément bornées.*

Soit $\Theta^{tra*} \subset \Theta^{tra}$ *l'ensemble des paramètres vérifiant :*

- $\mathbb{E}^{\mathsf{x}}[\alpha^b]$ *est une constante* $a_b > 0$
- β^c *est une variable aléatoire suivant une loi uniforme sur* $\mathcal{S}(\mathsf{x})$
- α^c *est une variable aléatoire suivant une loi uniforme sur* $\{1, \ldots, g \wedge \Delta_{\beta^c}\mathsf{x}\}$
- α^s *est une variable aléatoire suivant une loi uniforme sur* $\{1, \ldots, h \wedge \mathsf{x}^{|1}\}$
- $p_s(\mathsf{x}), p_b(\mathsf{x}), p_c(\mathsf{x})$ *sont constantes et*

$$\theta := -p_s\frac{1+h}{2} + p_b a_b - p_c\frac{1+g}{2} > 0. \tag{4}$$

Alors la chaîne de Markov $\mathbf{N}$ *est transiente.*

La condition veut dire que l'arrivée des ordres d'achat doit être supérieur à l'arrivée des annulations et l'arrivée des ordres de vente. Cela va impliquer le prix $\mathfrak{b}$ et le volume totale $\mathsf{n}^{|1}$ va tendre vers l'infini. Nous allons chercher des modèles haussiers plus réalistes.

Un modèle de transience plus réaliste

Nous définissons $\mathfrak{d}(\mathsf{x})$ le prix le plus bas possédant des volumes par

$$\mathfrak{d}(\mathsf{x}) = \begin{cases} \inf \mathcal{S}(\mathsf{x}) & \text{si } \mathcal{S}(\mathsf{x}) \neq \emptyset, \\ 0 & \text{si } \mathcal{S}(\mathsf{x}) = \emptyset. \end{cases}$$

Nous définissons pour tout $\mathsf{x} \in \mathcal{N}$, le processus support comme $\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x})$.

Dans un marché haussier, le processus bid tend vers l'infini, c'est la raison pour laquelle la chaîne de Markov $\mathbf{N}$ doit être transiente. Cependant, le processus support $(\mathfrak{b} - \mathfrak{d})$ ne doit pas explosé. Cela a un sens, les ordres limites achat disponibles dans le carnet d'ordres à des niveaux de prix éloignés du bid sont supprimés par les investisseurs car la probabilité d'être exécuté est faible. De plus, certains marchés peuvent avoir un mécanisme de suppression lorsque les niveaux de prix sont trop éloignés du bid. Nous cherchons un marché pour lequel les volumes sont contraints dans une rangée de prix.

Theorem 0.0.3. *Soit $\Theta^{tra1} \subset \Theta^{tra}$ l'ensemble des paramètres vérifiant les hypothèses suivantes :*

— *la variable aléatoire α^s est bornée*

— $p_s(\mathsf{x}) = 0$ *si* $\mathbb{Q}^{\mathsf{x}}[\alpha^s \geq \mathsf{x}^{|1}] = 0$, *et* $p_s(\mathsf{x})$ *est une constante sinon.*

— *la variable aléatoire β^b prend la forme :*

$$\beta^b = \mathfrak{b}(\mathsf{x}) - (1-\epsilon)\beta^{b,-} + \epsilon\beta^{b,+},$$

où $\beta^{b,-} \in [0, \mathfrak{b}(\mathsf{x})], \beta^{b,+} \in \mathbb{N}^*$ *sont indépendantes de* $\epsilon \in \{0,1\}$ *sous* $\mathbb{Q}^{\mathsf{x}}$, *et* $\beta^{b,-}, \beta^{b,+}$ *sont bornés par* b^- *et* b^+ *respectivement*

— *Sous* $\mathbb{Q}^{\mathsf{x}}$, β^c *est une variable aléatoire suivant une loi géométrique de paramètre* p *(le même* p *pour tout* x*) et conditionnée sur l'ensemble* $\mathcal{S}(\mathsf{x})$.

— *Pour une constante* $c > 0$, $\mathfrak{L}\mathfrak{b}(\mathsf{x}) \geq c$ *pour tout* x.

— *Pour des constantes* $g > 0, h > 0, c' > 0$, $\mathfrak{L}_g(\mathfrak{b} - \mathfrak{d})(\mathsf{x}) \leq -c'$ *for all* x *tel que* $\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) \geq h$.

Alors, le processus bid $\mathfrak{b}(\mathbf{N}_n)$ *tend vers l'infini (la chaîne de Markov* $\mathbf{N}$ *est transiente), alors que le processus support* $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ *revient vers de valeurs inférieurs à* h, *après une certaine durée.*

Proposition 0.0.3. *Soit* $\Theta^{tra1*} \subset \Theta^{tra1}$ *l'ensemble des paramètres tel que* $p_s(\mathsf{x})$, $p_b(\mathsf{x})$, $p_c(\mathsf{x})$ *vérifient :*

$$\begin{cases} -p_s(\mathsf{x})b_s + p_b(\mathsf{x})b_p - p_c(\mathsf{x})b_c = c_1 > 0, & \textit{pour } \mathsf{x}. \\ -p_s(\mathsf{x})b_s + p_b(\mathsf{x})b_p - p_c(\mathsf{x})(b_c + d_c) = c_2 < 0 & \textit{pour } \mathsf{x} \textit{ tel que } \mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) > h. \end{cases}$$

avec

- *le terme* b_s *correspond à la borne uniforme de* $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{T}^S(\mathsf{x}, \alpha^S))]$,
- *le terme* $b_p := b^+$ *correspondant à la borne de la variable aléatoire* β^{B+},
- *le terme* $b_c := \frac{e^{-1}}{-\ln q}$ *correspond à une borne pour le changement de prix* $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^o(\mathsf{x})$ *pour une annulation,*
- *le terme* $d_c := \mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x} \mid \beta^C = \mathfrak{d}(\mathsf{x})]$ *est une constante indépendante de* x *et correspond à la probabilité d'annuler la totalité du volume au niveau de prix* $\mathfrak{d}(\mathsf{x})$.

Dans l'ensemble Θ^{tra1*}, *le bid* $\mathfrak{b}(\mathbf{N}_n)$ *tend vers l'infini (la chaîne de Markov* $\mathbf{N}$ *est transiente), alors que le processus support* $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ *revient vers de valeurs inférieurs à* h*, après une certaine durée.*

Lorsque $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ le processus support est trop grand (il existe une valeur h tel que $\mathfrak{b}(\mathbf{N}_n) - \mathfrak{d}(\mathbf{N}_n) \geq h$), le drift doit être négatif pour faire revenir le processus vers des valeurs plus petites. Le terme d_c permet de rendre le générateur négatif. Ce terme correspond à l'annulation totale du volume au niveau de prix $\mathfrak{d}(\mathbf{N}_n)$, ce qui a pour but de réduire $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$.

Nous approfondissons la recherche de modèle plus réaliste en restreignant le processus volume H et le processus support $\mathfrak{b} - \mathfrak{d}$.

Theorem 0.0.4. *Supposons les hypothèses du théorème 0.0.3. les hypothèses du théorème 0.0.2, sauf la condition* $\mathfrak{L}H(\mathsf{x}) \geq \theta$ *qui est remplacé par : pour tout constante* $c' > 0, h' > 0$, $\mathfrak{L}H(\mathsf{x}) \leq -c'$ *lorsque* $x^{|1} > h'$. *Alors, la chaîne de Markov* $\mathbf{N}$ *est transiente et le processus support* $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ *et le processus volume* $H(\mathbf{N}_n)$ *reviennent vers de valeurs inférieurs à* h', *après une certaine durée.*

Proposition 0.0.4. *Nous supposons que* $a_b = \mathbb{E}^{\mathsf{x}}[\alpha^b]$, $\mathbb{E}^{\mathsf{x}}[\alpha^s]$ *sont des constantes positives, Nous ne modifions pas les variables aléatoires* $\alpha^s, \beta^{b,-}, \beta^{b,+}, \beta^c$ *fixé dans* **??**. *Pour les probabilités* $p_s(\mathsf{x}), p_b(\mathsf{x})$, *nous les choisissons tel que :*

$$\begin{array}{l} 0 < p_s(\mathsf{x}) < 1 \quad \textit{a constant}, \\ 0 < p_b(\mathsf{x}) \leq \frac{a_{pc}}{a_b}, \quad \textit{pour } \mathsf{x} \textit{ tel que } \mathsf{x}^{|1} > h'. \end{array}$$

avec $a_{pc} := p_c(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\widetilde{\alpha}^c]$ *que nous pouvons prendre constant. Alors, la chaîne de Markov* $\mathbf{N}$ *est transiente et le processus support* $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ *et le processus volume* $\mathbf{N}_n^{|1}$ *reviennent vers de valeurs inférieurs à* h', *après une certaine durée.*

0.0.2 Estimations et calibrations du modèle

Cette partie consiste à choisir les bons paramètres pour que le modèle reflète le marché. En particulier, nous travaillons ici avec des données du marché, venant du contrat du Future **Bund** négocié sur la bourse EUREX. Les données ont une précision de l'ordre de la milliseconde et ont été enregistrées entre mars 2013 et septembre 2013.

Les modèles de carnet d'ordres qui étudient les problématiques d'estimation et de calibration, se ramènent à un problème d'estimation. En effet, ils sont modélisés à l'aide de processus de Poisson et ils estiment les intensités des processus de Poisson à l'aide des données et d'une méthode de maximum de vraisemblance. Il n'y a donc aucun degré supplémentaire dans leur modèle et il n'y a pas de calibration à faire.

Notre problème de calibration consiste à trouver les paramètres dans l'ensemble des probabilités $\mathbb{Q}$ dans Θ^{rec}. Pour réduire l'ensemble des lois d'arrivées, nous allons nous placer dans le cas d'une chaine de Markov récurrente. Il s'agit d'un choix artificiel mais nous justifions ce choix par la littérature existante [24] et [2].

Les faits stylisés et les critères de liquidité donnent des restrictions aux modèles stochastiques de carnet d'ordres. Les bons modèles devraient être capable de capturer simultanément la plupart d'entre eux avec peu de paramètres. En se plaçant dans Θ^{rec}, nous étudions les faits stylisés et les critères de liquidité en imposant des lois.

Réduction de l'ensemble Θ^{market} par la récurrence Nous étudions dans cette partie différentes lois pour α^C et β^C. Soit Θ^{α^C} l'ensemble des probabilités tel que α^C dépend de l'état courant, c'est à dire α^C possède la forme :

$$\alpha^c = \mathbb{1}_{\{\epsilon^c=1\}} \Delta_{\beta^c} \mathsf{x} + \mathbb{1}_{\{\epsilon^c=0\}} \xi^c$$

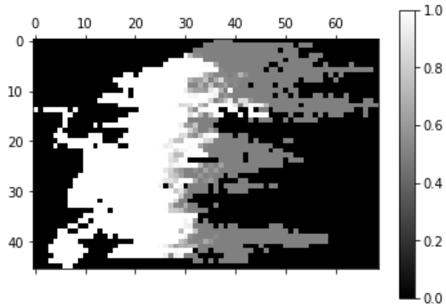
avec $\epsilon^c \in \{0, 1\}, \xi^c \in \{1, \ldots, 1 \vee (\Delta_{\beta^c} \mathsf{x} - 1)\}$, indépendant de β^c.

Lorsque α^C ne dépend pas de l'état du carnet ou lorsque α^C suit une loi uniforme et α^C dépend de l'état du carnet, la condition de récurrence n'est pas vérifiée.

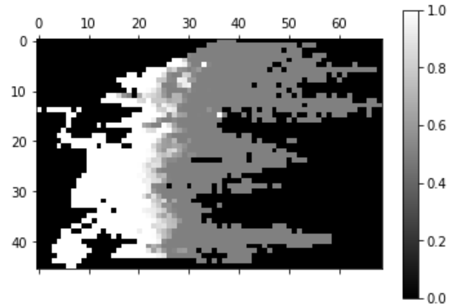
Lorsque α^C suit une loi binomiale et α^C dépend de l'état du carnet, nous vérifions le caractère récurrent de la chaîne. Les deux ensembles sont disjoints et sont séparés par une frontière. La frontière est différente selon le choix de β^C (Figure 2a, 4.7a et 4.7b). Il y a une indifférence au choix de β^C dans la recherche de paramètres rendant la chaîne récurrente. Cette étude nous a donné des restrictions sur le choix de la loi α^C et de l'importance de la dépendance par rapport à l'état courant.

Ainsi nous réduisons notre recherche de l'ensemble des probabilités dans $\Theta^{\alpha^C=bin} = \Theta^{\alpha^C} \cap \{\alpha^C \sim \text{une loi binomiale}\}$

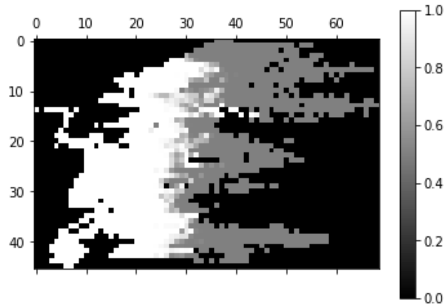
Recherche de l'ensemble Θ^{market} par les critères de liquidité Soit Θ^{α^S} l'ensemble des probabilités tel que α^S dépend de l'état courant, c'est à dire α^S possède



(a) β^C suit une loi binomiale, α^C une loi binomiale et $\mathbb{Q}$ dans Θ^{α^C}

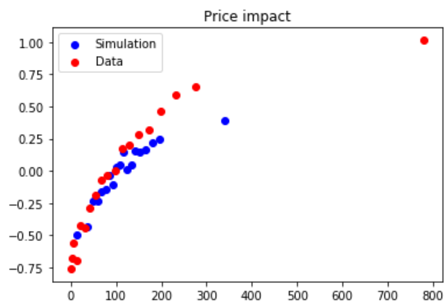


(b) β^C suit une loi géométrique, α^C une loi binomiale et $\mathbb{Q}$ dans Θ^{α^C}

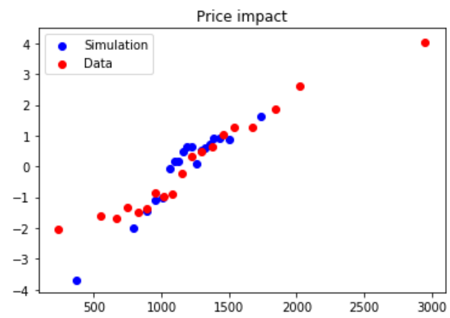


(c) β^C suit une loi lognormale, α^C une loi binomiale et $\mathbb{Q}$ dans Θ^{α^C}

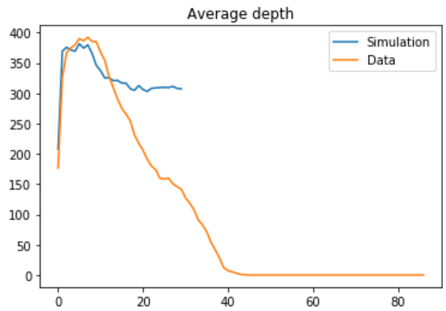
FIGURE 2 – Représentation des positions qui vérifient la condition de récurrence : les parties noires sont les positions non atteintes par les données du marché,la partie blanche correspond aux positions ne vérifiant pas la condition, la partie grise correspond aux positions vérifiant la condition. En abscisse, nous avons le volume totale $(\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p} - \mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min})/120$ et en ordonnée, le prix du bid $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}_{min}$



(a) Impact de prix avec $\mathbb{Q}$ dans Θ^{α^S} avec $\Delta t = 100$



(b) Impact de prix avec $\mathbb{Q}$ dans Θ^{α^S} avec $\Delta t = 1000$



(c) Profil moyen avec $\mathbb{Q}$ dans Θ^{α^S}

FIGURE 3 – Représentation graphique de critères de liquidité

la forme pour un certain niveau de prix l :

$$\alpha^S := \alpha^S(\mathsf{n}^{\mathfrak{b}(\mathsf{n})-l})$$

Cela veut dire que le support de la loi de α^S doit être fini et dépendre de $\mathsf{n}^{\mathfrak{b}(\mathsf{n})-l}$. Dans le cas de $l = 0$, la loi ne dépend que du volume disponible au niveau du bid.

Nous définissons le profil moyen comme la moyenne temporelle des volumes à chaque niveau de prix.

$$M(p) = \lim_{n\to\infty} \sum_{k=0}^{n} \Delta_{\mathfrak{b}(\mathsf{X}_k)-p}\mathsf{X}_k.$$

Nous prenons comme référence le prix du bid.

Dans le cas où α^S dépend de l'état du carnet (Figure 4.11b), la courbe empirique s'accorde bien pour les valeurs proches du bid avec le profil moyen calculé par les données du marché.

Nous définissons l'impact de prix, dans notre cas :

$$I_{ind}(\Delta t, q) := \mathbb{E}[r(t, \Delta t)|\delta_t = \text{ S }, q_t = q]$$

avec $\delta_t = \text{S}$ l'événement à l'instant t est un événement de marché arrivée et $q_t = q$, la taille du marché est q.

Nous obtenons une bonne concordance entre la courbe simulée et la courbe des données de marché. Cela peut s'expliquer notamment par la sélection de liquidité dû aux traders. Les traders choisissent d'envoyer des ordres de marché de grande taille lorsque la liquidité est élevée. En faisant dépendre α^S de l'état courant, nous caractérisons ce phénomène qui caractérise la concavité de la courbe de l'impact de prix. Nous retrouvons bien la concavité dans le cas α^S dépendant. Cette étude nous a montré l'importance de la dépendance par rapport à l'état courant de α^S.

Ainsi nous avons réduit notre recherche de l'ensemble des probabilités dans $\Theta^{\alpha^C=bin} \cup \Theta^{\alpha^S}$.

0.0.2.1 Problème de liquidation optimale dans le modèle de carnet d'ordres

Une caractéristique commune dans les modèles pour les problèmes de liquidation optimale est l'existence de l'impact de prix. Le marché peut être vu comme un jeu compétitif entre les apporteurs de liquidité envoyant des ordres limites et les preneurs de liquidité envoyant des ordres marché. Chaque participant cherche à acheter ou vendre de manière optimale en se basant sur le comportement global des autres participants. Les problèmes de liquidation optimale étudient la stratégie optimale pour les preneurs de liquidité.

Bertsimas et Lo [13] ont supposé que l'impact sur les prix était linéaire avec la taille des ordres marché. La stratégie optimale est d'exécuter à une vitesse constante la même quantité tout le long de l'horizon T. Almgren et Chriss [8] ont modélisé deux types d'impact sur les prix, un permanent et un instantané. Ils ont utilisé un critère de variance moyenne pour l'optimalité, et ont montré un compromis entre l'exécution rapide et lente.

En tradant à un rythme lent, le trader fait face à un faible impact sur les prix mais à une forte volatilité due au mouvement de prix de l'actif. Cependant, si il trade très rapidement, il risque d'avoir un impact de prix très élevé sur les prix mais est alors exposé à une faible volatilité des mouvements de prix.

Obizhaeva et Wang [53] ont mis en place un modèle qui génère un impact sur les prix à travers le carnet d'ordres, en particulier le profile moyen du carnet d'ordres. L'impact des prix résulte de la forme et de la profondeur du carnet. L'impact de prix n'est ni permanent ni instantané, mais décroît avec le temps. Ils ont trouvé la stratégie optimale où la profondeur du carnet d'ordres est constante.

Alfonsi et al. [7] et Predoiu et al. [56] ont généralisé le résultat. Ils ont trouvé la stratégie optimale sous le même critère et permettent à la forme du carnet d'ordres d'adopter des formes plus générales. Toutes les stratégies optimales de ces modèles ont un comportement singulier : il y a un pic au début et un autre à la fin de l'horizon d'exécution, mais entre le vitesse d'exécution est constante.

Récemment, dans un problème connexe, Abergel et al. [1] étudie la stratégie optimale pour les apporteurs de liquidité dans son modèle définit dans [2]. Ils caractérisent le problème à l'aide de la théorie des processus de décision markovienne . Ils utilisent une approche numérique et abordent le problème de grande dimension par une méthode de randomisation sur le contrôle et une méthode de quantification. Lorsque les intensités sont indépendantes de l'état courant, la stratégie optimale et la stratégie naïve, consistant à envoyer des ordres limites aux meilleurs prix, sont équivalentes. Cependant dans le cas où les intensités dépendent de l'état courant, la stratégie optimale fonctionne mieux que la stratégie naïve. Ils trouvent aussi qu'autoriser de placer des ordres limites au niveau du second meilleur prix donne de meilleurs résultats.

Dans cette thèse, pour le problème de liquidation optimale, nous utilisons deux approches très différentes : dans la première approche de manière similaire au problème d'Abergel et al. [1], nous allons étudier le problème de liquidation optimale dans un modèle stochastique de carnet d'ordres et dans une seconde approche, nous étendons le modèle de Predoiu et al. [56] en autorisant la dynamique du volume manquant a être stochastique.

Description du problème de liquidation optimale Dans cette étude, nous considérons la modélisation de carnet d'ordres définit dans la partie 0.0.1. Cependant, nous devons définir l'influence du trader stratégique sur ce modèle.
Pour le trader stratégique, une **stratégie de contrôle** $Q = (Q_k)_{k\in\{0,\dots,T\}}$ est un processus prévisible et non décroissant où, pour tout $k \in \{0, \dots, T\}$, Q_k est une variable aléatoire à valeurs dans $\mathbb{N}$ et $Q_0 = 0$.
Nous notons $\Delta Q_k := Q_{k+1} - Q_k$ la quantité d'actifs vendus au temps k, alors que Q_k est le nombre d'actions vendues par le trader stratégique jusqu'à l'instant k^-.
Á l'instant T, le trader stratégique n'a pas le choix, il doit vendre la plus grande quantité possible d'actifs pour obtenir le plus petit inventaire possible.

Lorsque le trader stratégique connaît l'état n, il peut interagir instantanément avec les ordres de vente. La **dynamique contrôlée** est définit par l'équation suivante :

$$\begin{aligned}\mathsf{N}^{|i}_{k+1} \;=\; & \check{\mathsf{N}}^{|i}_{k^+} - \mathbb{1}_{\{\delta_k=S\}}(\alpha^S_k \wedge \check{\mathsf{N}}^{|i}_{k^+}) \\ & -\mathbb{1}_{\{\delta_k=C\}}\mathbb{1}_{\{\beta^C_k\le i\}}(\alpha^C_k \wedge \Delta_{\beta^C_k}\check{\mathsf{N}}^{|i}_{k^+}) \qquad \text{pour tout } (i,k) \in \mathbb{N}^* \times \{0,\dots,T-1\} \\ & +\mathbb{1}_{\{\delta_k=B\}}\mathbb{1}_{\{\beta^B_k\le i\}}\alpha^B_k,\end{aligned} \tag{5}$$

où nous avons défini le processus représentant le carnet d'ordres juste après que le trader stratégique vend :

$$\check{\mathsf{N}}^{|i}_{k^+} = (\mathsf{N}^{|i}_k - \Delta Q_k)^+$$

Nous considérons également que le trader stratégique a l'obligation de respecter la contrainte de liquidation. Le nombre d'actions vendues par le trader stratégique doit être le plus proche possible de X à l'horizon T. Cependant, la liquidité peut baisser et le trader stratégique peut ne pas respecter la contrainte de liquidation. Pour $k \in \{1, T\}$, nous imposons qu'une stratégie admissible Q devrait satisfaire :

$$\begin{cases} Q_0 = 0 \\ 0 \leq \Delta Q_k \leq \mathsf{N}^1_{k-1} \\ \Delta Q_T = \mathsf{N}^{|1}_T \wedge (X - Q_T)(\text{ Contrainte de liquidation}) \end{cases}$$

Les ordres de vente du trader stratégique induisent le même mécanisme que les petits investisseurs qui vendent des ordres. Si, à l'instant k, l'état du carnet est $\mathsf{N}_k = \mathsf{n} \in \mathcal{N}$, et que le trader stratégique décide de vendre c_k alors le nouvel état induit par le trader stratégique est $\mathsf{T}^S(\mathsf{n}, c_k)$. Nous pouvons ensuite définir les processus d'état et leur dynamique.

Nous introduisons d'abord l'espace d'état :

$$\mathcal{Y} := \{0, \ldots, T\} \times \mathcal{N} \times \{0, \ldots, X\}$$

Nous définissons ensuite le processus d'état comme suit : $Y := (\tau, \mathsf{N}, Q)$ où τ représente l'instant et est tel que $\tau_k = k$ pour tout $k \in \{0, \ldots, T\}$, N est le processus associé à la stratégie Q et sa dynamique est donnée par l'équation (5.1).

$$\begin{aligned} \mathsf{N}^{|i}_{k+1} &= \check{\mathsf{N}}^{|i}_{k^+} - \mathbb{1}_{\{\delta_k = S\}}(\alpha^S_k \wedge \check{\mathsf{N}}^{|i}_{k^+}) \\ &\quad - \mathbb{1}_{\{\delta_k = C\}}\mathbb{1}_{\{\beta^C_k \leq i\}}(\alpha^C_k \wedge \Delta_{\beta^C_k}\check{\mathsf{N}}^{|i}_{k^+}) \qquad \text{pour tout } (i,k) \in \mathbb{N}^* \times \{0, ..., T-1\} \\ &\quad + \mathbb{1}_{\{\delta_k = B\}}\mathbb{1}_{\{\beta^B_k \leq i\}}\alpha^B_k, \end{aligned} \tag{6}$$

Où nous avons défini $\check{\mathsf{N}}^{|i}_{k^+} = (\mathsf{N}^{|i}_k - \Delta Q_k)^+$. Q est un processus représentant la quantité d'actifs vendus.
Pour $(k, \mathsf{n}) \in \{0, \ldots, T\}\ fois\ Nr$, nous notons $\mathsf{N}^{k,\mathsf{n},Q}$ la solution de l'équation (5.1) tel que $\mathsf{N}^{k,\mathsf{n},Q}_k = \mathsf{n}$.
Pour $(k, \mathsf{n}, x) \in \mathcal{Y}$, nous sommes maintenant en mesure de définir l'ensemble des stratégies admissibles par :

$$\begin{aligned} \mathcal{A}(k, \mathsf{n}, x) :=\{ & Q = (Q_j)_{j \in \{k, \ldots, T+1\}} : Q \text{ est un } \mathbb{F} - \text{processus prévisible et croissant s.t.} \\ & Q_k = x,\ \forall j \in \{k, \ldots, T-1\},\ 0 \leq \Delta Q_j \leq (\mathsf{N}^{k,\mathsf{n},Q})^{|1}_j \\ & \text{et } \Delta Q_T = (\mathsf{N}^{k,\mathsf{n},Q})^{|1}_T \wedge (X - Q_T)\} \end{aligned}$$

Pour $(k, \mathsf{n}, x) \in \{0, \ldots, T\} \times \mathcal{N} \times [0, \ldots, X]$, $\mathcal{A}(k, \mathsf{n}, x)$ est fini.

Soit $k \in \{0, \ldots, T\}$ et $\mathsf{n} \in \mathcal{N}$ l'état du carnet à l'instant k. Lorsque le trader stratégique envoie un ordre de vente de taille $q \geq 0$ à l'instant k, le trader stratégique

vend $q \wedge \mathsf{n}^{|\mathfrak{b}(\mathsf{n})}$ au prix $\mathfrak{b}(\mathsf{n})$.
Si $q < \mathfrak{b}(\mathsf{n})$, le gain est $q\mathfrak{b}(\mathsf{n})$. Le trader stratégique n'a pas d'impact instantanée sur le prix de l'offre dans ce cas.
Si $\mathfrak{b}(\mathsf{n}) - \mathfrak{b}^S(\mathsf{n}, q) = j > 0$, le trader stratégique vend au prix $\mathfrak{b}(\mathsf{n}) - p$ avec p allant de 0 à j . Pour chaque p allant de 0 à $j-1$, il vend $\Delta_{\mathfrak{b}(\mathsf{n})-p}\ mn$ et pour $p = z$, il vend $q - \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-z}$.

Par conséquent, nous sommes maintenant capables de définir la fonction de gain $\mathfrak{g}$ pour un ordre de vente de taille q pour un état du carnet n par :

$$\mathfrak{g}(\mathsf{n}, q) := \sum_{i=1}^{\infty} i(q \wedge \mathsf{n}^{|i} - \mathsf{n}^{|i+1})^+$$

Notons que

$$\mathfrak{g}(\mathsf{n}, q) := \sum_{i=1}^{\infty} i(\mathsf{n}^{|i} - \mathsf{n}^{|i+1}) - \sum_{i=1}^{\infty} i(\mathsf{T}^S(\ mn, q)^{|i} - \mathsf{T}^S(\mathsf{n}, q)^{|i+1})$$

Une intégration par parties donne $\sum_{i=1}^{\infty} i(\mathsf{n}^{|i} - \mathsf{n}^{|i+1}) = \sum_{i=1}^{\infty} \mathsf{n}^{|i}$.
Par conséquent, si nous définissons $\mathfrak{s}(\mathsf{n}) := \sum_{k=0}^{\infty} \mathsf{n}^{|i}$, nous pouvons réécrire $\mathfrak{g}$ de la façon suivante :

$$\mathfrak{g}(\mathsf{n}, q) = \mathfrak{s}(\mathsf{n}) - \mathfrak{s}(\mathsf{T}^S(\mathsf{n}, q))$$

Nous remarquons que $\mathfrak{g}$ est évidemment lié à l'état du carnet d'ordres

Comme dans [56], le coût d'un vente dépend du carnet, c'est-à-dire de la différence entre le volume cumulé sur l'état du carnet avant la vente et après la vente. Cependant, dans [56], il y a une hypothèse de martingale sur le prix du bid et de fortes hypothèse sont faites sur la forme du carnet d'ordres par rapport au prix du bid. Dans notre cas, le prix de référence de notre modèle de carnet d'ordres est 0.

Le trader stratégique a pour but de maximiser la valeur attendue du gain total obtenu à l'instant final T. Nous considérons la fonction valeur suivante v défini sur $\mathcal{Y}$ par,

$$v(k, \mathsf{n}, x) := \sup_{Q \in \mathcal{A}(k, \mathsf{n}, x)} J^Q(k, \mathsf{n}, x),$$

avec

$$J^Q(k, \mathsf{n}, x) := \mathbb{E}_{k,\mathsf{n},q}\Big[\sum_{j=k}^{T-1} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \Delta Q_s) + \mathfrak{g}\left(\mathsf{N}_T^{k,\mathsf{n},Q}, \Delta Q_T\right)\Big]$$

Rappelons que $\Delta Q_T = (X - Q_T) \wedge \left(\mathsf{N}_T^{k,\mathsf{n},Q}\right)^{|1}$.
Comme $\mathcal{A}(k, \mathsf{n}, x)$ est fini, le supremum est atteint et il existe une stratégie optimale. Nous avons donc,

$$v(k, \mathsf{n}, x) := \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} J^Q(k, \mathsf{n}, x)$$

Nous avons les conditions aux limites suivantes :

$$v(T, \mathsf{n}, x) = \mathfrak{g}(\mathsf{n}, x \wedge \mathsf{n}_T^{|1}) \qquad \text{et} \qquad v(k, \mathsf{n}, X) = 0$$

Nous pouvons établir le principe de programmation dynamique,

Theorem 0.0.5 (Le principe de programmation dynamique)**.**
Soit $(k, \mathsf{n}, x) \in \mathcal{Y}$. Pour tout temps d'arrêt τ prenant ses valeurs dans $\{k+1, \ldots, T\}$, nous avons :

$$v(k, \mathsf{n}, x) = \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} \{\mathbb{E}_{t,\mathsf{n},x}[\sum_{j=k}^{\tau-1} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \Delta Q_j) + v(\tau, \mathsf{N}_\tau^{k,\mathsf{n},Q}, Q_\tau)]\}$$

Résultats théoriques Nous définissons pour $\mathsf{n} \in \mathcal{N}$ et $c \geq 0$, la fonction $\mathfrak{b}^S$ par $\mathfrak{b}^S(\mathsf{n}, c) := \mathfrak{b}(\mathsf{T}^S(\mathsf{n}, c))$

Definition 0.0.1. *Nous définissons $\mathcal{A}_\mathfrak{b}$ l'ensemble des tailles des ordres marché vente qui conduit au même prix du bid, c'est à dire :*

$$\textit{For } \mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})], \mathcal{A}_\mathfrak{b}(\mathsf{n}, x) := \{c \in [0, x], \mathfrak{b}^S(\mathsf{n}, c) = \mathfrak{b}\}$$

En utilisant le principe de programmation dynamique pour $k = K - 1$, nous obtenons : pour $\mathsf{n} \in \mathcal{N}$, pour $x \in \mathbb{N}$

$$v(K-1, \mathsf{n}, x) = \max_{q \in [0, \ldots, x \wedge \mathsf{n}^{|1}]} \{\mathfrak{g}(\mathsf{n}, q) + \mathbb{E}[g(\mathsf{N}_{k+1}, x - q) | \mathsf{N}_k = \mathsf{T}^S(\mathsf{n}, q)]\}$$

et nous définissons : pour $\mathsf{n} \in \mathcal{N}$, pour $x \in \mathbb{N}$, pour $q \in \{0, \ldots, x\}$

$$J(K-1, \mathsf{n}, x, q) := \mathfrak{g}(\mathsf{n}, q) + \mathbb{E}[g(\mathsf{N}_{k+1}, x - q) | \mathsf{N}_k = \mathsf{T}^S(\mathsf{n}, q)]$$

Le résultat principal de cette partie est le théorème suivant :

Theorem 0.0.6. *Pour tout $\mathsf{n} \in \mathcal{N}$, pour tout $q \in \mathcal{A}_\mathfrak{b}$ avec $\mathfrak{b}^S(\mathsf{n}, q)$, la fonction $J(K-1, \mathsf{n}, x, .)$ est croissante en q. Alors, pour tout $\mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})]$, pour tout $q \in \mathcal{A}_\mathfrak{b}$,*

$$\arg\max_{q \in \mathcal{A}_\mathfrak{b}} J(K-1, \mathsf{n}, x, q) = \mathsf{n}^B - 1$$

Concrètement, nous montrons que chaque terme est croissant en q. Par exemple, pour le terme lié à la vente, nous avons le lemme suivant :

Lemma 0.0.2. *Pour tout* $\mathsf{n} \in \mathcal{N}$*, pour tout* $q \in \{0, \dots, x\}$*, nous avons*

$$J^S(\mathsf{n}, x, q) := p_s \mathbb{E}[\mathfrak{g}(\mathsf{T}^S(\mathsf{T}^S(\mathsf{n}, q), \alpha^S), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)] = A(\mathsf{n}, x) - A(\mathsf{n}, q)$$

avec $A(\mathsf{n}, q) := \sum_{i \le \mathfrak{b}^S(\mathsf{n},q)} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1 - q) + \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - q) + \mathbb{E}[\alpha^S] \right]$
De plus, pour tout $\mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})]$*, pour tout* $q \in \mathcal{A}_{\mathfrak{b}}$*,* $J^S(\mathsf{n}, x, q)$ *est croissant en* q*.*

Pour tout $\mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})]$, pour tout $q \in \mathcal{A}_{\mathfrak{b}}$, $A(\mathsf{n}, q)$ est décroissant pour q. Nous calculons de la même manière J^B et J^C.

Le résultat nous montre que l'investisseur doit toujours vendre jusqu'à un certain niveau de prix B, la taille $\mathsf{n}^B - 1$ de telle manière à éviter le coût supplémentaire lié à un changement de prix. La stratégie repose finalement sur la détermination du niveau de prix B.

0.0.2.2 Liquidation optimale dans un carnet d'ordres à un seul côté avec une résilience sur le volume

Description du problème de liquidation optimale Un agent financier veut vendre $\bar{X}$ lots d'un actif peu liquide sur l'intervalle $[0, T] \subset \mathbb{R}$. Soit $(A_t)_{t \ge 0}$ le prix de référence de l'actif que nous considérons comme une $\mathbb{P}$-martingale continu. Dans notre modèle, en l'absence de trading, le volume disponible à l'instant t sur l'intervalle de prix $[A_t, A_t + x)$ est $F(x)$. F est une fonction croissante et continu à gauche associé à la mesure infinie μ sur $[0, +\infty)$ de la manière suivante :

$$F(x) := \mu([0, x)), \quad \text{pour tout } x \ge 0. \tag{7}$$

Nous donnons 2 exemples de forme de carnet d'ordres sur les figures 5 et 4.

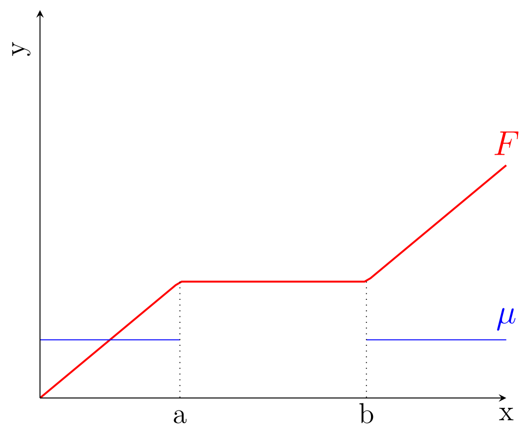


FIGURE 4 – Représentation d'un carnet d'ordres à volume constant possédant un gap entre entre les niveaux de prix a et b. En abscisse, nous représentons le prix et en ordonnée, le volume.

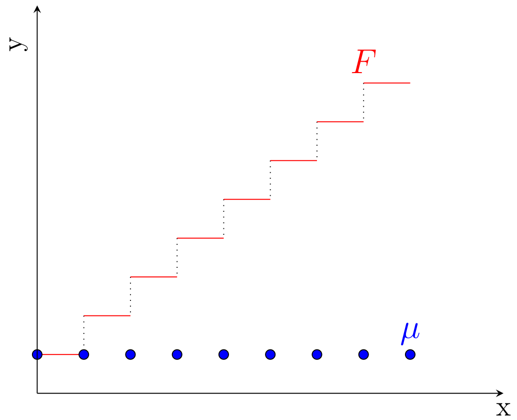


FIGURE 5 – Représentation d'un carnet d'ordres discret. En abscisse, nous représentons le prix et en ordonnée, le volume.

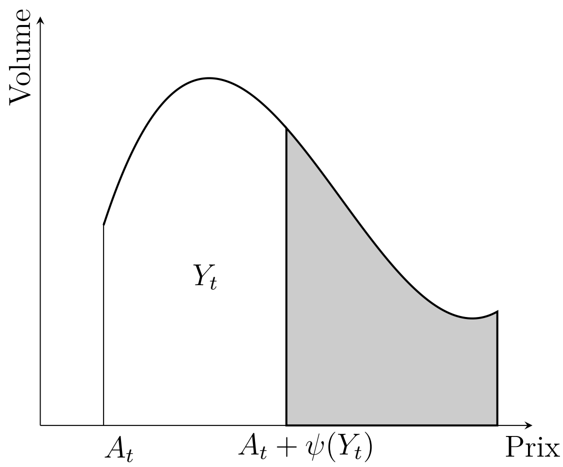


FIGURE 6 – Représentation de volume manquant (région blanche) et du volume disponible (région grise). La forme du carnet a pour référence A_t.

Les stratégies de l'agent. Les stratégies de l'agent sont données par des processus croissant continu à gauche et adapté $(X_t)_{0\leq t\leq T}$ avec $X_T = \overline{X}$. Nous supposons que $X_{0-} = 0$ et nous notons $\Delta X_t = X_t - X_{t-}$ le saut à l'instant t.

La dynamique du processus volume manquant (Y_t)**.** Nous supposons que la stratégie de l'investisseur a un impact sur le prix. Lorsque l'agent suit une stratégie X, nous supposons qu'à l'instant t, le prix de l'ask n'est plus le prix de référence mais est donnée par $A_t + D_t$ où $D_t := \psi(Y_t)$ (voir figure 6, avec Y_t représentant la dynamique du processus volume manquant définit par l'équation différentielle stochastique à sauts suivantes :

$$dY_t = dX_t - h(Y_{t^-})dt + \sigma(Y_{t^-})dW_t + \int_{\mathbb{R}} Y_{t^-} q(Y_{t^-}, z)\bar{M}(dt, dz); Y_{0-} = y. \quad (8)$$

la fonction réciproque de F continue à gauche ψ donnée par

$$\psi(y) := \sup\{a \geq 0 | F(a) < y\}, \;\; \text{pour } y > 0 \text{ et } \psi(0) := 0. \quad (9)$$

Coût de la stratégie. Nous définissons le coût de la stratégie $X = (X_t)_{0 \leq t \leq T}$ comme

$$\begin{aligned} C(X) := & \int_0^T (A_t + \check{D}_{t^-}) dX_t^c + \sum_{0 \leq t \leq T} [A_t \Delta X_t + (\Phi(Y_t) - \Phi(\check{Y}_{t-}))], \\ = & \int_0^T \psi(\check{Y}_{t-}) dX_t^c + \sum_{0 \leq t \leq T} (\Phi(Y_t) - \Phi(\check{Y}_{t-})) + \int_0^T A_t dX_t. \end{aligned}$$

Nous définissons la fonction valeur comme

$$v(t, x, y) := \inf_{X \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T \psi(\check{Y}_{s-}^{t,y,X}) dX_s^c + \sum_{t \leq s \leq T} (\Phi(Y_s^{t,y,X}) - \Phi(\check{Y}_{s-}^{t,y,X}))\right], \quad (10)$$

où $Y_s^{t,y,X}$ pour $t \leq s \leq T$ dénote la solution de (6.4) avec $Y_{t-}^{t,y,X} = y$ et l'ensemble des contrôles admissibles $\mathcal{A}(t, x)$ est donnée par :

$$\mathcal{A}(t, x) \triangleq \{X : X \nearrow;\; X_{t-} = x;\; X_T = \overline{X}\}. \quad (11)$$

La fonction valeur à la date terminale T est donnée par :

$$v(T, x, y) = \Phi(y + \overline{X} - x) - \Phi(y). \quad (12)$$

et la fonction valeur satisfait au bord

$$v(t, \overline{X}, y) = 0. \quad (13)$$

Résultats théoriques

Theorem 0.0.7. *La fonction valeur v est l'unique solution de viscosité continue sur $\mathcal{S}$ de l'inégalité variationnelle :*

$$\max\left(-\frac{\partial v}{\partial t} - \mathcal{L}v,\; -\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} - \psi\right) = 0, \quad (14)$$

satisfaisant les conditions de croissance suivantes :

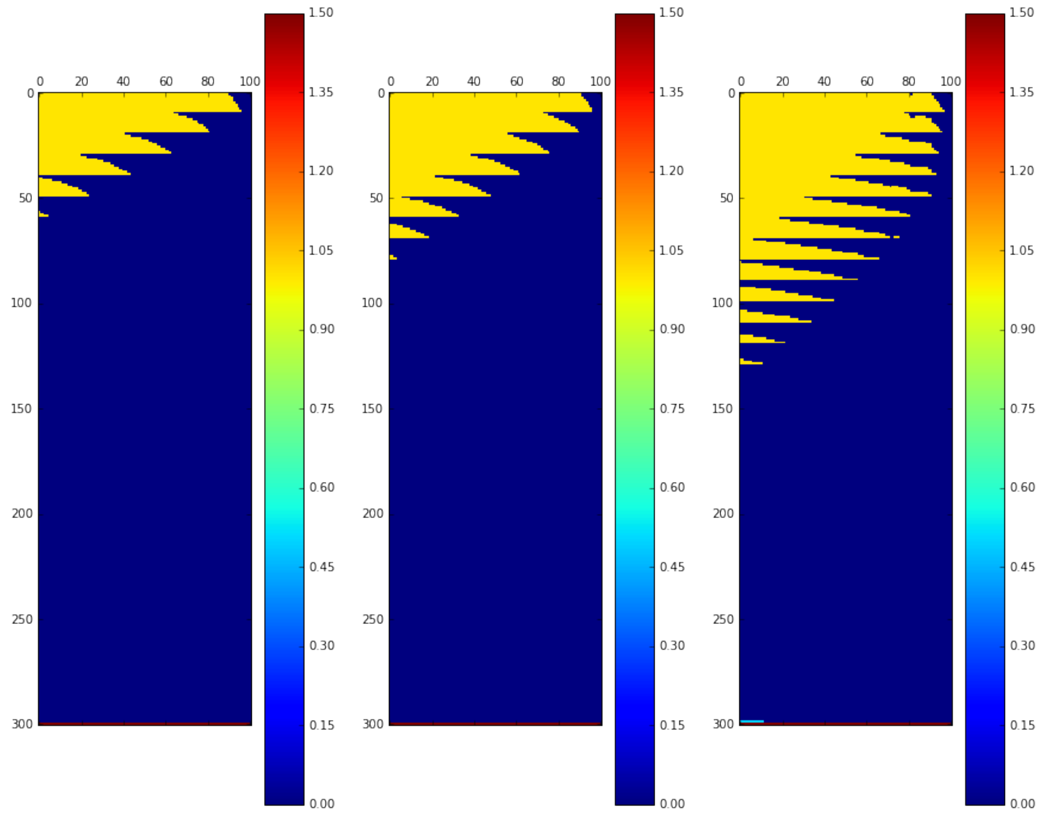
$$0 \leq v(t,x,y) \leq \Phi(y+\overline{X}-x)-\Phi(y) \ \ on\ [0,T)\times[0,\overline{X}]\times[0,+\infty), \quad (15)$$

et les conditions aux bords : $v(t,\overline{X},y)=0$ *and* $v(T,x,y)=\Phi(y+\overline{X}-x)-\Phi(y)$,

Résultats numériques Nous présentons quelques résultats numériques obtenus de (6.25). Nous avons implémenté une méthode de différences finies pour un problème de contrôle singulier pour une EDS à sauts basée sur [42].

Dans le cas du carnet d'ordre discret, la forme peut être déterminée par la fonction suivante :

$$\Phi(y)=\sum_{k=0}^{\infty}k(y-\frac{1}{2}k-\frac{1}{2})\mathbb{1}_{(k,k+1]}=\lfloor y\rfloor(y-\frac{1}{2}\lfloor y\rfloor-\frac{1}{2})$$



(a) t=0 (b) t=250 (c) t=450

FIGURE 7 – Il s'agit de l'évolution de la stratégie à adopter à différentes périodes si le carnet d'ordres est discret. En abscisse, nous représentons le nombre de lots à liquider. En ordonnée, nous représentons le nombre de volume manquant. Dans la région bleue, l'agent doit attendre. Dans la région jaune, l'agent doit acheter jusqu'à atteindre la région bleue. Le déplacement se fait suivant la direction $x=y$

Pour un carnet d'ordres discret, il y a des décalages de longueur de 1 ou plusieurs ticks entre deux prix consécutifs. Donc, dès que l'agent a consumé tout le volume à un certain prix, il y a un saut dans le coût d'achat de nouveaux lots. L'agent préfère attendre l'arrivée de nouveaux ordres limites vente pour remplir le décalage. Plus l'agent est proche de la maturité, moins elle doit attendre.

Organisation du document

Le manuscrit est organisé de la manière suivante :

— Le chapitre 1 vise à donner les concepts des marchés financiers auxquels nous faisons référence tout le long du mémoire. Nous présentons les différents faits stylisées et les critères de liquidité du marché. Puis nous donnons un panorama des modèles de carnet d'ordre en précisant les mérites et les manques. Nous finissons par une présentation des problèmes de liquidation optimale. Ce chapitre est rédigée en anglais.

— Dans l'amorce du chapitre 2, nous motivons l'intérêt de considérer un modèle de représentation par profondeur. Nous faisons face à la question représentation et unilatérale du carnet d'ordres. Nous sommes confrontés à la question de la modélisation du carnet d'ordres qui découle du problème initial de la liquidation optimale sous un point de vue à haute fréquence. Pour cela, nous considérons l'espace de la représentation et unilatérale du carnet d'ordres $\mathcal{N}$ et nous décrivons les opérateurs de transition induits par les événements. Ensuite, nous construisons la chaîne de Markov sur l'espace $\mathcal{N}$ à travers les opérateurs de transition après avoir spécifié le caractère aléatoire des événements. Nous spécifions et étudions l'influence des différents lois et des dépendances de l'état courant sur le prix et le volume du carnet. Nous simulons notre modèle. Ce chapitre est rédigée en anglais.

— Dans le chapitre 3, nous étudions l'irréductibilité, la récurrence et la transience de la chaîne de Markov que nous avons présenté au chapitre 2, à travers un outil mathématique original en utilisant une approche semi-martingale. Ce chapitre est rédigée en anglais.

— Le chapitre 4 est consacré au modèle d'estimation et la calibration à travers les données de marché. Nous considérons différentes arrivées d'événements dépendant de l'état et étudions leurs influences sur les faits stylisés et l'impact sur le marché. Nous présentons une calibration concrète et donnons une simulation pour différentes classes de modèles de stabilité du chapitre 3. Ce chapitre est rédigée en français.

— Après avoir étudié et corrigé le modèle, le chapitre 5 est consacré au problème de la liquidation optimale dans une modélisation LOB à haute fréquence que nous introduisons au chapitre 2. Ce chapitre est rédigée en anglais.

— Le chapitre 6 s'attaque à un problème de liquidation optimal en basse fréquence. Nous soulignons que ce chapitre est indépendant des autres. Nous introduisons le processus d'impact du volume manquant et nous définissons le problème optimal associé. Nous sommes amenés à étudier une équation de Hamilton Jacobi Bellman. Nous prouvons l'existence et l'unicité de la solution de viscosité. Enfin, nous proposons une approche numérique de la solution basée sur un schéma de différences finies et donnons une interprétation de la stratégie optimale. Ce chapitre est rédigée en anglais.

Chapitre 1

State of the art

Contents

This chapter aims at introducing financial market concepts and mathematical basic tools that will be applied along my thesis. One of our subject is the optimization of a financial asset trading program from a given position to zero in a given time which can be called optimal liquidation problem. Thereby, the question of financial asset dynamic modeling is primary and tackles at the same time the liquidity modeling. The liquidity can be study through different types of model : price model, volume model and limit order book model. Since we want to mimic as much as possible the financial reality, we validate the models by comparing the distribution of limit order book characteristics from the chosen model and the data.

Firstly, we give the financial context by introducing the organized markets and over the counter markets (OTC) by emphasizing their differences. Then, we explain the mechanism of a limit order book. We furnish financial interpretations on stylized facts and market liquidity criteria which involve the distribution of limit order book characteristics. We discuss the main existing limit order book modeling. At the end of this chapter, we present several classes of optimal liquidation problems.

1.1 Financial Market

1.1.1 Organized markets or over-the-counter markets

Financial markets are complex organizations with their own economic and institutional structures that play a critical role in determining how prices are determined. There are two basic ways to find and trade **financial assets** through a so-called over-the-counter (OTC) market or through an organized **markets** known as exchange. A market will refer to a set of financial assets and trading rules.

In OTC market, financial assets are non standardized. Most securities and derivatives involved in the financial crisis that began with a 2007 breakdown in the U.S. mortgage market were traded in OTC markets.

Exchanges, whether stock markets or derivatives exchanges, started as physical places where trading took place. Some of the best known include the Chicago Board of Trade (now part of the CME Group), which has been trading future contracts since 1851 or EUREX, which is the largest European futures and options market. An exchange centralizes bid and offer prices from market participants. Market participants can sell or buy at one of the quotes or reply with a different quote. When two parties find an agreement, the price at which the transaction is executed, is communicated throughout the market. The result is a level playing field that allows any market participant to buy as low or sell as high. The advances of electronic trading has eliminated the need for exchanges to be physical places. The London Stock Exchange and the NASDAQ Stock Market and EUREX are completely electronic. Some derivatives exchanges such as the CME Group, maintain both old-style pits and electronic trading.

OTC markets are less transparent and operate with fewer participants than exchanges. We only consider the organized markets for the rest of the thesis. Moreover, **traders** refer to direct market participants.

1.1.2 Mechanism of limit order book

According to their future believes of market price and their goals, traders can buy or sell financial assets through different orders.

For each financial asset, there is a **price grid** on which traders can place their orders. The smallest step on the grid between two consecutive **price levels** is the **tick value**. All price levels are multiple of the tick value.

A **limit order** is an order characterized by the **order sign** (buy or sell), the **order size** and the **price level**. A patient trader may have a belief on the future price of the assets and wants to buy a quantity at a specific price level. Thus, he will send a buy limit order with specific size at specific price level.

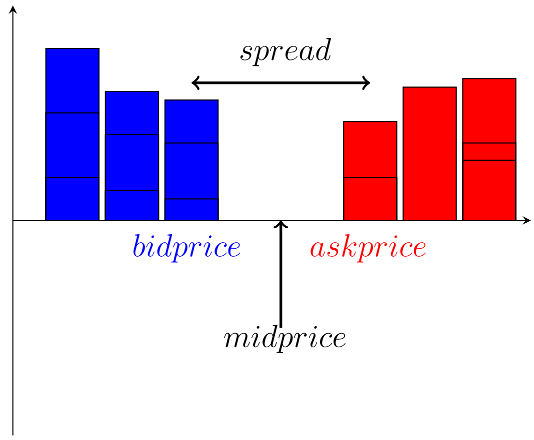


FIGURE 1.1 – Representation of a state of the limit order book. On the x-axis, we have the price levels and the volume is the second coordinate.

The **limit order book** is the set of all available limit orders for a particular financial asset. The limit order book contains each buy or sell limit orders for all price levels. For each price levels, there is a limit orders queue. For a specific price level, the limit order queue is the set of all available limit orders and the **volume** is the quantity of all available limit orders. The **total volume** is the quantity of all available limit orders for any price level. When a new limit order arrives at specific price level, the order is appended at the end of the limit order queue. The limit order book keeps the **sending timestamp** of any limit order received by the exchange.

The **bid price** is the highest price available for buy limit orders. The **ask price** is the lowest price available for sell limit orders. The bid price is always strictly smaller than the ask price. The **midprice** is the average between the bid price and the ask price. The **spread** is the difference between the ask price and the bid price. The difference is at least one tick, the smallest value between two consecutive prices. If the spread is strictly superior at one tick, trader can send buy (resp. sell) limit order inside the spread, it will increase the bid price (resp. decrease the ask price).

Depending on the market and trader subscription, the exchange will typically publish the volume for each p price levels (typically $p = 1, 5, 10, 20, \ldots$) deeper than the bid price or ask price.

A **cancel order** is an order characterized by the order sign (buy or sell), the order size and the price level. A trader can cancel his own existing limit order in the limit order book with a cancel order. Therefore, it will modify the limit order queue at this specific price level. Trader can modify their own existing limit order by increasing or decreasing the limit order size or modifying the price level. It is the same as considering simultaneous new limit order and new cancel order. When trader cancels their own limit order and the limit order is alone in the bid price queue (resp. the ask price queue), the bid price (resp. the ask price) decrease (resp. increase).

A **market order** is an order characterized by the order sign (buy or sell) and the order size. An impatient trader can have a strong belief of the future price of the particular financial asset and he wants to sell a quantity without specific price level. Thus, he will send a sell market order with the specific size.

A **trade** occurs when a market order is sent. A trade is characterized by the order size and the **last trade price**. The last trade price is the bid price (resp. ask price) if the order sign is sell (resp. buy) for the market order. The order size for sell (resp. buy) market order will be executed against the volume in the bid price (resp. ask price). If it is not enough, the remaining sell market order size will be executed against the volume in the following price levels until all sell (resp. buy) market order size is filled. It will decrease the bid price (resp. ask price). The last trade price is often the graphical representation of the "price". In the literature, "price" could refer to the bid price, the ask price, the last trade price or the midprice.

An **event** is a limit order sending, market order sending or cancel order sending. There are two different sources of information of the state of the market :

— All of trades (market order sending) are notified with a delay depending on multiple factors. A last trade price, a size and a timestamp are given.

— There is an update limit order book snapshot. Between, there could have been more than one event depending on multiple factors.

Multiple factors could be software location between the exchange, brokers, high activity. There can be a delay between the trade notification and the limit order book modification by the trade notification.

Limit order book mechanism explain at the lowest level the price formation.

Order Matching Algorithm

The pro-rata order matching algorithm fills orders according to price level, order size and time. A market order size is multiplied by each resting order's pro-rated percentage to calculate allocated trade size. All fills are rounded down to the nearest integer ; if an allocated trade size is less than two, it is rounded down to zero. An order's pro-rata percentage is calculated by taking order size divided by volume at a certain price level. Excess size, which occur as a result of the rounding down of the original allocated trade size, may be allocated with First-in, First-out order matching algorithm.

The First-in, First-out (**FIFO**) algorithm uses price and time as the only criteria for executing an order. The limit orders available in the ask price (or in the bid price) have execution priorities. At a specific price level, the limit orders with the oldest sending timestamp have execution properties. FIFO is often used (like EUREX).

We will consider FIFO order matching algorithm for the rest of the thesis.

1.2 Empirical observations in limit order book

In this section, we will discuss about statistical observations of the **characteristics** of a limit order book such as the price or the volume. Since one of our motivation is the limit order book modeling, we need to confront the distribution of characteristics from financial data and the the distribution of characteristics from simulated model in order to validate the model.

Since we don't have the theoretical quantities such as the distribution of price increments, we study the empirical quantities. In order to ensure that we can use different periods in the financial data, we need to check the stationary of the time series. A time series (X_n) has **stationary property** if the joint distribution of any subset of variables $X_1, X_2, \ldots, X_n$ is not affected by a shift in time. Moreover, if we want the convergence of the empirical quantity, we need to check the **ergodic property** of the time series. Given f, a measurable mapping of Ω into $\mathbb{R}$, define the sample averages

$$\langle f \rangle_n = \frac{1}{n} \sum_{i=1}^{n} f(X_i)$$

A time series will be said to have the ergodic property with respect to a measurable mapping f if the sample average $\langle f \rangle_n$ converges almost everywhere as $n \to \infty$ (Chapter 6 of [36]).

Many statistics exhibited **power law tail** [35] such as the distribution of price increments (Maslov [49], Cont [22]), the distribution of market order size (Maslov [49]) or the distribution of price level placement (Bouchaud [15], Gu [37]). A random variable X with f_X probability distribution have a power-law tail with exponent α if $f_X(x) \sim x^{-\alpha}$ as $x \to \infty$.

1.2.1 Stylized facts

Stylized fact is a term well known to physicists in the field of econophysics, the application of methods of physics to problems in economics and finance.

It was in 1961, the economist Nicholas Kaldor originally introduced the term "stylized facts" about theories of economic growth. He said that a theory should begin from a summary of the relevant and important facts which requiring explanation. It means the facts first and the theory after. Hence, Kaldor said that theorists should work from "a stylized view of the facts". They should "concentrate on broad tendencies, ignoring individual detail". Broad tendencies are facts deserving attention, and establishing such facts in finance and economics has been a primary achievement of econophysics. Many such facts have been established with appreciable precision only in the past $5-10$ years.

The first study of stylized fact dealt with the distribution of price increment were done by Benoit Mandelbrot [48], in 1963, with only a few thousand data points by proposing to extend Gaussian law by a family of stable laws for the last trade price increments. Last trade price data where the first to be available, stylized facts on price increments are more established than stylized facts on events information.

These stylized facts give restrictions on stochastic models who attempt to reproduce the financial reality. Good models should be to capture simultaneously most of them with few parameters. We will give the most studies stylized facts based on Cont [22] and Gould [35].

1.2.1.1 Stylized facts on price increments

Without any precision about the price, the price will refer to last trade price, bid price, ask price or mid price. The price increment is the difference between the price P evaluated at time t and at time $t + \Delta t$, i.e. $r(t, \Delta t) := P_{t+\Delta t} - P_t$ and Δt be a **time window** such as 100 milliseconds, 1 minute or 2 days.

Heavy tails of distribution of price increments The stylized fact is that distribution of price increments are not normally distributed. The distribution of price increments has a narrower central part and fatter tails than the normal distribution. This empirical regularity was noted already by Mandelbrot [48]. Since then, a whole range of different distributions have been suggested but there is no consensus on the exact form of the tails according to Chakraborti et al. [18]. Guillaume et al. [39] and Gopikrishnan et al. [34] used more than 200 million data points spanning half a century to establish a strong case for an inverse cubic-power law tail of stock-market distribution of price increments, at least asymptotically for large values corresponding to big market movements.

At different time window Δt, the unconditional distribution of price increments $r(t, \Delta t)$ exhibit a power law tail with an exponent $\alpha > 2$ that ensure a finite variance according most of studies such as Maslov [49] and Cont [22].

It exhibits different shapes at different time windows Δt and for Δt big enough it follow a Gaussian law called aggregational Gaussianity (Cont [22]).

By fixing the time window Δt, we explain the behavior of the distribution of price increments. Then, we want to study the time dependency of price increments. Thereby we define a long-memory process X if in the limit $\tau \to \infty$, $X(\tau) \sim \tau^{-\gamma} L(\tau)$ where $0 < \gamma < 1$ and $L(\tau)$ is a slowly varying function at infinity such as $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$, for all $t \in \mathbb{R}^+$. The degree of long-memory is given by the exponent γ, the smaller γ, the longer the memory. Long-memory is often discussed with the Hurst exponent H, $H = 1 - \gamma/2$. Short-memory processes have $H = 1/2$.

Absence of significant value in autocorrelation of price increments We define the autocorrelation of price increments,

$$C(s, \tau, \Delta t) := corr\left(r(s+\tau, \Delta t), r(s, \Delta t)\right).$$

The first and perhaps most obvious stylized fact of financial time series is that price increments are not significantly autocorrelated. Assuming that price increments were significantly autocorrelated, one could make correct predictions of price increments at least on average, which could be used by traders to make profitable investment decisions. When autocorrelations are exploited, market prices should equilibrate in such a way that the autocorrelations disappear.

This stylized fact was discovered by Fama [28] and often called "efficient market hypothesis". The efficient market hypothesis states that financial markets makes efficient use of all available information, thereby instantly incorporating any "hidden value" of an asset into its price. If the efficient market hypothesis is strictly true, there is no chance of making money : the future price increments of exchange-traded financial assets are always unpredictable.

Cont [22] claim that the autocorrelation of last trade price increments have small negative values, which it is due to the fact that the last trade price bounces between the bid-ask spread over a certain period of time [17].

The absence of autocorrelation does not seem to hold systematically when the time scale Δt is increased : weekly and monthly returns do exhibit some autocorrelation. However given that the sizes of data sets are inversely proportional to Δt, the statistical evidence is less conclusive and more variable from sample to sample.

Volatility clustering The absence of autocorrelation of price increments give a partial information of the behavior of the time dependency of price increments. It gives the absence of linear dependency. We should at least look at the non linear dependency by studying autocorrelation of square price increments absolute price increments. We define the autocorrelation of square price increments

$$C_2(s, \tau, \Delta t) := corr\left(r^2(s+\tau, \Delta t), r^2(s, \Delta t)\right)$$

or absolute price increments

$$C_0(s, \tau, \Delta t) := corr\left(|r(s+\tau, \Delta t)|, |r(s, \Delta t)|\right).$$

It is important because it says that price increments are not independently distributed. In 1963, Mandelbrot [48] wrote that : "large price changes are not isolated between periods of slow change" but "large changes tend to be followed by large changes of either sign and small changes tend to be followed by small changes". In other words, the autocorrelation of absolute or squared returns is positive.

This statistical fact has since been verified in many financial markets, and it has been quantitatively refined. According to Cont [22], several authors have remarked that the autocorrelation in squared returns decays like a power law, with a coefficient $\beta \in [0.2, 0.4]$. The quantitative results may be dependent on the time step Δt.

This is sometimes referred to as a long memory effect or volatility clustering. There are several possible explanations for volatility clustering, including the arrival of external news and the strategic splitting of orders by traders. The strategic splitting of orders can be explained by optimal liquidation problem (see section 1.4).

In a nutshell, the price increment of a time series should have a power law tail distribution, should not have autocorrelation and are not independently distributed.

1.2.1.2 Stylized facts on order flow

Long memory on order flow We discuss the statistical properties of order flow by considering the time series of order signs. Specifically, we consider the symbolic time series (ε_t) obtained in event time by replacing buy orders with $+1$ and sell orders with -1, irrespective of the order size. We can do it for market orders, limit orders or cancel orders, all of which show very similar behavior.

We reduce these series to ± 1 rather than analyzing the signed series of order sizes directly in order to avoid problems created by the large fluctuations in order size. Fluctuations in order size are heavy tailed and have long-memory themselves, so statistical averages based on them converge only slowly. The essential behavior is captured by the series of order signs.

We denote the autocorrelations of order signs

$$C_\varepsilon(s, \tau, \Delta t) := corr\left(\varepsilon(s+\tau, \Delta t), \varepsilon(s, \Delta t)\right)$$

This is sometimes interpreted as a sign of long-range dependence. In Lillo [47], the autocorrelation function decays roughly as a power law with an exponent of 0.6, corresponding to a Hurst exponent $H = 0.7$ observed in the London Stock Exchange. This implies that the signs of future orders are quite predictable from the signs of past orders. Bouchaud [16] measured in the Paris Stock Exchange, a larger interval of $\gamma \in [0.2, 0.7]$, $H \in (0.65, 0.9)$.

Long memory is also observed if the signs of all orders, including both limit and market orders, are taken together. In contrast, if we assign a negative sign for a buy cancel order, corresponding to the fact that price movements can be downward, then the combined sequence of signs for market orders, limit orders, and cancel order does not show long-memory.

1.2.2 Market liquidity criteria

Liquidity has many different aspects. Kyle [44] gives three criteria : tightness, depth, resilience. We focus on the **distribution of the volume at the best price**, **price impact**, **resilience** and depth represented by **mean average depth profile**.

Volume at the best price If we denote the volume available at the bid price $\mathfrak{v}$ at t and the distribution of $\mathfrak{v}$, $f_{\mathfrak{v}}$, Bouchaud [15] found that $f_{\mathfrak{v}}$ follow a Gamma distribution with $\gamma = 0.7$.

Price impact The notion of price impact is something that all traders ask when they want to know the effect of the execution of a large sell or buy orders. There is no clear consensus on the definition of price impact. Bouchaud [14] gives a good review on the question of the price impact. We follow the explanation between individual and aggregate price impact. We give the notion of transient and permanent price impact.

Individual price impact Individual price impact can be understand as the expectation price change after Δt induced by a single transaction q. We can define the price impact I_{ind} as

$$I_{ind}(\Delta t, q) := \mathbb{E}[r(t, \Delta t)|\delta_t = \text{ M }, q_t = q]$$

with $\delta_t = \text{ M }$ the event at time t which is a market event arrival and $q_t = q$, the market order size is q.

If we assume that we can decompose the individual price impact as $I_{ind}(\Delta t, q) := L(\Delta t)S(q)$, we can study the influence of Δt and q separately.

Instantaneous price impact S is the instantaneous price impact, it means the price change just after the transaction q. By fixing Δt, many studies found that S is concave in q. Lillo [47] found a power law with exponent 0.5 for small size and 0.2 for large size. For financial asset in London Stock Exchange, Farmer [29] found that a good fitting independently of q was an exponent 0.3. Weber [61] said that the concave shape of the function is very surprising : concave price impact would theoretically be an incentive to make large trades as they would be less costly than many small ones. In contrast, a convex price impact would encourage a trader to split large trade into several smaller ones, which is what actually happens. The concave observation can be explain at least by the selective liquidity taking [31]. Selective liquidity taking means that agents condition the size of their transactions on available volume in limit order book, making large transactions when available volume is high and small transactions when it is low. Concretely, we can write, for

Δt fixed,

$$R = \mathbb{E}[r(t)|\delta_t = \text{ M }, q_t = q] = \mathbb{P}[r(t) > 0|\delta_t = \text{ M }, q_t = q]\mathbb{E}[r(t)]$$

At this point, we need to prove that $\mathbb{P}[r(t) > 0|\delta_t = \text{ M }, q_t = q]$ is concave in q. In [14], they assume a correlation between the volume available at the bid price $\mathfrak{v}$ at t and the market order size q. The distribution of $\mathfrak{v}$ is $f_{\mathfrak{v}}$. If a trader wants to sell more that $\mathfrak{v}$, she will exactly sell $\mathfrak{v}$ and if want less than $\mathfrak{v}$, she will sell q. We can rewrite the conditional probability with the distribution of $\mathfrak{v}$

$$\mathbb{P}[r(t) > 0|\delta_t = \text{ M }, q_t = q] = \sum_{i=0}^{q} f_{\mathfrak{v}}(i)$$

Moreover, $f_{\mathfrak{v}}$ is decreasing in q (volume at the best price should follow a Gamma distribution) then $\mathbb{P}[r(t) > 0|\delta_t = \text{ M }, q_t = q]$ is concave.

After talking about the influence of q on price impact, we talk about the influence of Δt.

Lagged impact function : L is the lagged impact function. In order to explain the lagged impact function, we use a simple model of the price impact by trade q_k and a random shock which represent the price change due to limit and cancel order.

If $\mathbb{E}[r|\epsilon q] = \epsilon f(q)$, we define ν_k such as $\mathbb{E}[\nu_k] = 0$ and $\mathbb{E}[\nu_k^2] = \sigma^2$, a permanent price impact model is defined as

$$P_n = \sum_{k<n} \epsilon_k f(q_k, \mathsf{n}_k) + \sum_{k<n} \nu_k$$

Then :

$$L(\Delta t) = \mathbb{E}[\epsilon_n (P_{n+\Delta t} - P_n)] = \mathbb{E}[f]$$

In the case of $S(q) = Cq^{0.3}$ (Farmer et al. [29]), the individual price impact is $I_{ind}(\Delta t, q) = \mathbb{E}[f]Cq^{0.3}$.

And a transient price impact model is defined as, with $\alpha \in (0,1)$

$$P_n = \sum_{k<n} \alpha^{n-k-1} \epsilon_k f(q_k, \mathsf{n}_k) + \sum_{k<n} \nu_k$$

$$L(\Delta t) = \alpha^{\Delta t - 1}\mathbb{E}[f]$$

In this case, $I_{ind}(\Delta t, q) = C\alpha^{\Delta t - 1}\mathbb{E}[f]q^{0.3}$

In [30], Farmer et al. claims that price impact is permanent but depends on the state of the market but for Bouchaud et al., [16] price impact is state-independent, temporary and decays as a power law. Bouchaud et al. [14] answers that two models are equivalent in the case where the average impact function is history dependent by taking account the long memory of the order signs.

Aggregate price impact We define aggregate price impact. For a sequence of N successive transactions beginning at time t, let $Q_N = \sum_{i=1}^{N} \epsilon_{t+i} q_{t+i}$ be the aggregate volume and $R_N = \sum_{i=1}^{N} r_{t+i}$ be the aggregate price increment. The average price impact conditioned on volume is

$$R(Q, N) := \mathbb{E}[R_N | Q_N = Q]$$

i.e. it is the expected price increment associated with a signed volume fluctuation Q. We write $R(Q, N)$ to emphasize that this can depend both on the signed trading volume imbalance Q and the number of transactions N. Bouchaud et al. make this statements :

— the slope of the linear region decreases with N,
— the shape and the scale of the aggregate price impact change with the aggregation scale,
— at short time scales, the function is significantly non-linear and
— at large aggregation scales the market impact becomes close to linear, and the slope of the impact decays with the aggregation scale.

Bouchaud et al. conclude that we can't compare aggregate impact curves with different scales.

Taking account on the question of arbitrage, Gatheral [33], the expectation of trading costs should be positive in order to respect non dynamical arbitrage. They model the price dynamics such as

$$P_t = \sum_{i<t} S(q_i) G(t-i) + noise$$

with the price impact such as a transient price impact model with S the instantaneous price impact function and G the general decay kernel ($G(t) = \alpha t$ in Bouchaud [14]). Exponential decay kernel is only compatible with linear instantaneous price impact. In the case of S and G power law, they found a relation between the exponent of market impact and the exponent of decay.

Cont et al. [23] study the price impact of all events i.e. limit orders, market orders and cancel orders. They found a linear relation between price changes and bid-ask volume imbalance with a coefficient proportional to the inverse of the depth.

Resilience Kyle [44] propose a definition for resilience as "the speed with which prices recover from a random, uninformative shock". However, for Large [45] no shock is fully unexpected. He defines the resilience through the impact of aggressive market order (which change the best price) intensity of aggressive limit order (which change the best price) arrival. Concretely, he defines the intensity of a point processes at time t conditional on its natural filtration up to but not including time $s \leq t$, $\mathcal{F}_s$, as,

$$\lambda(t|\mathcal{F}_s) = \lim_{\delta t \to 0} \frac{\mathbb{P}[N(t+\delta t) > N(t)|\mathcal{F}_s]}{\delta t},$$

where $N(t)$ is the number of events to have occurred up to and including time t. If a shock of type m (aggressive market event) is time-invariant, denoted ϵ_m, happens at time s, then his impact on these intensities λ_r (limit aggressive event) at a later time t can be defined as the function G_{rm}

$$G_{rm}(t-s) = \lambda_r(t|\mathcal{F}_s, \epsilon_m) - \lambda_r(t|\mathcal{F}_s).$$

G_{rm} measures the resilience. Large [45] propose a log-likelihood estimation method for the intensities.

The concept of resilience is widely use in optimal liquidation problem for price impact model such as [8], [53], [21], [4] and for the shadow order book model [5], [7], [6], [3]

Mean relative depth profile We will define the mean relative depth profile or the average shape. We define the $d_t(i)$ the volume available at the level price $\mathfrak{b}_t - i$.

The mean relative depth profile is the time averaging volume for each level price $\mathfrak{b}_t - i$:

$$\bar{d}(i) = \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K} d_k(i).$$

Through different studies on a wide range of markets, the mean relative depth profile exhibit a hump shape. With "zero-intelligence" simple models, Maslov [50], Challet [19] and Bouchaud [15] reproduced this empirical observations. Ioane Muni Toke [60] build an explicit formula for the shape of a continuous price order book model. Even Weber [61] study the price impact through the mean relative depth profile.

In some optimal liquidation problem [7], [3], [56], [32], they use the order book shape in order to model the non linear price impact. In our knowledge, there are no studies who link the order book shape and the mean relative depth profile.

1.3 Challenges of modeling the dynamic of limit order book

What is a limit order book model ? We need to highlight the difference with a price or volume model.

Under price model, financial assets are driven directly by a price process and the price evolve without knowing the mechanism of order book directly. Black-Scholes model describe the price dynamic of a financial asset by a geometric Brownian motion.

Under volume model such as Alfonsi et al. [7], Predoiu et al. [56], financial assets are driven by a reference price process, a volume process and a fixed order book shape. The reference price process can be modeled by a semi-martingale process. The volume process is the remaining volume related to the fixed order book shape. The order book shape links the best price evolution and the remaining volume evolution.

Limit order book model describe the mechanism of limit order book and the price dynamic is a consequence of event arrival and the mechanism of limit order book. In this case, the model is richer for example but more complicated.

The first difficulty is the complexity of the space state. In price model, there is one dimension which is the price scale as opposed to LOB model, there are the price scale multiply by the volume scale.

Remark 1.3.1. *More precisely, in order to explain completely the mechanism, we should retain all limit orders with their size at each price.*

The second difficulty is the complexity of the transition between the market configurations induced by market orders and cancel orders which are mechanically state dependent. Since most financial assets with limit order book can always find counterpart for market orders, it is weakly state dependent for the size of market order. Since, we can cancel orders which are attached to a limit order available in the limit order book, we should retrieve the limit order associated. It leads to a model describe in 1.3.1. In the case where all event orders size are constant and the same for each event, we get around the problem. All previous models Maslov [49], Challet [19], Cont [24], Abergel [2] fall in this class of model. The third difficulty is related to the time structural of event properties such as the duration between 2 events.

1.3.1 Formalism

When we design a model with limit order book mechanism, we need to specify : How do we define the price level grid ? How are the main characteristics especially the best prices derived ? how do we model the event arrival randomness ?

The price level grid refers to the interval of the price and the reference price of the grid. For example, in Bak's model [9] the interval of the price is $[0, n] \subset \mathbb{N}$ and the reference price of the grid is 0. The main characteristics are the total volume and the best prices. For example, in Cont et al. [24], they can derive the best prices with the actual state of the limit order book. As opposed to the previous one, Abergel et al. [2] derive the best prices with stochastic differential equations. The most important part is the event arrival randomness. We need to specify timestamp, event type, price level and size randomness. In the case of the Poisson processes for each price level, as in Cont et al. [24], the duration between two consecutive events which determine timestamp randomness, are modeled by exponential law. The price level randomness is included in the fact that for each price level, there is a Poisson process. In the limit order book modeling literature, size is constant.

After specifying the three questions, we need to validate the model by studying the simulated model with respect to the stylized facts and market liquidity criteria. This validation express that the characteristics of the limit order book evolve in the good way with respect to the event arrival randomness. For example, we study the price increment distribution or the average shape of the limit order book.

We will explain for each model how they design and how they check the model effectiveness.

1.3.2 Classical limit order book models

As our knowledge, Bak et al.[9] is the pioneer of limit order book modeling.

Bak (1997) They do a simple model in order to study the stylized facts. The dimension is finite $n < \infty$. The price position is relative to 0.

The number of sell (and buy) orders available remain constant to N. The total volume remains constant N can be a good approximation for a limit order book at equilibrium.

At each time t,

— Step 1 : We choose uniformly one order in $\{1, \dots, N\}$.

— Step 2 : The chosen order, for example a sell order, at price $p(t)$ is canceled and submitted at a new price $p \pm 1$ with probabilities $\frac{1 \pm D}{2}$.

— Step 3 : If the new price of sell order takes place at the price $p(t)$ of existing buy order, there is a transaction at price $p(t)$.

— Step 4 : In order to keep the total volume constant, there are one sell order submitted uniformly at price between $\{p(t), \dots, n\}$ and one buy order submitted uniformly at price $\{1, \dots, p(t)\}$.

The simulated autocorrelation of last trade price increments exhibit a Hurst exponent $H = 1/4 < 1/2$ at long time scale.

By modifying the after transaction rule, the sell order is submitted at the range of price p where $N^p \neq 0$. This new rule gives some intelligence by assuming that traders mimic others traders. In this case, the simulated autocorrelation of last trade price increments exhibit a Hurst exponent for short time scale $H > 1/2$ and at long time scale $H = 1/2$ which is similar to a Brownian process. They found a heavy tail for last trade price increment distribution.

They found that giving state dependency to the limit order placement f^L_β gives better results on the last trade price increments at long time scale.

In this model, the drawbacks are :

- the non existence of market orders,
- the simultaneity of limit and cancel event,
- the constancy of total volume N,
- the non focus in market liquidity criteria and
- the constancy of event sizes.

Maslov models Maslov [49] propose a simple model which can be seen as a multidimensional Markov Chain. The dimension is finite with $n = 2 \times \varDelta$ dimensions with $\varDelta$ the support of the f^L_β. The price position is relative to the last trade price.

Traders can send (buy and sell) limit orders with a price placement probability distribution f^L_β.

With a simple model, they can exhibit 2 stylized facts such as fat tails of the distribution of last trade price increments, the autocorrelation function last trade price increments exhibit a Hurst exponent $H = 1/4$, the autocorrelation function of absolute last trade price increments exhibit a Hurst exponent $H = 3/4$ ($\gamma = 1/2$).

The drawbacks are :

- the non existence of market orders,
- the price position relativity to the last trade price,
- the non focus in market liquidity criteria,
- and the constancy of event sizes.

Challet and Stinchcombe models Challet and Stinchcombe [19] propose a model as an infinite-multidimensional Markov Chain with $n = \infty$ dimensions. The price position reference is relative to 0.

At each time t, traders choose with constant probability δ send (buy or sell with the same probability) limit orders with a price placement probability distribution which is centered Gaussian with variance $\sigma_X(t) = Kg(t) + C$ with $g(t)$ the spread and K and C some constant.

They introduce cancellation event, limit order can be cancel with constant probability ν and with a uniform cancel price. We made the remark that market orders are modeled by limit orders which cross the opposite best price. The model has no over-diffusive behavior at short-time but under-diffusive with $H = 1/4$ and tends to $H = 1/2$ at large-time.

It is clear that the over-diffusive behavior at short-time that can be seen in all market can be produced by the phenomena of mimic in limit order placement like the second model of Bak [9].

In this model, the drawbacks are the constant order size. They don't model market and they focus only on stylized facts and not in market liquidity criteria.

One year later, Challet and Stinchcombe [20] extend their model by testing different type of changes of rate and separate in the model the market (rate α) and the limit orders (rate δ). We will give some examples of type of changes in order to understand the underlying mechanism which give this behavior.

- At each time step, with probability p, all rates δ, α are random variable and are redrawn independently at random.
- Rates are defined with random variable ε who can takes values $\{-1, 1\}$ with probability p, $\alpha(t) = \alpha_0 + \alpha_1\varepsilon(t)$ and $\delta(t) = \delta_0 - \delta_1\varepsilon(t)$.

By randomizing the rates, they find that the last trade price has an over-diffusive behavior at short-time and a diffusive behavior at long time.

The drawbacks are :

- the constancy of event sizes and
- the non focus in market liquidity criteria.

Daniels, Smith, Farmer models In [26], [59], [29], they model the event order flow as a six dimensional Poisson processes (sign $\times$ event type) with :

- arrival of a new market order with intensity μ with constant size,
- arrival of a new limit order with intensity α with constant size and uniform price placement in the range of price $\mathfrak{b}(t) < p < \infty$,and similarly for bids on $-\infty < p < a(t)$,
- cancellation of an existing limit order in the limit order book with intensity δ with size 1.

The dimension is infinite. The price position reference is relative to 0. The price levels are logarithm then price can be negative.

By taking another direction, the mean-field theory, they study some characteristics such as the depth profile, the spread, the slope of depth profile and the price diffusion rate at short term and at long term. The drawback is the constancy of event sizes.

Preis 2006 Preis et al. [57] assume in their model that there are a fixed number of traders. There are N_A liquidity takers which send market orders modeled by Poisson processes with intensity μ. A fixed number of traders N_A of liquidity providers send limit orders which are modeled by Poisson processes with intensity α. Liquidity providers can cancel their orders which are modeled by Poisson processes with intensity ν.

- Price placement is simulated by uniform law and exponential law.
- When limit event happens, there is a probability $q_{provider}$ (resp. $1 - q_{provider}$) to have a buy (resp. sell) order.
- When market event happens, there is a probability q_{taker} (resp. $1 - q_{taker}$) to have a buy (resp. sell) order.

The event size is 1. They define the total volume recursively as :

$$N(t+1) = (N(t) + \alpha N_A) - (N(t) + \alpha N_A)\delta - \mu N_A$$

. At equilibrium, the total volume is : $\frac{N_{eq}}{N_A}\delta = \alpha(1-\delta) - \mu$. They show that a market trend i.e. an asymmetric order flow of any type, leads to a Hurst exponent $H < 1/2$ for small time, $H > 1/2$ for medium time and converge to $H = 1/2$ at large time for the price increment. Price increments don't exhibit fat-tailed distributions. They randomize q_{taker} as a random walk and modify their price placement by taking account the randomness of q_{taker}. In this case, they find the stylized fact of fat tails.

The drawbacks are :

- the constancy of event sizes and
- the non focus in market liquidity criteria.

Cont (2010**)** In Cont et al. [24], the authors model the limit order book as a continuous time Markov process $X_t := (X_1(t), X_2(t), ..., X_n(t))$, where $-X_p(t)$ (resp. $X_p(t)$) is the total sell (resp. buy) volume at price p, for $p \in \{1, \ldots, n\}$. For all $t > 0$, the bid and ask prices are defined :

$$b_t = \sup\{p, X_p(t) < 0\} \text{ and } a_t = \inf\{p, X_p(t) > 0\}$$

They assume that all orders are of unit size. The dimension is finite $(= 2n+2)$. The price position reference is relative to 0.

The intensity of limit orders at level p is $\lambda(p) := \frac{k}{p^\alpha}$ (follow a power law). These limit orders are canceled at rate $\theta(p) := \theta x_p$. Market orders arrive according to a Poisson process of intensity μ. Under these assumptions, the process (X_t) is Markovian and ergodic, and several quantities of interest such as transition probabilities for prices, distribution of duration are computed using Laplace transform techniques. The simulated average depth profile is almost the same as the data one.

The drawbacks are :

- the constancy of event sizes and

— the non focus in market liquidity criteria.

Abergel (2013) Abergel et al. [2] consider the dynamics of a general order book under the assumption of Poissonian arrival times for market orders, limit orders and cancellations. The dimension is finite ($= 2 \times 2K$). It follows a finite moving frame

They assume that each side of the order book is fully described by a finite number of limits K, ranging from 1 to K ticks away from the best available opposite quote. We will use the notation $X(t) := (a(t); b(t)) := (a_1(t), \dots, a_K(t); b_1(t), \dots, b_K(t))$, where $a := (a_1, \dots, a_K)$ designates the ask side of the order book and a_i the number of shares available i ticks away from the best opposite quote, and $b := (b_1, \dots, b_K)$ designates the bid side of the book. They adopt the representation of a finite moving frame. They assume that all orders are constant size q. They impose constant boundary conditions outside the moving frame of size $2K$. If the intensity of cancellation is state-dependent and non null, the continuous (resp. discrete) time process X is stable and converge to is stationary state at an exponential (resp. geometric) rate. If the intensity of cancellation is state-independent, the intensity of market and cancellation are superior to the intensity of limit order, they have the same conclusions. Empirically, they exhibit the first non negative lag and the fast decay for the autocorrelation of price increments. The simulated average depth profile is almost the same as the data one. By appending the last event $e(t)$ to the order book $(X(t))$, the time process $Y(t) := (X(t), e(t))$ is stable with the same condition as before. By rescaling and centering the price increment processes, they prove a limit theorem on a Brownian motion by the functional central limit theorem.

The drawbacks are :

— the constancy of event sizes,

— the non focus in market liquidity criteria and

— the independence of $(2 \times 2K)$ Poisson processes.

Huang [41] extend the model of Abergel [2] by adding a price reference and assuming the state dependencies of all intensities. In this model, under negative drift conditions and boundness of the market intensity, the Markov chain is ergodic. They study empirically the dependence between the state by taking the third best bid volume and the third best ask volume and the intensity.

1.4 Optimal liquidation

Optimal liquidation problem deals with the optimization of a trading program from a given position to zero in a given time interval $[0, T]$ with $T \in \mathbb{R}^+ \cup +\infty$. A trader takes the decision to buy or to sell a large number of shares, he needs to implement his decision on trading platforms. If he trades too fast, he will suffer from market impact and liquidity costs on his price. But on the other hand if he trades too slow, he will suffer from a large risk penalization, the adverse selection (the "fair price" will have time to change in the opposite way).

We need to optimize the trade schedule, how to split a large number of shares over smaller number of shares to be executed in a given time interval. It requires a cost function for immediate costs due to current trading decisions, transient cost and permanent cost i.e. impact of current decisions on future prices. We need to consider the different costs with respect to the relevant time scale. At long-time scale, we need to consider the permanent cost.

In order to solve the optimal liquidation problem, we need to model how the strategy X impacts the price dynamic P and the strategy cost C which depends to the price dynamic and the strategy. In the case of continuous time framework, the strategy X can be decomposed with a discrete part and a continuous part. In the case of optimal liquidation problem, a strategy is admissible if $X_0 = \bar{X} > 0$ and $X_T = 0$ which means that the initial shares are executed in the given time interval. Optimal liquidation problem consists of :

Finding an admissible strategy X which minimize $\mathbb{E}[C([0,T],P,X)]$

with $C([0,T],P,X)$ the total cost i.e. the strategy cost between $[0,T]$. In the discrete time framework, the instants are indexed by $\mathbb{N}$. In this case, the strategy X is merely the choice of the quantities $x_0,\dots,x_T$ with $\sum_{k=0}^{T} x_k = \bar{X}$.

1.4.1 Price impact model

We will begin by some basic model of optimal liquidation with market orders. In our knowledge, the pioneers were Bertsimas and Lo, in 1998 [13]. They model a linear permanent price impact. They established the dynamical programming principle and found in several cases, in a backward recursive way the optimal strategy. Almgren and Chriss [8] add a linear temporary impact. In this case, the strategy cost is quadratic function on the execution rate.

They build an efficient frontier for the optimal execution problem by minimizing the price impact and the variance. The phenomena of the resilience was added by Obizhaeva and Wang [53]. They established the HJB equation of the optimal execution problem and found an explicit value function quadratic on the remaining share and the resilience. Recently, Alfonsi and Blanc [4] model the market orders and cancellation orders by Hawkes Processes in order to capture the auto-excitation phenomena of market orders. They also found an explicit value function quadratic on the remaining share, the resilience and the intensity of Hawkes processes.

In [32], they model the linear permanent price impact with a time-dependent, deterministic depth and an exponential resilience of the book. They determine optimal portfolio liquidation strategies. In specific cases, they can state the optimal strategy in closed form. In [21], they model the linear permanent price impact with a stochastic depth (q_t) which follow a one dimensional finite Markov chain and a exponential resilience. The optimal order execution policy is solved by a Markov Chain Decision processes algorithm.

1.4.2 Volume impact model with limit order book shape

Under volume impact model, financial assets are driven by a reference price process, a volume process and a fixed order book shape. The reference price process can be modeled by a semi-martingale process. $(A_t)_{t\geq 0}$ be the reference price of a financial asset, which we assume to be a semi-martingale.

The volume process is the remaining volume related to the fixed order book shape. The limit order book shape is a limit theoretical limit order book shape mean reverting after market order. He is defined relative to a theoretical reference price. The order book shape links the best price evolution and the remaining volume evolution.

In [7], they introduce the notion of limit order book shape in order to introduce the dynamics of the limit order book at middle or low frequency for the optimal control liquidation problem. Let the limit order book shape characterized by the cumulative function F or μ.

In the absence of trading, the total volume at time t in the price interval $[A_t, A_t + x)$ is $F(x)$. F is a non-decreasing and left-continuous function associated to an infinite measure μ on $[0, +\infty)$ in the following way :

$$F(x) := \mu([0, x)), \quad \text{for all } x \geq 0. \tag{1.1}$$

In [7], the shadow order book density is price continuous. They found an explicit optimal strategy. In [56], the density can be price continuous, discontinuous or price discrete. They show the existence of two types of optimal strategy according to the convexity of a certain function depending on the density and the resilience.

1.4.3 Optimal execution with Market and Limit orders

Guilbaud [38] propose a framework for studying optimal market making policies with limit and marker orders in a bid-ask spread modeling. They want to manage inventory risk, adverse selection risk and execution risk. They model the cash, the number of shares. The randomness which represent the other market order is modeled by Cox processes. They formulate the control problem and they characterize in terms of quasi-variational system by dynamic programming methods. They use numerical approach based in finite scheme. They exhibit distinct regions for the choice of limit or market order according the spread and the number of shares. They found that it is better to send market orders when approach the maturity.

Jacquier [43] propose a framework to study the optimal market making policies with limit and marker orders in a Level 1 limit order book for large-tick stocks, with the spread equal to one tick and unit size for orders. They formulate the control problem as a semi Markov decision process. The numerical approach for the dynamic programming, conclude that order limit is preferred when far to the maturity, less latency, price decrease and volume imbalance.

1.4.4 Optimal execution in limit order book

Abergel [1] propose a framework for studying optimal market making policies with limit orders in the context of the limit order book. They characterize the problem with Markov Decision Processes theory. For the numerical approach, a high-dimension problem arise and they tackle the problem with a control randomization method and quantization method. When the intensity is state independent, the optimal strategy and the naive strategy (which consists to put limit orders in best prices) are equivalent. However in the case of state dependent, optimal strategy performs better than naive. They find allowing to put in the second best price performs better.

Première partie

Limit order book modeling with depth representation

Chapitre 2

Limit order book modeling

Contents

Introduction

This chapter is devoted to the theoretical foundations of our limit order book model.

Limit order book modeling consists to find a dynamical system which describe as much as possible the mechanism of limit order book (see the subsection 1.3.1 for explanation).

One of the reason we use the depth representation is the optimal liquidation problem in a limit order book modeling framework. One of a quantity we are concerned is the gain of a sell market order. The expression of the gain is easier to use in the case of depth representation.

For example, suppose the limit order book configuration state denoted s in the bid side is : the bid price b is 100 and the volume v_0 is 10, the second best price is 99 and the volume v_1 is 5, the thirst best price is 96 and the volume v_2 is 10 and there is no volume at other price.

LOB

b	v_0		v_1		v_2	...	...
100	10	99	5	98	10	...	0

FIGURE 2.1 – Array LOB represents a configuration state s for each price level, the volume associated

CLOB

b	n_0		n_1		n_2	...	...
100	10	99	15	98	25	...	25

FIGURE 2.2 – Array CLOB represents a configuration state s for each price level, the cumulative volume associated

The trader wants to sell with a market order with a size $q = 20$. The gain of this transaction is : $b \times (q - v_0) + (b - 1) \times (q - v_0 - v_1) + (b - 2) \times (q - v_0 - v_1) = 1985$. If we consider the depth representation of the same limit order book configuration state is : the bid price $\mathfrak{b}$ is 100 and the cumulative volume $n_0 = v_0$ is 10, the second best price is 99 and the cumulative volume $n_1 = v_1 + v_0$ is 15, the third best price is 98 and the cumulative volume $n_2 = v_2 + v_1 + v_0$ is 25 and the cumulative volume for other price is 25.

The gain of this transaction can be written in term of cumulative representation elements (n_i), $(n_0 + n_1 + n_2 + \ldots + n_{99}) - (n_0 - q)^+ - (n_1 - q)^+ - (n_2 - q)^+ - \ldots (n_{99} - q)^+$. The gain of a transaction with size q is the sum of the cumulative representation elements minus the sum of the cumulative representation elements after the transaction q. Moreover, the cumulative representation elements after the transaction q are just the positive part of the difference between the cumulative representation elements before the transaction q and the size q. For optimal liquidation problem involving shadow order book which is a density, they use the same consideration by taking the cumulative density in order to compute the gain of a transaction [56]. The gain operator is use in the chapter 5.

The second reason is the selective liquidity taking. According to [14], the price impact should be concave and one of the mean reason is the state dependency of the market order size to the actual limit order book state. A trader sell high when the liquidity is high and sell low when the liquidity is low [31]. The trader watches the first levels $K + 1$ of the state $v_0 + v_1 + \ldots + v_K$ which is n_K.

We relax the constant order size assumption. The first reason is the mechanism of the limit order book which allow any order size. Since one of our subject is the optimal liquidation problem, the trader should send any order size in order to increase the set of his strategy.

We emphasize that we model one side of the limit order book, the bid side. Thus, the reference price is 0 and the interval of the price is $\mathbb{N}$. The main characteristics are derived through the actual state of the limit order book.

We present in the section 2.1, the mathematical definition of the depth representation space $\mathcal{N}$, the characteristics such as the bid price, the different transition operators induced by sell order, buy order and cancel order. In the section 2.2, we define the Markov chain on $\mathcal{N}$ by constructing the transition probabilities. In the section 2.3, we discuss about the distribution law since most of microstructure statistical literature. Furthermore, the mechanism of limit order book gives constraint on cancel size and sell size, we talk about state dependence of the random variables. In the section 2.4, we give an algorithm for simulating trajectories of our model and we talk about the influence of the bid price and the total volume against the distribution law and the state dependence of the random variables.

2.1 Orderbook's representation

As explained in the introduction, only the bid side of an order book, that we continue to call by abuse of language order book, is studied hereafter. In this case, the state of an order book at a given time is completely determined by the buy order volume at each of the price levels, or equivalently by the buy depth at the different price levels, which constitutes a decreasing sequence of integer numbers, called the depth of market. We introduce the space $\mathcal{N}$:

$$\mathcal{N} = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{N}, a_i \geq a_{i+1} \text{ and for some index } k,\, a_k = 0\}.$$

The interpretation of the space $\mathcal{N}$ is clear. The index $k \in \mathbb{N}^*$ represents the different price level, while any element of $\mathcal{N}$ represents a particular situation of the order book in form of the depth of market : the difference $n_k - n_{k+1}$, for $k \in \mathbb{N}^*$, represents the volume at the price level k and, in particular, n_1 represent the total volume of the order book.

Lemma 2.1.1. *The space $\mathcal{N}$ is denumerable.*

Proof : The space $\mathcal{N}$ is clearly imbedded into a subset of $\cup_{k=1}^{\infty} \mathbb{N}^k$. □

The space $\mathcal{N}$ is very different from the usual infinite product space $(\mathbb{N}^*)^\infty$. To be distinct, we employ a different system to denote the coordinates : for $\mathsf{n} \in \mathcal{N}$, we denote by $\mathsf{n}^{|k}$ the value of n at the coordinate $k \in \mathbb{N}^*$.

2.1.1 The characteristics of an order book

The characteristics of order book are functions defined on the space $\mathcal{N}$. The most important characteristic of an order book is the bid price function $\mathfrak{b}$:

$$\mathfrak{b}(\mathsf{n}) = \max\{k \in \mathbb{N}^* : \mathsf{n}^{|k} > 0\} = \sum_{k=1}^{\infty} 1\!\!1_{\{\mathsf{n}^{|k} > \mathsf{n}^{|k+1}\}},$$

with $\max \emptyset = 0$. Associated with the bid price function $\mathfrak{b}$, we introduce the second best bid function :

$$\mathfrak{b}^{\circ}(\mathsf{n}) = \sum_{k<\mathfrak{b}(\mathsf{n})} 1\!\!1_{\{\mathsf{n}^{|k} > \mathsf{n}^{|k+1}\}}.$$

We define the support of order book :

$$\mathcal{S}(\mathsf{n}) = \{k \in \mathbb{N}^* : \mathsf{n}^{|k} > \mathsf{n}^{|k+1}\}, \ \mathsf{n} \in \mathcal{N},$$

and accordingly $\sigma(\mathsf{n}) = \sharp\mathcal{S}(\mathsf{n})$. Notice that $\mathfrak{b}(\mathsf{n}) = \max \mathcal{S}(\mathsf{n})$. We will also consider the scope of order book $\mathfrak{b}(\mathsf{n}) - \mathfrak{d}(\mathsf{n})$, where

$$\mathfrak{d}(\mathsf{n}) = \inf \mathcal{S}(\mathsf{n}) \ \text{ if } \mathcal{S}(\mathsf{n}) \neq \emptyset, \ \text{ and } \mathfrak{d}(\mathsf{n}) = 0 \ \text{ if } \mathcal{S}(\mathsf{n}) = \emptyset.$$

Notice that, contrary to the usual convention, here $\inf \emptyset = 0$.

We define the total volume of order book $\mathfrak{h}$:

$$\mathfrak{h}(\mathsf{n}) = \mathsf{n}^{|1}$$

The elements n of $\mathcal{N}$ are decreasing functions on $\mathbb{N}^*$, stopped at zero. For any real function f on $\mathbb{N}^*$, the integral

$$\textstyle\int_0^\infty f(s) d\mathsf{n}_s = \sum_{k=1}^\infty f(k) \Delta_k \mathsf{n}$$

is a well-defined function on $\mathcal{N}$, where

$$\Delta_k \mathsf{n} = \mathsf{n}^{|k} - \mathsf{n}^{|k+1}, \ k \in \mathbb{N}^*,$$

is the volume at the price level k. For example, if $f(k) = k$, the integral

$$\mathfrak{s}(\mathsf{n}) = \textstyle\int_0^\infty s d\mathsf{n}_s = \sum_{k=1}^\infty k \Delta_k \mathsf{n}$$

represents the total valuation of the order book.

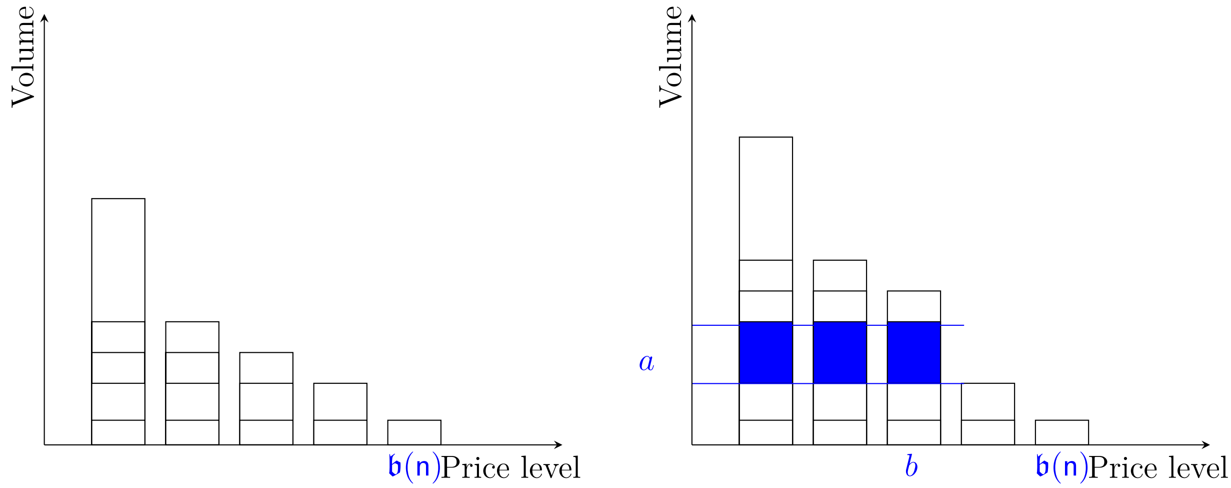


FIGURE 2.3 – A representation of the transition between two states induced by a buy event at the price level b of a size a

2.1.2 The basic transformations on an order book

When a buy order at the price level $b \in \mathbb{N}^*$ of size $a \in \mathbb{N}^*$ is added, the order book changes from $\mathsf{n} \in \mathcal{N}$ to a new state $\mathsf{T}^B(\mathsf{n}, a, b)$ (Figure 2.3) : for $k \in \mathbb{N}^*$,

$$\mathsf{T}^b(\mathsf{n}, a, b)^{|k} = \begin{cases} \mathsf{n}^{|k} + a, & k \leq b, \\ \\ \mathsf{n}^{|k}, & k > b. \end{cases}$$

We call T^b the buy operator on the space $\mathcal{N}$. We employ the expression

$$\mathsf{n} +^b a := \mathsf{T}^b(\mathsf{n}, a, b).$$

Similarly, we define the sell operator T^S :

$$\mathsf{T}^S(\mathsf{n}, a) = ((\mathsf{n}^{|1} - a)^+, (\mathsf{n}^{|2} - a)^+, (\mathsf{n}^{|3} - a)^+, \ldots),$$

$\mathsf{n} \in \mathcal{N}, a \in \mathbb{N}^*$, representing the modification of the order book after a market order of size a (Figure 2.4). Notice that one can want to sell with an arbitrary size a, but only up to $\mathsf{n}^{|1}$ quantity can be executed. We introduce the notation

$$\mathsf{n} -^* a := \mathsf{T}^S(\mathsf{n}, a).$$

Again, we define the canceling operator T^C : for $\mathsf{n} \in \mathcal{N}, a \in \mathbb{N}^*, b \in \mathbb{N}^*$, for

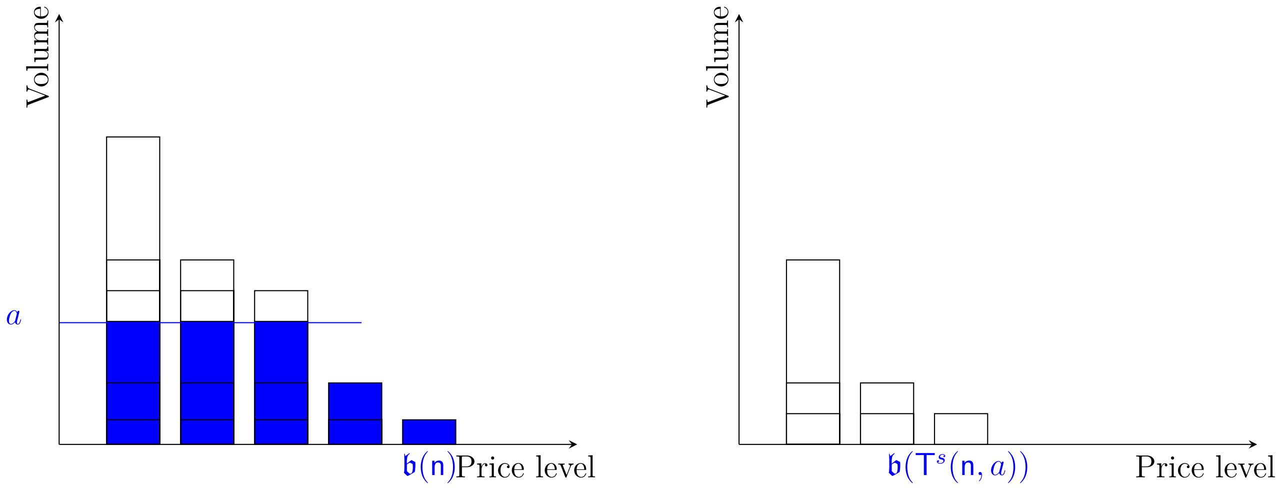


FIGURE 2.4 – A representation of the transition between two states induced by a sell event of a size a

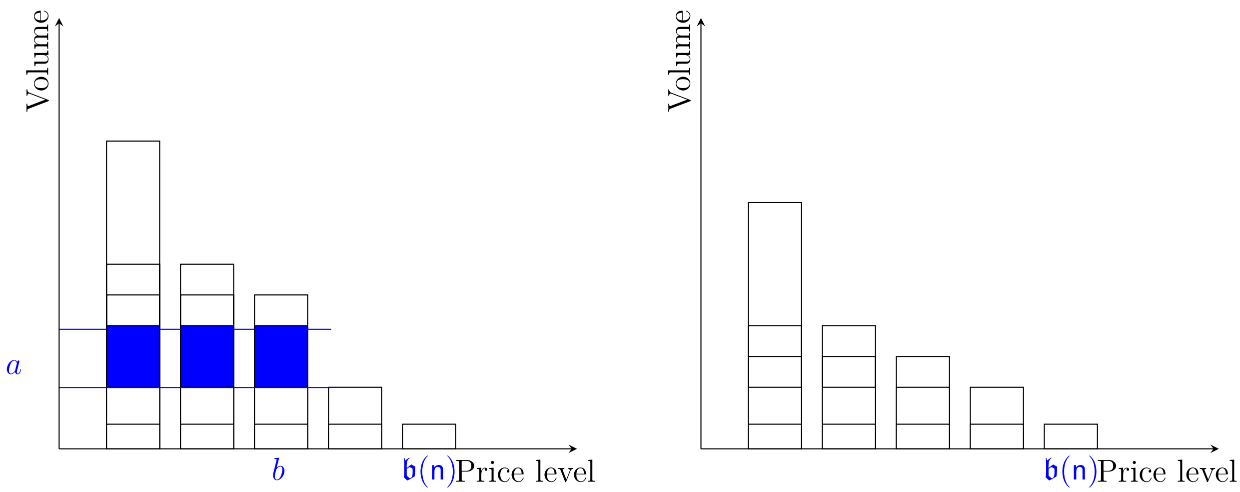


FIGURE 2.5 – A representation of the transition between two states induced by a cancel event at the price level b of a size a

$k \in \mathbb{N}^*$,

$$\mathsf{T}^C(\mathsf{n}, a, b)^{|k} = \begin{cases} \mathsf{n}^{|k} - a \wedge (\mathsf{n}^{|b} - \mathsf{n}^{|b+1}), & k \leq b, \\ \\ \mathsf{n}^{|k}, & k > b, \end{cases}$$

representing a cancellation order at the price level b of size a (Figure 2.5). We introduce the notation

$$\mathsf{n} -^b a := \mathsf{T}^C(\mathsf{n}, a, b).$$

Notice that one can only cancel existing orders. From the view of modeling, a cancel order can be formulated with an arbitrary size a, but only the quantity $\mathsf{n}^{|k} - a \wedge (\mathsf{n}^{|b} - \mathsf{n}^{|b+1})$ of orders is canceled.

2.2 The dynamic of the order book

An order book changes along with the arrivals of new orders.

2.2.1 The evolution of the order book

Assumption 2.2.1. *We suppose that the new orders arrive along a discrete timeline. At each point of the timeline, only one of the events can occur : nothing, sell order, buy order or cancel order.*

The points in timeline will be represented by the integers $\mathbb{N}$. At each point $n \in \mathbb{N}$ of the timeline, let δ_n be the event indicator taking values in the set of signs $\{N, S, B, C\}$ representing the four events. When an order arrive effectively at time n, we denote the order size by $\alpha_n^S, \alpha_n^B, \alpha_n^C \in \mathbb{N}^*$ corresponding respectively to a sell order, a buy order and a cancel order. The valuation levels of a buy order or a cancel order are denoted respectively by $\beta_n^B, \beta_n^C \in \mathbb{N}^*$. Recall that all sell orders considered in this thesis are market orders without limit of price level.

The arrivals of the new orders are random events. We fix from now on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{A}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure on $\mathcal{A}$. The quantities

$$\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C$$

are henceforward random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

The new orders modify the order book. Let $\mathbf{N}_0 \in \mathcal{N}$ be the initial state of the order book. We define recursively

$$\mathbf{N}_{n+1} = \begin{cases} \mathbf{N}_n, & \text{if } \delta_{n+1} = N, \\ \mathsf{T}^S(\mathbf{N}_n, \alpha_{n+1}^S), & \text{if } \delta_{n+1} = S, \\ \mathsf{T}^B(\mathbf{N}_n, \alpha_{n+1}^B, \beta_{n+1}^B), & \text{if } \delta_{n+1} = B, \\ \mathsf{T}^C(\mathbf{N}_n, \alpha_{n+1}^C, \beta_{n+1}^C), & \text{if } \delta_{n+1} = C, \end{cases} \tag{2.1}$$

for $n \in \mathbb{N}$.

The process $\mathbf{N} = (\mathbf{N}_n)_{n\in\mathbb{N}}$ represents the evolution of the order book over the time. Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ denote the natural filtration generated by the sequence of random variables $(\delta_n, \alpha_n^S, \alpha_n^B, \alpha_n^C, \beta_n^B, \beta_n^C)$ for $n \in \mathbb{N}$.

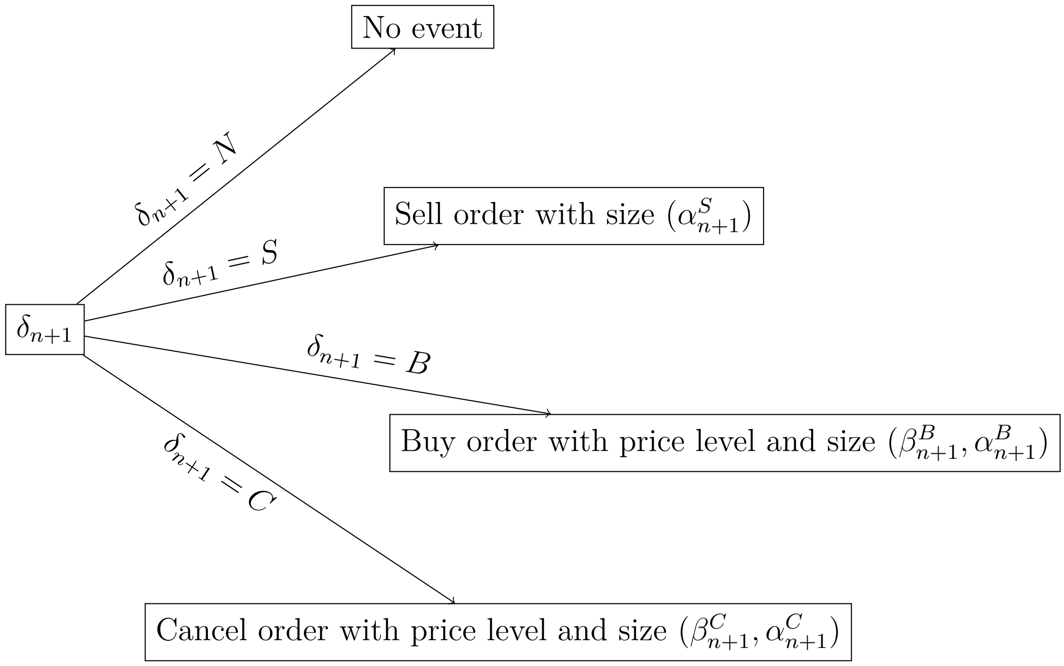


FIGURE 2.6 – At each instant n, we simulated the random variables δ_{n+1}, α^S_{n+1}, α^B_{n+1}, α^C_{n+1}, β^B_{n+1}, β^C_{n+1}

2.2.2 The randomness specification : a Markovian setting

After having settled down the formalism of the order book, the modeling consists in choosing the probability distribution of the family of the random variables δ_n, α^S_n, α^B_n, α^C_n, β^B_n, β^C_n for $n \in \mathbb{N}$. In this thesis, the evolution of the order book is simply modeled by a Markov chain.

More precisely, we suppose that there exists a probability measure kernel $\mathbb{Q}^{\mathsf{n}}[A]$ from $\mathcal{N}$ to $\{N, S, B, C\} \times (\mathbb{N}^*)^5$ (equipped with the discrete σ-algebra $\mathcal{D}$) such that

$$\begin{aligned} & \mathbb{P}[(\delta_{n+1}, \alpha^S_{n+1}, \alpha^B_{n+1}, \alpha^C_{n+1}, \beta^B_{n+1}, \beta^C_{n+1}) \in A \,|\mathcal{F}_n] \\ = \; & \mathbb{P}[(\delta_{n+1}, \alpha^S_{n+1}, \alpha^B_{n+1}, \alpha^C_{n+1}, \beta^B_{n+1}, \beta^C_{n+1}) \in A \,|\mathbf{N}_n] \\ = \; & \mathbb{Q}^{\mathbf{N}_n}[(\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C) \in A], \end{aligned}$$

for $A \in \mathcal{D}, n \in \mathbb{N}$, where $(\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C)$ denotes (in form of the coordinates) the identity map on the space $\{N, S, B, C\} \times (\mathbb{N}^*)^5$. Under this assumption, the process $\mathbf{N}$ is a Markov chain taking values in $\mathcal{N}$. As usual, we introduce the transition probabilities

$$\mathsf{P}(\mathsf{x}, \mathsf{y}) = \mathbb{P}[\mathbf{N}_{n+1} = \mathsf{y} \,|\mathbf{N}_n = \mathsf{x}], \;\; \mathsf{x}, \mathsf{y} \in \mathcal{N}.$$

Note that the above conditioning is independent of $n \geq 0$. We have

$$\begin{aligned} \mathsf{P}(\mathsf{x}, \mathsf{y}) \;\; := \;\; & \mathbb{Q}^{\mathsf{x}}[\delta = N] \mathbb{1}_{\{\mathsf{x}=\mathsf{y}\}} \\ & + \mathbb{Q}^{\mathsf{x}}[\delta = S, \mathsf{x} -^* \alpha^S = \mathsf{y}] + \mathbb{Q}^{\mathsf{x}}[\delta = B, \mathsf{x} +^{\beta^B} \alpha^B = \mathsf{y}] + \mathbb{Q}^{\mathsf{x}}[\delta = C, \mathsf{x} -^{\beta^C} \alpha^C = \mathsf{y}]. \end{aligned}$$

We introduce equally the Markov chain generator (also called drift when applied to a function) $\mathfrak{L}$: for any real function F on the space $\mathcal{N}$,

$$\mathfrak{L}F(\mathsf{x}) = \sum_{\mathsf{y}\in\mathcal{N}} \mathsf{P}(\mathsf{x},\mathsf{y})(F(\mathsf{y}) - F(\mathsf{x})),\ \mathsf{x}\in\mathcal{N}, \tag{2.2}$$

whenever the sum is absolutely convergent. We have

$$\begin{aligned}\mathfrak{L}F(\mathsf{x}) &= \mathbb{E}^{\mathsf{x}}[\mathbb{1}_{\{\delta=S\}}(F(\mathsf{x} -^{*} \alpha^S) - F(\mathsf{x}))] + \mathbb{E}^{\mathsf{x}}[\mathbb{1}_{\{\delta=B\}}(F(\mathsf{x} +^{\beta^B} \alpha^B) - F(\mathsf{x}))] \\ &\quad + \mathbb{E}^{\mathsf{x}}[\mathbb{1}_{\{\delta=C\}}(F(\mathsf{x} -^{\beta^C} \alpha^C) - F(\mathsf{x}))],\end{aligned}$$

where $\mathbb{E}^{\mathsf{x}}$ denote the expectation under the probability distribution $\mathbb{Q}^{\mathsf{x}}$.

In fact, we will not need to always work in the above general setting. Various complementary assumptions can be introduced in different situations for different discussions. Actually, we will suppose from now on the following assumption.

Assumption 2.2.2. *For every $\mathsf{x}\in\mathcal{N}$, under the probability measure $\mathbb{Q}^{\mathsf{x}}$, the random variable δ is independent of $\alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$.*

Let

$$p_N(\mathsf{x}) = \mathbb{Q}^{\mathsf{x}}[\delta = N],\ p_S(\mathsf{x}) = \mathbb{Q}^{\mathsf{x}}[\delta = S],\ p_B(\mathsf{x}) = \mathbb{Q}^{\mathsf{x}}[\delta = B],\ p_C(\mathsf{x}) = \mathbb{Q}^{\mathsf{x}}[\delta = C].$$

Under Assumption 2.2.2, the transition function $\mathsf{P}(\mathsf{x},\mathsf{y})$ and the generator $\mathfrak{L}$ take the following forms :

$$\begin{aligned}\mathsf{P}(\mathsf{x},\mathsf{y}) &= p_N(\mathsf{x})\mathbb{1}_{\{\mathsf{x}=\mathsf{y}\}} + p_S(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{x} -^{*} \alpha^S = \mathsf{y}] \\ &\quad + p_B(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{x} +^{\beta^B} \alpha^B = \mathsf{y}] + p_C(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\mathsf{x} -^{\beta^C} \alpha^C = \mathsf{y}],\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\mathfrak{L}F(\mathsf{x}) &= p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{x} -^{*} \alpha^S)] - F(\mathsf{x})) \\ &\quad + p_B(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{x} +^{\beta^B} \alpha^B)] - F(\mathsf{x})) + p_C(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[F(\mathsf{x} -^{\beta^C} \alpha^C)] - F(\mathsf{x})).\end{aligned} \tag{2.4}$$

2.3 More about the randomness specification

One must look the random variables $\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$ as parameters of the model introduced in Section 2.2. This section is devoted to the question how these parameters should be chosen. It is a second fundamental question, besides the definition of the model, on the order book modeling. We will demonstrate that the simple model defined in Section 2.2 is rich enough to possess the various important properties of an order book.

2.3.1 A review of the literature

There are many empirical studies in the literature on the order book. These studies reveal a number of features that characterize the order book. A good model is one that reproduces the same properties. This requirement conditions the choice of the random variables $\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$ in our model. This section is devoted to a discussion on the empirical properties of the order book and the issues. We emphasize that some studies come from statistical point of view and other from order book modeling. Since most of studies use Poisson processes, the random variable δ is the equivalent of the intensity of the Poisson processes.

In the subsection 1.2.1, we talk about the long memory of order signs. In this case, we need to add a time structure on orders flow. In order to mimic these stylized fact, the literature consider multivariate Hawkes processes. In order to keep a simple model, we don't add time structure on δ.

Specification of δ : arrival event time Since most study in limit order book modeling are Poisson processes (continuous timeline), the duration between two consecutive events with the same type, follows exponential law [24], [2].

In our model (discrete timeline), assuming that for all $\mathsf{x} \in \mathcal{N}$, $p_N(\mathsf{x}) := p_N$, $p_S(\mathsf{x}) := p_S$, $p_B(\mathsf{x}) := p_B$, $p_C(\mathsf{x}) := p_C$, the duration Δt between two consecutive events with the same type, follows a geometric law. In the case of two consecutive sell events, we have :

$$\mathbb{P}[\Delta t = j] = \mathbb{P}[\delta_{k+j} = S | \delta_{k+j-1} \neq S, \ldots, \delta_{k+1} \neq S, \delta_k = S] = (1 - p_S)^{j-1} p_S$$

Then, Poisson processes model are equivalent to our model since geometric law is the discrete version of exponential law.

Specification of α^B : limit order size In limit order book modeling, most study consider constant size [49], [59], [24], [2]. In Abergel et al. [2], they use log-normal distribution for the simulation.

However, statistical study find power law tail distributions for limit order size. Maslov [50] find a power law tail and the log-normal distribution can be a good fit. Bouchaud et al. [15] get a uniform law in log size.

In order to explain the influence of the limit order size distribution and the shape of the limit order book, Muni [60] consider for the discrete case a geometric law and the continuous case an exponential law.

Assumption 2.3.1 (Specifications of α^B)**.** *We will look for α^B with a log-normal distribution, binomial distribution or a geometric distribution.*

Specification of β^B : limit price placement Moreover, they use independent Poisson processes for each price level. In Cont [24], the intensity is a function of the distance between price level and Bid Price (or the Ask Price). However in Abergel [2], they consider Poisson processes with constant intensity.

In Smith [59], they use one Poisson process for limit order and they consider Uniform Law for order placement respect to the bid price and the ask price. In statistical studies, Bouchaud [15], [62], Gu [37] found a power law tail for the order placement. It can be understand as the optimistic belief that large price movement could happen.

But Bouchaud et al. [15], in the last part of his paper, propose an a priori distribution based on empirical data.

Most of reference on order book modeling use the opposite best price for the limit price level. As we consider bid part of the order book, we don't have the ask price. We will use the bid price as reference for the buy price level.

Since there is no constraint on the limit order size, we propose α^B is a r.v $\in \mathbb{N}^*$.

β^B is a bid price dependent random variable. The event $\{\beta^B \leq 0\}$ should be independent of the spread and the event $\{\beta^B > 0\}$ is constraint by the spread, we separate the event. Furthermore, the event $\{\beta^B > 0\}$ increase the bid price.

Assumption 2.3.2 (Specifications of β^B)**.** *We propose : $\epsilon^B \in \{0,1\}$ and $\beta^{B-} \in \{0,\dots,v \wedge (\mathfrak{b}(\mathsf{n})-1)\}$ and $\beta^{B+} \in \mathbb{N}^*$,*

$$\beta^B := \mathfrak{b}(\mathsf{n}) - \mathbb{1}_{\{\epsilon^B=0\}}\beta^{B-} + \mathbb{1}_{\{\epsilon^B=1\}}\beta^{B+}.$$

We will look for ϵ^B with a Bernoulli distribution and for (β^{B-},β^{B+}) a log-normal distribution, a binomial distribution, a geometric distribution.

The random variable ϵ^B explains the choice of increase the bid price. β^{B-} (resp. β^{B+}) gives the distribution of price placement inside the book (resp. inside the spread).

Specification of α^S : market order size In limit order book modeling, like limit order event, most study consider constant size ([49], [59], [24], [2]). In Abergel [2], they use log-normal distribution for the simulation.

However, statistical study find power law tail distributions for limit order size. Maslov [50] find a power law tail and the log-normal distribution can be a good fit. Bouchaud [15] get a uniform law in log size. Challet [19] find sizes are clustering on round amounts such as 10, 100, and 1000.

The previous studies are unconditional laws. As we said in the subsection about price impact, the selective liquidity taking is one of reasons about the concavity of the instantaneous price impact with respect to the market order size. In this case, we need to add some state dependencies on the size distribution. In this case, we need to consider random variable with finite support. For example, if we consider that random variable which represent market order size follow a uniform law and all traders consider the l first levels of depth when sending market orders then the support of the random variable is $[\![1, \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}]\!]$.

Assumption 2.3.3 (Specifications of α^S)**.** *We propose two specifications for α^S :*

- *In the case of state independence of α^S, α^S is a r.v. $\in \mathbb{N}^*$ and we will look for α^S with a log-normal distribution, a binomial distribution, a geometric distribution.*
- *In the case of state dependence of α^S, α^S is a r.v. $\in \{1, \ldots, 1 \wedge \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}\}$. We will look for α^S with a uniform distribution $\in \{1, \ldots, 1 \wedge \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}\}$ or a binomial distribution with parameters $(\mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}, p)$.*

Specification of β^C : cancel order placement Most of reference on order book modeling [24], [2] use the bid price as reference. So we will follow these idea. Then β^B is state dependent random variable.

Assumption 2.3.4 (Specifications of β^C)**.** *We propose β^{C-} is a r.v. $\in \{0, \ldots, v \wedge (\mathfrak{b}(\mathsf{n}) - 1)\}$,*

$$\beta^C(\mathsf{n}) := \mathfrak{b}(\mathsf{n}) - \beta^{C-}$$

with a log-normal distribution, a binomial distribution or a geometric distribution.

Specification of α^C : cancel order size In limit order book modeling, like limit order event, most study consider constant size [49], [59], [24], [2]. In [2], they use log-normal distribution for the simulation.

Most of studies don't study the cancel order size law, they studies instead the rate of cancellation since cancel event arrival are modeled by Poisson processes. Concretely, in the case of the rate of cancellation dependent of the state, Cont et al. [24] estimate the rate such as :

$$\hat{\lambda}(i) = \frac{N_c(i)}{TQ_i} S_c$$

with T : total trading time in the sample, $N_c(i)$: the number of times of cancel event at a distance of i to the bid price, S_c : the average size of cancel order and

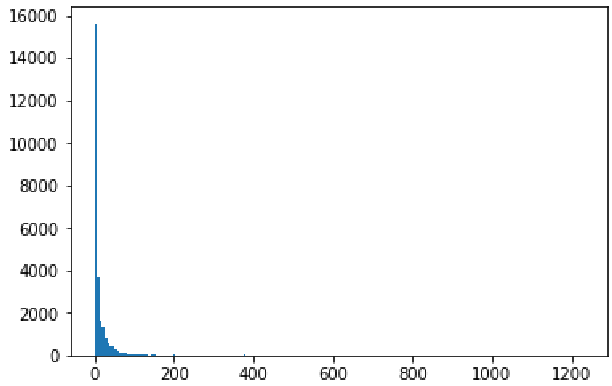


FIGURE 2.7 – Histogram of cancel size unconditionally to the state, the $x-$axis is the size and the $y-$axis is the occurrence in the data

Q_i : the average volume at a distance of i to the bid price. They need to estimate the average shape of the order book. Abergel et al. [2] used the same approach for the rate of cancellation estimation.

Bayer et al. [12] considered a stochastic model for the dynamics of limit order book. They describe the model as a continuous time models with state-dependent price dynamics. The cancel order sizes are described by a sequence of i.i.d. random variable $(\omega_i^C)_{i\in\mathbb{N}}$. The random variables (ω_i^C) take values in $[0, 1]$ and describe the proportions of cancellations. The change after a cancellation of buy volume is state-dependent, have $(\omega_i^C)v(t, x)$ at instant t and v is the volume at price x.

We want to assume a sequence of random variable (ξ_i^C) such as :

$$\alpha_i^C(\mathsf{n}) := \xi_i^C \Delta_{\mathfrak{b}}\mathsf{n}$$

At this point, we didn't mention the data we use in the chapter 4. But for this subparagraph, we use the data for the approach (we refer to the chapter 4 for more information on financial market data). With data, we draw histogram of cancel size unconditionally to the state and histogram of volume at the bid. In order to have the sample, we extracted all size $(\hat{\alpha^C})_i$ cancel event happen to the bid.

With the same data, we draw histogram of cancel size conditionally to the volume at the bid. In order to have the sample, we extracted all size $(\hat{\alpha^C})_i$ cancel event happen to the bid and all bid volume $\Delta_{\mathfrak{b}}\mathsf{x}$ just before the cancel and we compute the variable :

$$\xi_i^C := \frac{\hat{\alpha^C}}{\Delta_{\mathfrak{b}}\mathsf{x}}$$

We notice a non negligible frequency for $\xi_i^C = 1$. This value means that $\Delta_{\mathfrak{b}}\mathsf{x}$ is entirely canceled. It probably suggest that this value come from another variable.

From our analysis, the probability that $\Delta_{\beta^C}\mathsf{x}$ is entirely canceled should be an independent parameter from the other aspects of α^C.

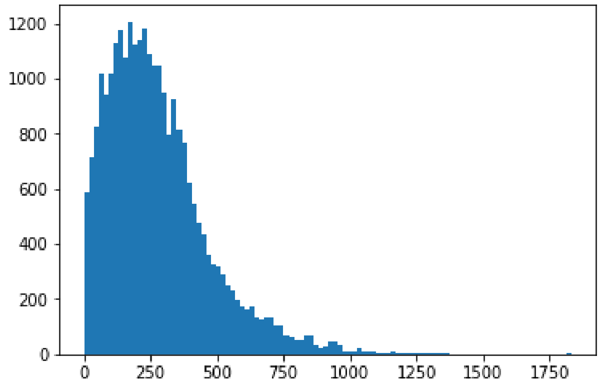


FIGURE 2.8 – Histogram of volume at the bid, the $x-$axis is the volume at the bid price and the $y-$axis is the occurrence in the data

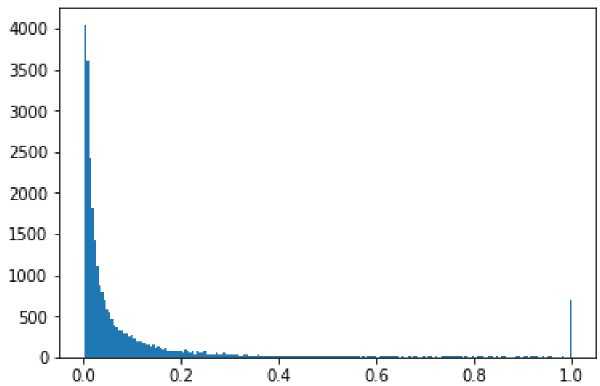


FIGURE 2.9 – Histogram of cancel size conditionally to the volume at the bid price, the $x-$axis is the ratio between size and the volume at the bid price, the y axis is the occurence in the data

Assumption 2.3.5 (Specifications of α^C)**.** *We propose two specifications for α^C :*

- *In the case of state independence of α^C, α^C is a r.v. $\in \mathbb{N}^*$ and we will look for α^C with a log-normal distribution, a binomial distribution or a geometric distribution.*
- *In the case of state dependence of α^C, suppose that, under $\mathbb{Q}^{\mathsf{x}}$, there exist independent r.v.'s $\epsilon^C \in \{0,1\}, \xi^c \in \{1,\ldots,1 \vee (\Delta_{\beta^C}\mathsf{x}-1)\}$, independent of β^C, such that α^C is decomposed as follows*

$$\alpha^C = \mathbb{1}_{\{\epsilon^C=1\}}\Delta_{\beta^C}\mathsf{x} + \mathbb{1}_{\{\epsilon^C=0\}}\xi^c$$

  We will look for ϵ^C with a Bernoulli distribution and for ξ^C an uniform distribution $\in \{1,\ldots,1 \vee (\Delta_{\beta^C}\mathsf{x}-1)\}$ or a binomial distribution with parameters $(\Delta_{\beta^C}\mathsf{x}-1, p)$.

To conclude the specifications, we look for level price and size (in the case of state independent) random variables with log-normal distribution, binomial distribution or geometric distribution. In the case of state dependent, we look for size random variables with uniform distribution or binomial distribution.

2.3.2 The homogenization

The following general result on discrete Markov chain, which links the asymptotic of a function on $\mathcal{N}$ with its drift, will be applied in the next chapter.

Lemma 2.3.1. *Let F be a function such that $\mathfrak{L}F$ is well-defined. Let*

$$X_n = (F(\mathbf{N}_n) - F(\mathbf{N}_{n-1}) - \mathfrak{L}F(\mathbf{N}_{n-1})),$$
$$e(\mathbf{N}_{n-1}) = \mathbb{E}[X_n^2 \,|\mathcal{F}_{n-1}], \quad v_m = \textstyle\sum_{k=1}^m e(\mathbf{N}_{k-1}) = \sum_{k=1}^m \mathbb{E}[X_k^2 \,|\mathcal{F}_{k-1}].$$

Suppose that the X_n are square integrable.

i. Suppose that there exists a constant $0 < c < \infty$ such that $e(\mathbf{N}_{n-1}) \geq c$ for all n. Then,

$$\liminf_{n\to\infty} \frac{F(\mathbf{N}_n)}{v_n} = \liminf_{n\to\infty} \frac{1}{v_n}\sum_{k=1}^n \mathfrak{L}F(\mathbf{N}_{k-1}),$$
$$\limsup_{n\to\infty} \frac{F(\mathbf{N}_n)}{v_n} = \limsup_{n\to\infty} \frac{1}{v_n}\sum_{k=1}^n \mathfrak{L}F(\mathbf{N}_{k-1}),$$

almost surely.

ii. Suppose that $\sum_{n=1}^\infty \frac{1}{n^2}\mathbb{E}[X_n^2 \,|\mathcal{F}_{n-1}] < \infty$. Then

$$\liminf_{n\to\infty} \frac{F(\mathbf{N}_n)}{n} = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \mathfrak{L}F(\mathbf{N}_{k-1}),$$
$$\limsup_{n\to\infty} \frac{F(\mathbf{N}_n)}{n} = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \mathfrak{L}F(\mathbf{N}_{k-1}),$$

almost surely.

Proof : Let

$$\mathbf{M} = \sum_{k=1}^{n} X_n,$$

which is a square integrable martingale, whose predictable quadratic variation is the process $(v_m)_{m\in\mathbb{N}}$ $(v_0 = 0)$. Under the conditions of the lemma, the strong law of large number theorem for martingales (cf. Hall and Heyde Theorem 2.18 [40]) is applicable. We have respectively

$$\text{either} \quad \frac{\mathbf{M}_n}{v_n} \to 0 \quad \text{or} \quad \frac{\mathbf{M}_n}{n} \to 0$$

almost surely. The lemma is proved, because $F(\mathbf{N}_n) = F(\mathbf{N}_0)+\mathbf{M}_n+\sum_{k=1}^{n} \mathfrak{L}F(\mathbf{N}_{n-1})$.
□

2.3.2.1 The basic idea

Lemma 2.3.1 is a useful result, because the drift sequence $(\mathfrak{L}F(\mathbf{N}_n) : n \in \mathbb{N})$ can be much less random than the initial sequence $(F(\mathbf{N}_n) : n \in \mathbb{N})$ is. The lemma transcribes a random data into a functional representation.

That said, the application of Lemma 2.3.1 depends on the computation of the drift function $\mathfrak{L}F(\mathsf{x})$, which may appear laborious. Actually, the space $\mathcal{N}$ is a new subject in the literature of order book. No general analysis theory exists yet for the space $\mathcal{N}$ and little is known with the three transformations $\mathsf{x} -^{*} a, \mathsf{x} -^{b} a, \mathsf{x} +^{b} a$. One may feel ineffective in dealing with the computation of $\mathfrak{L}F(\mathsf{x})$ and one may question the benefit of a model based on the space $\mathcal{N}$.

This section is devoted to a detailed discussion on that question. Clearly, the computation of a drift function in general can be ineffective. However, we notice that the drift functions are expectations under $\mathbb{Q}^{\mathsf{x}}$. The model defined in Section 2.2 possesses a so flexible system of parameters that, for a given function F, it is very likely to find a specification of the parameters $\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$ which makes the drift function $\mathfrak{L}F(\mathsf{x}) = c$ constant with respect to x, a case where the computation of $\mathfrak{L}F(\mathsf{x})$ becomes trivial.

The idea is therefore to leave out the general question, and instead, to ask the question for some specific functions F and to look for parameters δ,α^S,α^B, α^C, β^B, β^C which provide the drift functions $\mathfrak{L}F(\mathsf{x})$ with desired properties. Recall that the ultimate criterion of a good model is its relevance with respect to the applications in practice. The functions F are determined by the context and the search of the parameters respects the market reality.

A drift functions $\mathfrak{L}F(\mathsf{x})$ is decomposed into several parts, each of which is a function of x and has its own economical meaning. See (2.4). They should be considered separately. At the question of a simple and clear expression $\mathfrak{L}F(\mathsf{x})$, the idea is to have the aforementioned parts to become constant functions, whenever it is possible regarding to the economic meaning and the model specification. We call such an idea a homogenization.

The homogenization aims to reduce the erratic state, while to focus at the essential characteristics of the model. This idea has always existed in the literature of order book under various different forms. The best example is, instead of a direct modeling of the bid price $\mathfrak{b}(\mathbf{N}_n)$, to model it synthetically by a continuous Itô process. For other examples, we can mention the assumption of shadow book density, or that of a unit size for all new orders. Indeed, the point is to question the relevance of the homogenization with respect to the order book reality. In the case of our model, the homogenization inherits the economic interpretation of the functions F, and even, put some new insight on the order book.

To illustrate the idea, let us consider the drift function $\mathfrak{L}V(\mathsf{x})$ for the function V defined by

$$V(\mathsf{x}) = \mathsf{x}^{|1} + \mathfrak{b}(\mathsf{x}),\ \mathsf{x} \in \mathcal{N}.$$

This function V will be used in the proof of the recurrence of the Markov chain $\mathbf{N}$ in the next chapter, where one should estimate if the set $\{\mathsf{x} \in \mathcal{N} : \mathfrak{L}V(\mathsf{x}) > 0\}$ is a finite set.

2.3.2.2 An analysis of the homogeneity of $\mathfrak{L}V(\mathsf{x})$

We compute the drift $\mathfrak{L}V(\mathsf{x})$ with (2.4) and look at its dependence on the variable x.

$$\begin{aligned} \mathfrak{L}V(\mathsf{x}) = \quad & p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[(\mathsf{x}^{|1} - \alpha^S)^+ - \mathsf{x}^{|1}] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x} -^* \alpha^S) - \mathfrak{b}(\mathsf{x})] \ ) \\ + \quad & p_b(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\alpha^B] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] \ ) \\ + \quad & p_C(\mathsf{x})(-\mathbb{E}^{\mathsf{x}}[\alpha^C \wedge \Delta_{\beta^C}\mathsf{x}] \ + \mathbb{Q}^{\mathsf{x}}[\beta_c = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}^\circ(\mathsf{x}) - \mathfrak{b}(\mathsf{x})) \ ). \end{aligned} \tag{2.5}$$

The drift $\mathfrak{L}V(\mathsf{x})$ contains three terms : one of them is positive and the two others are negative. To simplify our discussion, let

$$\begin{aligned} \widetilde{\alpha^S} &= \mathsf{x}^{|1} - (\mathsf{x}^{|1} - \alpha^S)^+ = \alpha^S \wedge \mathsf{x}^{|1}, \\ \tilde{\alpha}^C &= \alpha^C \wedge \Delta_{\beta^C}\mathsf{x}, \end{aligned} \tag{2.6}$$

so that

$$\begin{aligned} \mathfrak{L}V(\mathsf{x}) \ = \ & -p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\widetilde{\alpha^S}] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] \ ) \\ & +p_b(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\alpha^B] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] \ ) \\ & -p_C(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] \ + \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \ ). \end{aligned} \tag{2.7}$$

A perfect homogeneity of $\mathfrak{L}V(\mathsf{x})$ would be the case where $\mathfrak{L}V(\mathsf{x})$ is a constant function on $\mathcal{N}$ outside of a finite set. This can be possible from the mathematical viewpoint. But the corresponding parameterization may correspond to no economical reality. We now analyze each of the terms in (2.7) to find reasonable setting of $\delta, \alpha^S, \alpha^B, \alpha^C, \beta^B, \beta^C$ to make $\mathfrak{L}V(\mathsf{x})$ homogeneous.

i. We begin with the easiest one : If the random variable α^B has the same law under $\mathbb{Q}^{\mathsf{x}}$ for any x, which is reasonable setting, the term $\mathbb{E}^{\mathsf{x}}[\alpha^B]$ will be independent of x.

ii. For the term $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})]$, we can suppose, by definition, that β^B takes the form :
$$\beta^B = \mathfrak{b}(\mathsf{x}) - (1 - \epsilon^B)\beta^{B-} + \epsilon^B \beta^{B+},$$
where the random variables $\beta^{B-} \in [0, \mathfrak{b}(\mathsf{x})], \beta^{B+} \in \mathbb{N}^*$ are independent of the random variable $\epsilon \in \{0, 1\}$ under $\mathbb{Q}^{\mathsf{x}}$. Then,
$$\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] = \mathbb{Q}^{\mathsf{x}}[\epsilon = 1]\mathbb{E}^{\mathsf{x}}[\beta^{B+}].$$
Hence, if the random variables ϵ^B, β^{B+} have the same law under $\mathbb{Q}^{\mathsf{x}}$ for any x, the term $\mathbb{Q}^{\mathsf{x}}[\epsilon^B = 1]\mathbb{E}^{\mathsf{x}}[\beta^{B+}]$ will be independent of x.

iii. If α^S is a bounded random variable (which is a natural assumption with respect to the market reality), for $\mathsf{x}^{|1}$ big enough, $\mathbb{E}^{\mathsf{x}}[\widetilde{\alpha^S}] = \mathbb{E}^{\mathsf{x}}[\alpha^S]$, which is a constant if the probability distribution of α^S is defined independent of x.

iv. It is not illegitimate, with respect to the market reality, to define the quantity
$$\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x}))$$
independent of x, whenever $\mathsf{x} \neq 0$. In fact, the probability $\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}]$ is the probability of the total cancellation of the bid, for which empirical studies exist, and it depend naturally the gap $(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x}))$.

v. Clearly, each of the above quantities
$$\mathbb{E}^{\mathsf{x}}[\alpha^S],\ \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x})),$$
etc., constitutes a statistic test that can be calibrated from the market. Notice that the gap $(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x}))$ is mostly equal to one unit in a liquid market.

vi. However, whatever we do, the terms such as $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)]$ or $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$ depend always on x. For example, $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$ can vary from 1 to infinity, according to x.

Hence, to make the homogeneity, we should adjust the probabilities $p_S(\mathsf{x})$, $p_B(\mathsf{x})$, $p_C(\mathsf{x})$ so that, for example,
$$-p_C(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] \ + \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x}))\ )$$
becomes constant for all but a finite number of x. Such adjustment is possible under reasonable setting. Actually, as $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] \geq 1, \forall \mathsf{x} \neq 0$, for any negative value $-1 \leq a \leq 0$, for any $\mathsf{x} \neq 0$, we can find $p_S(\mathsf{x})$, $p_B(\mathsf{x})$, $p_C(\mathsf{x})$ to make $\mathfrak{L}V(\mathsf{x}) = a$. The idea of such an adjustment is not completely absurd, because traders, who have viewed the market configuration x, can modify their strategies, that leads to the modification of the probabilities $p_S(\mathsf{x})$, $p_B(\mathsf{x})$, $p_C(\mathsf{x})$.

Look at the value $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)]$ in particular. In an absolute sense, this value can be very big. But, the big value happens only when the market has a singular configuration (for example, a market with a very small total volume $\mathsf{x}^{|1}$ and a big gap $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})$). The following result gives some ideas about the size of $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)]$.

Proposition 2.3.1. *If there exists a $m > 1$ such that $\mathbb{E}^{\mathsf{x}}[(\alpha^S)^m]$ is bounded by a constant $c > 0$, then, for any positive real m' such that $\frac{m}{m'} > 1$, the quantity $\mathbb{E}^{\mathsf{x}}[(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S))^{m'}]$ is bounded on $\mathcal{N}$.*

Proof : Actually,

$$\begin{aligned} & \mathbb{E}^{\mathsf{x}}[(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S))^{m'}] \\ = \; & \mathbb{E}^{\mathsf{x}}[(\textstyle\sum_{i=1}^{\mathfrak{b}(\mathsf{x})} 1\!\!1_{\{\alpha^S \geq \mathsf{x}^{|i}\}})^{m'}] \leq \mathbb{E}^{\mathsf{x}}[(\alpha^S)^m](\textstyle\sum_{i=1}^{\mathfrak{b}(\mathsf{x})} \frac{1}{(\mathsf{x}^{|i})^{m/m'}})^{m'} \leq c \textstyle\sum_{i=1}^{\infty} \frac{1}{i^{m/m'}}. \end{aligned}$$

□

2.3.3 Statistics and calibrations

Before a thorough study of calibration in a later chapter, we make a quick discussion about the model statistics. Under suitable specification, the law of large number theorem holds for the model, which lays the theoretical basis of the statistics. Consider the quantity $\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)]$, which represents the prediction of the price impact after a sell order α^S.

Proposition 2.3.2. *Suppose the above homogeneity assumption on β^B in Section 2.3.2.2. Suppose the assumption on α^S in Proposition 2.3.1. Suppose that $\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x}))$ is constant for all but a finite number of x. Suppose constant probabilities $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$. Then, the average value $\frac{1}{n}\sum_{j=1}^n \mathbb{E}^{\mathbf{N}_j}[\mathfrak{b}(\mathbf{N}_j) - \mathfrak{b}(\mathbf{N}_j -^* \alpha^S)]$ converges, as soon as $\frac{\mathfrak{b}(\mathbf{N}_n)}{n}$ converges.*

Proof : We give only a sketch of the proof. In the case of a positive recurrent Markov chain $\mathbf{N}$ the result is immediate. Consider the case of a non positive recurrent Markov chain. We can then apply Lemma 2.3.1 to link the limit of $\frac{\mathfrak{b}(\mathbf{N}_n)}{n}$ to that of the average of the drift sequence. The drift part of $\mathfrak{b}(\mathbf{N}_n)$ is divided into three parts : the one that we consider in this proposition, the one corresponding to "buy", and the one corresponding to "cancellation". Clearly the "buy" part converges correctly. The ergodic theorem implies that the occupation time at each x is negligible face to $n \uparrow \infty$. Hence, the "cancellation" part converge also. □

Notice that, in a liquid market, the average value $\frac{1}{n}\sum_{i=1}^n \mathbb{E}^{\mathbf{N}_i}[\mathfrak{b}(\mathbf{N}_i) - \mathfrak{b}(\mathbf{N}_i -^* \alpha^S)]$ is expected to be very small. With the same consideration, the gap $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})$ is very often equal to 1 in a liquid market.

Now, consider another quantity $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$. This quantity can not be homogeneous in general. Let us compute it under concrete specifications. In the case where α^C is uniform on $\{1,\ldots,\Delta_{\beta^C}\mathsf{x}\}$, we have

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \mathbb{E}^{\mathsf{x}}[\frac{1+\Delta_{\beta^C}\mathsf{x}}{2}].$$

If moreover β^C is proportional to $\Delta\mathsf{x}$, we have

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \frac{1}{2} + \frac{1}{2\mathsf{x}|^1}\sum_{i=1}^{\infty}(\Delta_i\mathsf{x})^2.$$

In the case where β^C follows the geometric distribution(p) from the level 1 to the infinity, we have $(q = 1-p)$

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \frac{1}{2} + \frac{1}{2}p\sum_{i=0}^{\infty}q^i\Delta_{i+1}\mathsf{x}.$$

In the case where, instead, β^C follows the geometric distribution(p) from the level $\mathfrak{d}(\mathsf{x})$ to the infinity, we have

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \frac{1}{2} + \frac{1}{2}p\sum_{i=0}^{\infty}q^i\Delta_{i+\mathfrak{d}(\mathsf{x})}\mathsf{x}.$$

Or in the case where β^C follows the geometric distribution(p) backwardly from the level $\mathfrak{b}(\mathsf{x})$ to the $-\infty$, we have

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \frac{1}{2} + \frac{1}{2}p\sum_{i=0}^{\mathfrak{b}(\mathsf{x})-1}q^i\Delta_{\mathfrak{b}(\mathsf{x})-i}\mathsf{x}.$$

In any case, $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$ is a function of the $\Delta_i\mathsf{x}$ which is too erratic to be homogenized. That said, in spite of the non homogeneity, the parameter calibration about α^C remains possible. As an example, suppose that the bid price tends to the infinity and β^C follows the geometric distribution(p) from the level $\mathfrak{d}(\mathsf{x})$ to the infinity. Then, we can write an informal identity

$$\lim_{n\to\infty}\tfrac{1}{n}\sum_{j=1}^{n}\mathbb{E}^{\mathbf{N}_j}[\tilde{\alpha}^C] = \tfrac{1}{2} + \tfrac{1}{2}p\sum_{i=0}^{\infty}q^i\lim_{n\to\infty}\tfrac{1}{n}\sum_{j=1}^{n}\Delta_{i+\mathfrak{d}(\mathbf{N}_j)}\mathbf{N}_j.$$

This formula links the statistic on α^C to the statistic $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\Delta_{i+\mathfrak{d}(\mathbf{N}_j)}\mathbf{N}_j$, which has been examined in the literature and has been proved to exhibit a simple distribution profile.

To end this section, notice that, instead of considering $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$, we can consider the product value

$$a_{pc} := p_C(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] \tag{2.8}$$

which may be reasonably homogeneous. Hence, the a_{pc} becomes an autonomous parameter corresponding to the average cancellation : average in the expected volume and also average in the time.

2.4 Simulations

We propose the algorithm 1 based on the subsection 2.3.1. For the specification of each random variable, we invite to read for α^S the assumption 2.3.3, α^C the assumption 2.3.5, β^C the assumption 2.3.4, α^B the assumption 2.3.1 and β^B the assumption 2.3.2. We emphasize that this section is only dedicated to the simulation. The law parameters are arbitrarily chosen. We refer to the chapter 4 for the calibration and the estimation.

We begin by initialize the depth n, the maximal price and the set of law parameters $\mathbf{p} := (\mathbf{p}^E, \mathbf{p}^S, \mathbf{p}^{PC}, \mathbf{p}^{VC}, \mathbf{p}^{PB}, \mathbf{p}^{VB})$.

— Time step $N = 100000$

— Maximal price $K = 400$

— Initial price $P = 200$

— Depth $\Delta_k \mathsf{n} = 300$ for $k \in \{1, \dots, 200\}$

We simulate the random variables $\{\delta, \alpha^S, \alpha^C, \epsilon^C, \beta^C, \alpha^B, \epsilon^B, \beta^{B-}, \beta^{B+}\}$.

We begin by plotting some paths of the Bid price $\mathfrak{b}$ and the total volume $\mathfrak{h}$ with simple parameters.

Law parameters

— α^S is state independent,

— α^C is state independent,

— $(\alpha^S, \alpha^C, \alpha^B)$ follow a binomial law with parameters $(40, 0.5)$,

— (β^{B-}, β^C) follow a binomial law with parameters $(10, 0.2)$,

— (β^{B+}) follow a binomial law with parameters $(5, 0.05)$

— $\mathbb{Q}[\delta = C]\mathbb{Q}[\epsilon^C = 0] = 0.04$,

— $\mathbb{Q}[\delta = C]\mathbb{Q}[\epsilon^C = 1] = 0.3$,

— $\mathbb{Q}[\delta = B]\mathbb{Q}[\epsilon^B = 0] = 0.3$,

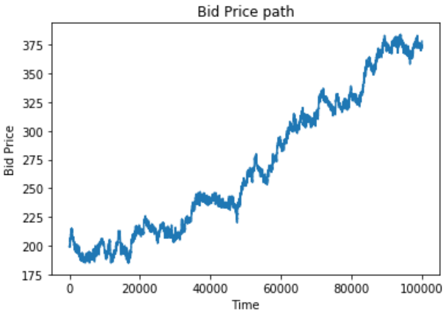
— $\mathbb{Q}[\delta = B]\mathbb{Q}[\epsilon^B = 1] = 0.04$,

— $\mathbb{Q}[\delta = S] \in \{0.06, 0.065, 0.07\}$

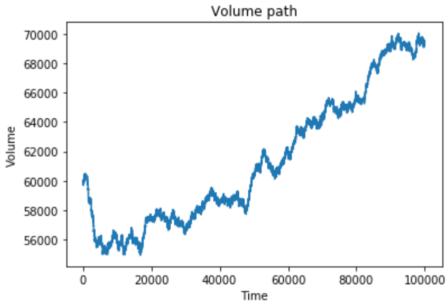
Empirically, we find that by changing $\mathbb{Q}[\delta = S]$ by amount in the order of 0.05, the bid price and the total volume change the direction. It means that the model is really sensitive to the parameters. By fixing all parameters and letting free $\mathbb{Q}[\delta = S]$, it's not easy to find neutral bid price path and total volume path. With $\mathbb{Q}[\delta = S] = 0.65$ in figures 2.11a and 2.11b, the bid price path and volume price

Data: N : time step, K : maximal price, δ : Simulate the vector δ_i (size N),
 n : initial vector of depth (size K)
Result: B : vector (size N), B^* : vector (size N), V : vector (size N)
initialization : $\mathsf{N}[0] \leftarrow \mathsf{n}$, $B[0] \leftarrow \mathfrak{b}(\mathsf{N}[0])$, $V[0] \leftarrow \mathfrak{v}(\mathsf{N}[0])$;
for $i \in [\![1, N]\!]$ **do**
  switch *the value of* δ_i **do**
    case *N*
      $\mathsf{N}[i] \leftarrow \mathsf{N}[i-1]$;
    case *B+*
      $a \leftarrow$ Simulate α^B;
      $g \leftarrow$ Simulate β^{B+};
      $b \leftarrow$ Compute β^B with g and $B[i-1]$;
      $\mathsf{N}[i] \leftarrow$ Compute the new state with a,b and T^B;
    case *B-*
      $a \leftarrow$ Simulate α^B;
      $g \leftarrow$ Simulate β^{B-};
      $b \leftarrow$ Compute β^B with g and $B[i-1]$;
      $\mathsf{N}[i] \leftarrow$ Compute the new state with a,b and T^B;
    case *C+*
      $g \leftarrow$ Simulate β^{C-};
      $b \leftarrow$ Compute β^C with g and $B[i-1]$;
      $a \leftarrow$ Simulate α^C;
    case *C-*
      $g \leftarrow$ Simulate β^{C-};
      $b \leftarrow$ Compute β^C with g and $B[i-1]$;
      $a \leftarrow$ Compute $\Delta_b \mathsf{n}$;
      $\mathsf{N}[i] \leftarrow$ Compute the new state with a,b and T^C ;
    case *S*
      $a \leftarrow$ Simulate α^S;
      $\mathsf{N}[i] \leftarrow$ Compute the new state with a and T^S;
    case *S dep*
      $a \leftarrow$ Simulate α^S with $\mathsf{n}^{|B[i-1]-l}$;
      $\mathsf{N}[i] \leftarrow$ Compute the new state with a and T^S;
  endsw
  $B[i] \leftarrow$ Compute the Bid Price of $\mathsf{N}[i]$;
  $V[i] \leftarrow$ Compute the the total Volume of $\mathsf{N}[i]$;
end

Algorithm 1: Bid Price and the total volume simulation

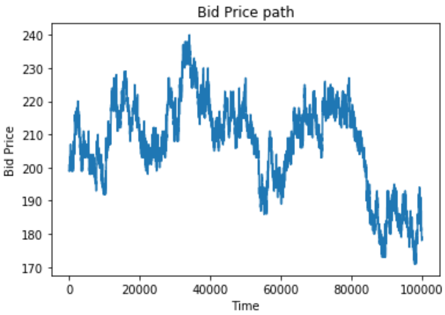


(a) Bid price path

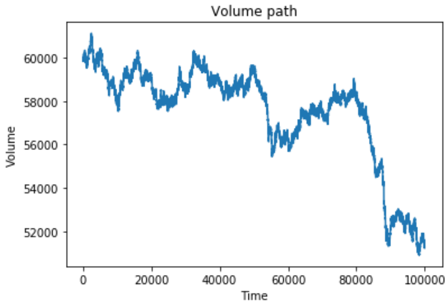


(b) Total volume path

FIGURE 2.10 – Path simulation for $\mathbb{Q}[\delta = S] = 0.06$

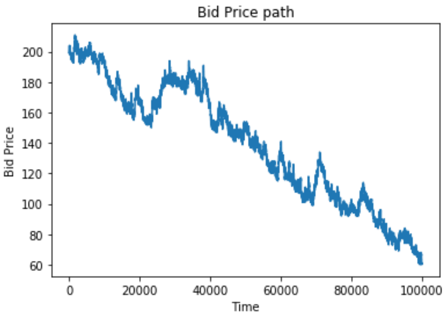


(a) Bid price path

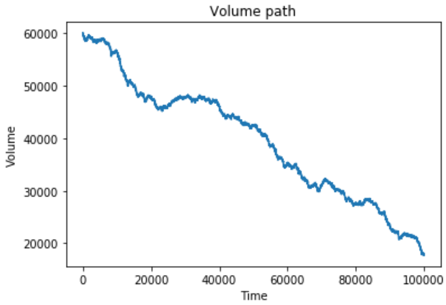


(b) Total volume path

FIGURE 2.11 – Path simulation for $\mathbb{Q}[\delta = S] = 0.065$

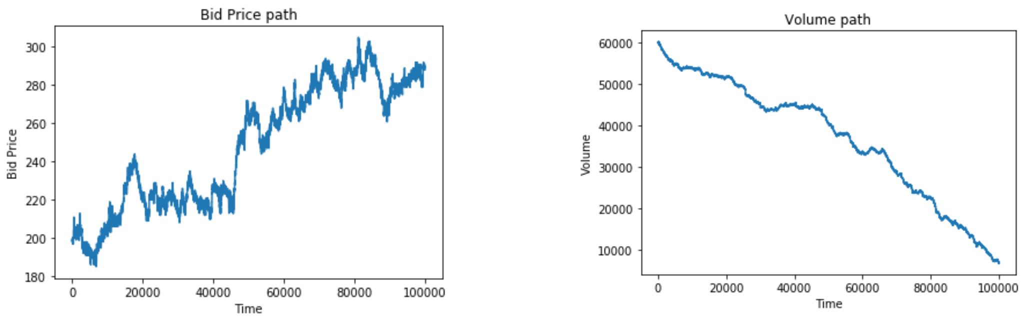


(a) Bid price path



(b) Total volume path

FIGURE 2.12 – Path simulation for $\mathbb{Q}[\delta = S] = 0.07$



(a) Bid price path

(b) Total volume path

FIGURE 2.13 – Path simulation for law parameters 2

path seems to be neutral. The choice parameters to have a neutral bid price path and volume price path are linked to the drift $\mathfrak{L}\mathfrak{b}$ and $\mathfrak{L}\mathfrak{h}$.

Moreover, in figures 2.10a and 2.10b, 2.13a and 2.13b, the monotony of the bid price path and volume price path seems to be linked. $\mathbb{Q}[\delta = S]$ is involved in the drift $\mathfrak{L}\mathfrak{b}$ and $\mathfrak{L}\mathfrak{h}$.

However, by writing the drift $\mathfrak{L}\mathfrak{b}$ and $\mathfrak{L}\mathfrak{h}$,

$$\begin{aligned} \mathfrak{L}\mathfrak{b}(\mathsf{x}) &= -p_S\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^{*} \alpha^S)] \\ &\quad +p_B\mathbb{Q}[\epsilon^B = 1]\mathbb{E}^{\mathsf{x}}[\beta^{B+}] \\ &\quad -p_C\mathbb{Q}[\beta^C = \mathfrak{b}(\mathsf{x}), \epsilon^C = 1](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x})). \end{aligned}$$

$$\mathfrak{L}\mathfrak{h}(\mathsf{x}) = -p_S\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + p_B\mathbb{E}^{\mathsf{x}}[\alpha^B] - p_C\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C].$$

We notice that in order to unbind the evolution of the bid price and the total volume, we can choose $\mathbb{Q}[\epsilon^B = 1]\mathbb{E}^{\mathsf{x}}[\beta^{B+}]$ big and $\mathbb{E}^{\mathsf{x}}[\alpha^B]$ small. For example, **Law parameters 2**

— α^S is state independent,

— α^C is state independent,

— α^C follow a binomial law with parameters $(40, 0.5)$,

— α^B follow a binomial law with parameters $(10, 0.5)$

— β^C follow a binomial law with parameters $(10, 0.2)$,

— β^{B+} follow a binomial law with parameters $(20, 0.05)$

— $\mathbb{Q}[\delta = C] = \mathbb{Q}[\delta = B]$,

— $\mathbb{Q}[\delta = S] = 0$

But we noticed that the choice of parameters is not realistic since we choose $p_S = 0$. In the next chapter 3, we discuss about the choice of parameters for a bullish market (the bid price increase) and the total volume should not explode.

Conclusion

We designed a limit order book model with one side of the limit order book, the bid side. Moreover, the reference price is 0 and the interval of the price is $\mathbb{N}$. The main characteristics are derived through the actual state of the limit order book. We represented the state of the limit order book by the depth n. We constructed a theory around the space of depth $\mathcal{N}$ by defining operators of $\mathcal{N}$. Through the Markovian framework, we described the transition probabilities and the Markov chain generator. We specified the space of the random variables δ, α^S, α^B, α^C, β^B, β^C by using the empirical studies in the literature on the limit order book. We explained the concept of homogeneity and gave a lemma 2.3.1, we use in chapter 3.

Chapitre 3

The fundamentals of the Markov chain N

Contents

Introduction

The model introduced in the previous chapter possesses all the features and the flexibility for the study of limit order book. This chapter deals with the fundamentals of the Markov chain.

Concretely, we study the usual properties of a Markov chain. Particularly, we study the long term behavior, i.e. the classification of Markov chain. The classification of Markov chain on general state is well studied in Meyn and Tweedie [52]. In our case, we refer to the book of Menshikov [51] for definitions and properties about discrete-time denumerable state Markov chain.

The question of positive recurrence is well studied in the limit order book modeling literature. In Abergel [2], the Foster-Lyapunov approach is used for proving the positive recurrence of the Markov chain and the exponential rate of convergence to the stationary state. The Lyapunov functions used are : $V(\mathsf{n}) = z^{|\mathsf{n}|_1 + q}$ with $z > 1$ and q is the change quantity per event. In the limit order book modeling literature, the question of transience is never mentioned. In our model, we emphasize that we study different cases of transience.

In the section 3.1, we characterize the space of parameters which lead to the irreducibility of the Markov chain. In the section 3.2, we prove that the Markov chain can be recurrent and we exhibit a partial characterization of the space of parameters which leads to the recurrence. In the section 3.3, we prove that the Markov chain can be transient and exhibit in the same fashion a partial characterization of the space of parameters. The naive characterization of transience gives that main characteristics of limit order book especially the bid price $\mathfrak{b}$ and the total volume $\mathfrak{h}$ tends to infinity when time goes to infinity. In the section 3.4, we introduce the scope which is the difference between the bid price and the smaller price which contains non zero volume. We give another characterization of transience by bounding the scope process. In the same manner in the section 3.5, we bound simultaneously the volume process and the scope process in order to give a more realistic transient market.

3.1 The irreducibility

Theorem 3.1.1. *Suppose*

i. $\mathbb{Q}^{\mathsf{x}}[\beta^B = \mathfrak{b}(\mathsf{x}) - 1] > 0$ *if* $\mathfrak{b}(\mathsf{x}) > 1$, *and* $\mathbb{Q}^{\mathsf{x}}[\beta^B = \mathfrak{b}(\mathsf{x})] > 0$, *if* $\mathfrak{b}(\mathsf{x}) > 0$, *and* $\mathbb{Q}^{\mathsf{x}}[\beta^B = \mathfrak{b}(\mathsf{x}) + 1] > 0$, *for any* x.

ii. $\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathsf{x}] > 0$ *for any* $\mathsf{x} \in \mathcal{S}(\mathsf{x})$.

iii. $\mathbb{Q}^{\mathsf{x}}[\alpha^B = 1] > 0$ *and* $\mathbb{Q}^{\mathsf{x}}[\alpha^C = 1] > 0$, *for any* x.

iv. $p_N(\mathsf{x}) > 0, p_B(\mathsf{x}) > 0, p_C(\mathsf{x}) > 0$ *for any* x.

Then, the Markov chain **N** *is irreducible and aperiodic.*

Proof : The Markov chain **N** is aperiodic, because $p_N(\mathsf{x}) > 0$ for all x. As for the irreducibility, we can prove that every x communicates with the null element of $\mathcal{N}$, and vice versa. Actually, for any non null element x, the conditions of the theorem allow, with a positive probability, chaining the cancellations till the annihilation of the total volume. Inversely, starting from a null element, beginning with the first valuation level, by suitable combinations of buys and cancellations, we can construct any element x. For example, to have $(1, 1, 0, 0, 0, \ldots)$, we combine, with positive probability, the sequence of the following orders :

i. buy 1 size at value 1,
ii. buy 1 size at value 2,
iii. cancel 1 size at value 1.

The theorem is thus proved. □

Remark 3.1.1. *In fact, for the purpose of order book modeling, we need not always to have the irreducibility on the whole space* $\mathcal{N}$.

3.2 The recurrence

The recurrence is a frequently asked question about the order book models. One of the reasons of studying the recurrence is that the limit theorems under recurrence provide the theoretical basis for the eventual calibration methods and for the study on the stylized facts. For the model **N**, it is so flexible that there is no difficulty for **N** to have the recurrence. We will illustrate this feature by some examples.

We consider the function V introduced in Section 2.3.2.1 :

$$V(\mathsf{x}) = \mathsf{x}^{|1} + \mathfrak{b}(\mathsf{x}).$$

Clearly, for any integer $N > 0$, $\{\mathsf{x} \in \mathcal{N} : V(\mathsf{x}) \leq N\}$ is a finite set. The next result follows from Menshikov Theorem 2.5.2 [51]. It is the Foster-Lyapunov criterion.

Lemma 3.2.1. *Suppose that the random variables α^B, β^B are integrable under $\mathbb{Q}^{\mathsf{x}}$ for any x. The drift $\mathfrak{L}V(\mathsf{x})$ is well-defined. If the set $\{\mathsf{x} \in \mathcal{N} : \mathfrak{L}V(\mathsf{x}) > 0\}$ is a finite set, the Markov chain* **N** *is recurrent.*

Recall the formula (2.7) :

$$\begin{array}{rcl} \mathfrak{L}V(\mathsf{x}) & = & -p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] \ ) \\ & & +p_B(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\alpha^B] \ + \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] \ ) \\ & & -p_C(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] \ + \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \ ). \end{array}$$

With this formula, it is easy to find model specifications which make the set $\{\mathsf{x} \in \mathcal{N} : \mathfrak{L}V(\mathsf{x}) > 0\}$ finite. For example, we can simply define the random variables under $\mathbb{Q}^{\mathsf{x}}$ so that

$$p_B(\mathsf{x}) = 0, \text{ whenever } V(\mathsf{x}) \text{ becomes big.} \tag{3.1}$$

The condition (3.1) erases the only positive term of $\mathfrak{L}V(\mathsf{x})$. It is not very realistic. Look at now another model specification :

i. Under $\mathbb{Q}^{\mathsf{x}}$, the random variable (α^B, β^B) is uniformly bounded and has a probability distribution independent of x, so that the quantity $\mathbb{E}^{\mathsf{x}}[\alpha^B] \ + \mathbb{E}^{\mathsf{x}}[\beta^B]$ is a constant D.

ii. The probabilities $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$ also are positive constants.

iii. Under $\mathbb{Q}^{\mathsf{x}}$, the random variable α^C is uniform on $\{1, \ldots, \varDelta_{\beta^C}\mathsf{x}\}$ and the random variable β^C distributed proportionally to $\varDelta\mathsf{x}$.

Under these conditions, the Markov chain **N** is essentially recurrent. Actually, under the conditions, the reduced space $\{\mathsf{x} \in \mathcal{N} : \mathfrak{b}(\mathsf{x}) \leq K\}$, where K is the upper bound of β^B, is absorbing for **N**. Moreover, according to the discussion in Section 2.3.2.1,

$$\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = \frac{1}{2} + \frac{1}{2\mathsf{x}^{|1}} \sum_{i=1}^{\infty} (\varDelta_i \mathsf{x})^2.$$

Hence, for $\mathsf{x}^{|1}$ big enough,

$$p_B(\mathsf{x})D \leq p_C(\mathsf{x}) \left(\frac{1}{2} + \frac{1}{2\mathsf{x}^{|1}} \sum_{i=1}^{\infty} (\Delta_i \mathsf{x})^2 \right),$$

(recalling that $p_C(\mathsf{x})$ is by assumption a positive constant,) which implies that the set $\{\mathsf{x} \in \mathcal{N} : \mathfrak{L}V(\mathsf{x}) > 0, \mathfrak{b}(\mathsf{x}) \leq K\}$ is finite. Applying Lemma 3.2.1 to $\mathbf{N}$ as a Markov chain in the reduced space, we obtain its recurrence in the reduced space.

We can change to another scenario.

i. The random variable β^B is bounded.
ii. There is a constant K such that the probabilities $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$ also are positive constants, whenever $\mathfrak{b}(\mathsf{x}) \leq K$. Otherwise, $p_S(\mathsf{x}) = 1$ (which defines a market resistance level).
iii. Under $\mathbb{Q}^{\mathsf{x}}$, the random variable α^C is uniform on $\{1, \ldots, \Delta_{\beta^C}\mathsf{x}\}$ and the random variable β^C distributed proportionally to $\Delta \mathsf{x}$.

Again, the Markov chain $\mathbf{N}$ can be proved essentially recurrent in this scenario in its value space.

To end the discussion on the recurrence, we notice that the random variables such as α^C, β^B, etc., have simple economical meaning, whose probability distributions can be calibrated directly from market data. The various limit theorems under the recurrence ensure the validity of the calibration.

3.3 The transience

As for the recurrence, the Foster-Lyapunov criterion (Theorem 2.5.8 [51]) can be applied to establish the transience . However, for the proof of transience, the Foster-Lyapunov criterion is neither convenient nor intuitive. In this section, we will follow a different, a more direct, and somewhat more intuitive approach. We apply Lemma 2.3.1.

Corollary 3.3.1. *Let F be a function on $\mathcal{N}$, which satisfies the second condition in Lemma 2.3.1. Suppose in addition that $\mathbf{N}$ is irreducible and, for some constant $0 < \theta < \infty$, $\mathfrak{L}F(\mathsf{x}) \geq \theta$ uniformly in x. Then, the Markov chain $\mathbf{N}$ is transient.*

Proof : By Lemma 2.3.1,

$$\liminf_{n \to \infty} \frac{F(\mathbf{N}_n)}{n} \geq \theta > 0,$$

which can not happen, if the Markov chain $\mathbf{N}$ is recurrent. □

3.3.1 The volume function

We now consider the volume function $\mathfrak{h}(\mathsf{x}) = \mathsf{x}^{|1}$. Recall (2.6) for the definitions of $\tilde{\alpha}^S$ and $\tilde{\alpha}^C$. If the random variables α^B is integrable under $\mathbb{Q}^{\mathsf{x}}$ for any x, the drift $\mathfrak{L}H(\mathsf{x})$ is well-defined and

$$\mathfrak{L}\mathfrak{h}(\mathsf{x}) = -p_S(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\alpha^B] - p_C(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]. \tag{3.2}$$

Following Lemma 2.3.1 and Corollary 3.3.1, we consider the process

$$\begin{aligned}
\mathbf{N}_n^{|1} &= \mathbf{N}_0^{|1} + \textstyle\sum_{k=1}^n (\mathbf{N}_k^{|1} - \mathbf{N}_{k-1}^{|1}) \\
&= \mathbf{N}_0^{|1} + \textstyle\sum_{k=1}^n (-1\!\!1_{\{\delta_k=S\}}\tilde{\alpha}_k^S + 1\!\!1_{\{\delta_k=B\}}\alpha_k^B - 1\!\!1_{\{\delta_k=C\}}\tilde{\alpha}_k^C) \\
&= \mathbf{N}_0^{|1} + \textstyle\sum_{k=1}^n (-1\!\!1_{\{\delta_k=S\}}\tilde{\alpha}_k^S + 1\!\!1_{\{\delta_k=B\}}\alpha_k^B - 1\!\!1_{\{\delta_k=C\}}\tilde{\alpha}_k^C - \mathfrak{L}H(\mathbf{N}_{k-1})) \\
&+ \textstyle\sum_{k=1}^n \mathfrak{L}H(\mathbf{N}_{k-1}).
\end{aligned}$$

The process is divided into two parts. The first part is a martingale :

$$\begin{aligned}
\mathbf{M} &= \mathbf{N}_0^{|1} + \textstyle\sum_{k=1}^n X_k, \quad \text{where} \\
X_k &= (-1\!\!1_{\{\delta_k=S\}}\tilde{\alpha}_k^S + 1\!\!1_{\{\delta_k=B\}}\alpha_k^B - 1\!\!1_{\{\delta_k=C\}}\tilde{\alpha}_k^C - \mathfrak{L}H(\mathbf{N}_{k-1})).
\end{aligned}$$

As in Lemma 2.3.1 let

$$e(\mathbf{N}_{n-1}) = \mathbb{E}[X_n^2 \,|\mathcal{F}_{n-1}].$$

Theorem 3.3.1. *Suppose that* $\mathbf{N}$ *is irreducible. Suppose that the expectations* $\mathbb{E}^{\mathsf{x}}[(\alpha^S)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^B)^2]$, $\mathbb{E}^{\mathsf{x}}[(\alpha^C)^2]$ *are uniformly bounded. Suppose that, for a constant* $\theta > 0$, $\mathfrak{L}\mathfrak{h}(\mathsf{x}) \geq \theta$ *uniformly. Then,*

$$\liminf_{n\to\infty} \frac{\mathbf{N}_n^{|1}}{n} > 0.$$

In particular, the chain $\mathbf{N}$ *is transient.*

Proof : We compute

$$\begin{aligned}
e(\mathbf{N}_{n-1}) &= \mathbb{E}[(-1\!\!1_{\{\delta_n=S\}}\tilde{\alpha}_n^S + 1\!\!1_{\{\delta_n=B\}}\alpha_n^B - 1\!\!1_{\{\delta_n=C\}}\tilde{\alpha}_n^C - \mathfrak{L}H(\mathbf{N}_{n-1}))^2 \,|\mathcal{F}_{n-1}] \\
&= \mathbb{E}^{\mathsf{x}}[(-1\!\!1_{\{\delta=S\}}\tilde{\alpha}^S + 1\!\!1_{\{\delta=B\}}\alpha^B - 1\!\!1_{\{\delta=C\}}\tilde{\alpha}^C - \mathfrak{L}H(\mathsf{x}))^2] \\
&\quad \text{for } \mathsf{x} = \mathbf{N}_{n-1}, \\
&= \mathbb{E}^{\mathsf{x}}[((-1\!\!1_{\{\delta=S\}}(\tilde{\alpha}^S - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S]) + 1\!\!1_{\{\delta=B\}}(\alpha^B - \mathbb{E}^{\mathsf{x}}[\alpha^B]) - 1\!\!1_{\{\delta=C\}}(\tilde{\alpha}^C - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C])) \\
&+ (-(1\!\!1_{\{\delta=S\}} - p_S(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + (1\!\!1_{\{\delta=B\}} - p_B(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\alpha^B] - (1\!\!1_{\{\delta=C\}} - p_C(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]))^2] \\
&= \mathbb{E}^{\mathsf{x}}[(-1\!\!1_{\{\delta=S\}}(\tilde{\alpha}^S - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S]) + 1\!\!1_{\{\delta=B\}}(\alpha^B - \mathbb{E}^{\mathsf{x}}[\alpha^B]) - 1\!\!1_{\{\delta=C\}}(\tilde{\alpha}^C - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]))^2] \\
&+ \mathbb{E}^{\mathsf{x}}[(-(1\!\!1_{\{\delta=S\}} - p_S(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + (1\!\!1_{\{\delta=B\}} - p_B(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\alpha^B] - (1\!\!1_{\{\delta=C\}} - p_C(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C])^2] \\
&= p_S(\mathsf{x})\mathbb{E}^{\mathsf{x}}[(\tilde{\alpha}^S - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S])^2] + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[(\alpha^B - \mathbb{E}^{\mathsf{x}}[\alpha^B])^2] + p_C(\mathsf{x})\mathbb{E}^{\mathsf{x}}[(\tilde{\alpha}^C - \mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C])^2] \\
&+ \mathbb{E}^{\mathsf{x}}[(-(1\!\!1_{\{\delta=S\}} - p_S(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + (1\!\!1_{\{\delta=B\}} - p_B(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\alpha^B] - (1\!\!1_{\{\delta=C\}} - p_C(\mathsf{x}))\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C])^2].
\end{aligned}$$

By the assumption of the theorem, $e(\mathbf{N}_{n-1}) \leq D$ for some constant $D > 0$. Hence, Corollary 3.3.1 is applicable, which proves the theorem. □

Consider the following example. Let $h, g \in \mathbb{N}^*$. Suppose the assumptions in Theorem 3.1.1 and Theorem 3.3.1. Suppose, in addition, that $\mathbb{E}^{\mathsf{x}}[\alpha^B]$ is a constant $a_b > 0$ and that, under $\mathbb{Q}^{\mathsf{x}}$ for $\mathsf{x} \neq \mathsf{0}$, β^C is a uniform random variable on $\mathcal{S}(\mathsf{x})$ and α^C is a uniform random variable on $\{1, \ldots, g \wedge \Delta_{\beta^C}\mathsf{x}\}$, and α^S is a uniform random variable on $\{1, \ldots, h \wedge \mathsf{x}^{|1}\}$. Under this condition,

$$\begin{array}{rcl} \mathfrak{L}\mathfrak{h}(\mathsf{x}) & = & -p_S(\mathsf{x})\frac{1+h\wedge \mathsf{x}^{|1}}{2} + p_B(\mathsf{x})a_b - p_C(\mathsf{x})1\!\!1_{\{\mathsf{x}\neq\mathbf{0}\}}\frac{1}{\sigma(\mathsf{x})}\sum_{k\in\mathcal{S}(\mathsf{x})}\frac{1+g\wedge\Delta_k\mathsf{x}}{2} \\ & \geq & -p_S(\mathsf{x})\frac{1+h}{2} + p_B(\mathsf{x})a_b - p_C(\mathsf{x})\frac{1+g}{2}, \end{array}$$

where $\sigma(\mathsf{x})$ denotes the number of elements in $\mathcal{S}(\mathsf{x})$. Suppose that the probabilities $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$ are constant and are so that

$$\theta := -p_S\frac{1+h}{2} + p_B a_b - p_C\frac{1+g}{2} > 0. \tag{3.3}$$

Then, the Markov chain **N** is transient.

3.4 A more realistic transient model

The study of the recurrence and the transience of the Markov chain **N** is motivated by the natural idea that the recurrence corresponds to a flat or bear market, while the transience corresponds to a bullish market. In a bullish market, the bid $\mathfrak{b}(\mathbf{N}_n)$ tends to the infinity, that is why the Markov chain **N** must be transient. However, even in a bullish market, all the market indicators do not go to the the infinity. For example, the scope process $\mathfrak{b}(\mathbf{N}_n) - \mathfrak{d}(\mathbf{N}_n)$ should not explode. (See section 2.1 for the definition of $\mathfrak{d}(\mathsf{x})$.)

This section is devoted to a study of the parameterization which makes **N** a more realistic market model.

3.4.1 The bid $\mathfrak{b}(\mathbf{N}_n)$ and the scope $\mathfrak{b}(\mathbf{N}_n) - \mathfrak{d}(\mathbf{N}_n)$

Recall

$$\begin{array}{l} \mathcal{S}(\mathsf{x}) = \{k \in \mathbb{N}^* : \mathsf{x}^{|k} > \mathsf{x}^{|k+1}\}, \\ \mathfrak{d}(\mathsf{x}) = \inf \mathcal{S}(\mathsf{x}) \ \text{ if } \mathcal{S}(\mathsf{x}) \neq \emptyset, \ \text{ and } \mathfrak{d}(\mathsf{x}) = 0 \ \text{ if } \mathcal{S}(\mathsf{x}) = \emptyset. \end{array}$$

Let in addition

$$\begin{array}{l} \sigma(\mathsf{x}) = \sharp\mathcal{S}(\mathsf{x}), \\ \mathfrak{d}^\circ(\mathsf{x}) = \inf(\mathcal{S}(\mathsf{x}) \setminus \{\mathfrak{d}(\mathsf{x})\}) \ \text{ if } \mathcal{S}(\mathsf{x}) \setminus \{\mathfrak{d}(\mathsf{x})\} \neq \emptyset, \ \text{ and } \mathfrak{d}^\circ(\mathsf{x}) = 0 \ \text{ otherwise.} \end{array}$$

Our intention is to find a parameterization of the Markov chain $\mathbf{N}$ so that the bid $\mathfrak{b}(\mathbf{N}_n)$ tends to the infinity, while the scope $\mathfrak{b}(\mathbf{N}_n) - \mathfrak{d}(\mathbf{N}_n)$ remains, in some sense, recurrent. We write the drift

$$\begin{aligned}\mathfrak{L}\mathfrak{b}(\mathsf{x}) &= -p_S(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] \\ &\quad - p_C(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})).\end{aligned} \tag{3.4}$$

As for the function $\mathfrak{d}(\mathsf{x})$, we have

$$\begin{aligned}\mathfrak{d}(\mathbf{N}_{n+1}) =& \mathbb{1}_{\{\delta_{n+1}=S,\alpha^S<\mathsf{x}^{|1}\}}\mathfrak{d}(\mathsf{x}) + \mathbb{1}_{\{\delta_{n+1}=B\}}\mathfrak{d}(\mathsf{x}) \wedge \beta^B_{n+1} \\ &+ \mathbb{1}_{\{\delta_{n+1}=C\}}\mathbb{1}_{\{\beta^C_{n+1}=\mathfrak{d}(\mathsf{x}),\alpha^C\geq\Delta_{\beta^C}\mathsf{x}\}}\mathfrak{d}^\circ(\mathsf{x}) + \mathbb{1}_{\{\delta_{n+1}=N\}}\mathfrak{d}(\mathsf{x})\end{aligned}$$

so that

$$\begin{aligned}\mathfrak{L}\mathfrak{d}(\mathsf{x}) =& - p_S(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}]\mathfrak{d}(\mathsf{x}) + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})] \\ &+ p_C(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})).\end{aligned} \tag{3.5}$$

Putting together the two, we obtain the drifts

$$\begin{aligned}\mathfrak{L}(\mathfrak{b} - \mathfrak{d})(\mathsf{x}) \;=\; & -p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] - \mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}]\mathfrak{d}(\mathsf{x})) \\ & +p_B(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] - \mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})]) \\ & -p_C(\mathsf{x})(\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ & \qquad +\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x}))).\end{aligned} \tag{3.6}$$

Notice that the last term writes also as

$$\begin{aligned}&-p_C(\mathsf{x})(\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}) > \mathfrak{d}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ &\qquad +\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}) < \mathfrak{b}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x}))).\end{aligned}$$

In the next section, we will consider the increments

$$\begin{aligned}& (\mathfrak{b} - \mathfrak{d})(\mathbf{N}_{n+1}) - (\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n) \\ = \;& \mathbb{1}_{\{\delta=S\}}(\mathfrak{b}(\mathsf{x} -^* \alpha^S) - \mathfrak{b}(\mathsf{x})) - \mathbb{1}_{\{\delta=S\}}\mathbb{1}_{\{\alpha^S\geq\mathsf{x}^{|1}\}}(0 - \mathfrak{d}(\mathsf{x})) \\ & +\mathbb{1}_{\{\delta=B\}}((\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})) - (\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x}))) \\ & +\mathbb{1}_{\{\delta=C\}}(\mathbb{1}_{\{\beta^C=\mathfrak{b}(\mathsf{x}),\alpha^C\geq\mathsf{x}^{|\mathfrak{b}(\mathsf{x})}\}}(\mathfrak{b}^\circ(\mathsf{x}) - \mathfrak{b}(\mathsf{x})) - \mathbb{1}_{\{\beta^C=\mathfrak{d}(\mathsf{x}),\alpha^C\geq\Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x}\}}(\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x}))) \\ \leq \;& \mathbb{1}_{\{\delta=S\}}(\mathfrak{b}(\mathsf{x} -^* \alpha^S) - \mathfrak{b}(\mathsf{x})) - \mathbb{1}_{\{\delta=S\}}\mathbb{1}_{\{\alpha^S\geq\mathsf{x}^{|1}\}}(0 - \mathfrak{d}(\mathsf{x})) \\ & +\mathbb{1}_{\{\delta=B\}}((\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})) - (\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x}))) \\ & +\mathbb{1}_{\{\delta=C\}}\mathbb{1}_{\{\beta^C=\mathfrak{b}(\mathsf{x})>\mathfrak{d}(\mathsf{x}),\alpha^C\geq\mathsf{x}^{|\mathfrak{b}(\mathsf{x})}\}}(\mathfrak{b}^\circ(\mathsf{x}) - \mathfrak{b}(\mathsf{x})) \\ & -\mathbb{1}_{\{\delta=C\}}\mathbb{1}_{\{\beta^C=\mathfrak{d}(\mathsf{x})<\mathfrak{b}(\mathsf{x}),\alpha^C\geq\Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x}\}}(\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g) \\ =: \;& F_g(\mathsf{x}, \Phi)\end{aligned} \tag{3.7}$$

with $\mathsf{x} = \mathbf{N}_n$ and $g > 0$ a constant, where the δ denotes δ_{n+1}, etc., and Φ denotes the set of all these random variables. Taking the expectation of $F_g(\mathsf{x}, \Phi)$ under $\mathbb{Q}^{\mathsf{x}}$, we obtain the function

$$\begin{aligned}\mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathsf{x}) \;:=\; & -p_S(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] - \mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}]\mathfrak{d}(\mathsf{x}))\\ & +p_B(\mathsf{x})(\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] - \mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})])\\ & -p_C(\mathsf{x})(\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}) > \mathfrak{d}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^{\circ}(\mathsf{x}))\\ & \qquad +\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}) < \mathfrak{b}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^{\circ}(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g),\end{aligned} \tag{3.8}$$

considered as a truncation of the drift $\mathfrak{L}(\mathfrak{b}-\mathfrak{d})(\mathsf{x})$.

3.4.2 A result about the scope in a transient market

We want to fix parameterizations which make $\mathfrak{L}\mathfrak{b}(\mathsf{x})$ and $\mathfrak{L}(\mathfrak{b}-\mathfrak{d})(\mathsf{x})$ positive or negative. This can be easily done, if we impose no restriction on the probabilities $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$. But we should not let these probabilities depending too much of x, because this dependence will make impossible the calibration. Hence, in our search of the parameterization, we try to keep these probabilities as homogeneous as possible.

i. Suppose the assumption of Theorem 3.1.1.
ii. α^S is bounded.
iii. $p_S(\mathsf{x}) = 0$ if $\mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}] = 0$, and $p_S(\mathsf{x})$ is constant otherwise.
iv. β^B takes the form :

$$\beta^B = \mathfrak{b}(\mathsf{x}) - (1-\epsilon)\beta^{B-} + \epsilon\beta^{B-},$$

where the random variables $\beta^{B-} \in [0, \mathfrak{b}(\mathsf{x})], \beta^{B-} \in \mathbb{N}^*$ are independent of $\epsilon \in \{0,1\}$ under $\mathbb{Q}^{\mathsf{x}}$ (cf. Section 2.3.2.2), and β^{B-}, β^{B-} are bounded.
v. Under $\mathbb{Q}^{\mathsf{x}}$, β^C follows the law of a geometric(p) (same p for all x) random variable conditioned on the set $\mathcal{S}(\mathsf{x})$.

Notice that under the above conditions,

$$\begin{aligned}\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] &= \mathbb{Q}^{\mathsf{x}}[\epsilon = 1]\mathbb{E}^{\mathsf{x}}[\beta^{B-}],\\ \mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})] &= -\mathbb{Q}^{\mathsf{x}}[\epsilon = 0]\mathbb{E}^{\mathsf{x}}[(\beta^{B-} - (\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}))^+].\end{aligned}$$

Theorem 3.4.1. *Suppose the above conditions on the random variables $\alpha^S, \beta^B, \beta^C$. Suppose also the followings :*

i. For a constant $c > 0$, $\mathfrak{L}\mathfrak{b}(\mathsf{x}) \geq c$ for all x.
ii. For some constants $g > 0, h > 0, c' > 0$, $\mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathsf{x}) \leq -c'$ for all x such that $\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) \geq h$.

Then, the bid $\mathfrak{b}(\mathbf{N}_n)$ tends to the infinity (so that the Markov chain $\mathbf{N}$ is transient), while the scope process $(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n)$ turns back to values smaller than h, after whatever *long period.*

Proof : Notice that, by Theorem 3.1.1, $\mathbf{N}$ is irreducible. There are two processes $\mathfrak{b}(\mathbf{N}_n)$ and $(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n)$ to be studied. They can be both studied in the line fixed by Lemma 2.3.1 and Corollary 3.3.1. For the bid process, we have in fact

$$0 < \liminf_{n\to\infty} \frac{\mathfrak{b}(\mathbf{N}_n)}{n} \leq \limsup_{n\to\infty} \frac{\mathfrak{b}(\mathbf{N}_n)}{n} < \infty. \tag{3.9}$$

But we will not give the proof of (3.9).

We now consider the scope process

$$\begin{aligned}(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n) &= (\mathfrak{b}-\mathfrak{d})(\mathbf{N}_0) + \textstyle\sum_{k=1}^n((\mathfrak{b}-\mathfrak{d})(\mathbf{N}_k) - (\mathfrak{b}-\mathfrak{d})(\mathbf{N}_{k-1}))\\ &\leq (\mathfrak{b}-\mathfrak{d})(\mathbf{N}_0) + \textstyle\sum_{k=1}^n F_g(\mathbf{N}_{k-1}, \Phi_k)\end{aligned}$$

according to (3.7). Consider the martingale increments

$$X_k = F_g(\mathbf{N}_{k-1}, \Phi_k) - \mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_{k_1}).$$

We introduce

$$\begin{aligned}\mathbf{u}_1 &= (\mathfrak{b}(\mathsf{x} -^* \alpha^S) - \mathfrak{b}(\mathsf{x})),\\ \mathbf{u}_2 &= (\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})) - (\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})),\\ \mathbf{u}_3 &= 1\!\!1_{\{\beta^C=\mathfrak{b}(\mathsf{x})>\mathfrak{d}(\mathsf{x}),\alpha^C\geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}\}}(\mathfrak{b}^\circ(\mathsf{x}) - \mathfrak{b}(\mathsf{x})) - 1\!\!1_{\{\beta^C=\mathfrak{d}(\mathsf{x})<\mathfrak{b}(\mathsf{x}),\alpha^C\geq \Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x}\}}(\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g,\end{aligned}$$

and

$$\begin{aligned}w_1 &= \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x} -^* \alpha^S) - \mathfrak{b}(\mathsf{x})],\\ w_2 &= \mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] - \mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})],\\ w_3 &= \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}) > \mathfrak{d}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}^\circ(\mathsf{x}) - \mathfrak{b}(\mathsf{x}))\\ &\qquad - \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}) < \mathfrak{b}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g,\end{aligned}$$

so that $w_i = \mathbb{E}^{\mathsf{x}}[\mathbf{u}_i]$ and

$$\mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathsf{x}) = p_S(\mathsf{x})w_1 + p_B(\mathsf{x})w_2 + p_C(\mathsf{x})w_3.$$

With $\mathsf{x} = \mathbf{N}_{k-1}$, we have :

$$\mathbb{E}[X_k^2 |\, \mathcal{F}_{k-1}] = \mathbb{E}^{\mathsf{x}}[(1\!\!1_{\{\delta=S\}}\mathbf{u}_1 - p_S(\mathsf{x})w_1 + 1\!\!1_{\{\delta=B\}}\mathbf{u}_2 - p_B(\mathsf{x})w_2 + 1\!\!1_{\{\delta=C\}}\mathbf{u}_3 - p_C(\mathsf{x})w_3)^2].$$

Notice that, under the conditions of the theorem, the expectations $\mathbb{E}^{\mathsf{x}}[(\mathbf{u}_i)^2]$ are uniformly bounded. (For the computation of $\mathbf{u}_3$, see next section.) (cf. Hall and

Heyde Theorem 2.18 [40]] is applicable. We obtain

$$\limsup_{n\to\infty} \frac{(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n)}{n} = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_{n-1}).$$

If the values $(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n) \geq h > 0$ for all but a finite number of $n \in \mathbb{N}$, by the assumption of the proposition, we would have

$$0 \leq \liminf_{n\to\infty} \frac{(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_n)}{n} = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathfrak{L}_g(\mathfrak{b}-\mathfrak{d})(\mathbf{N}_{k-1}) \leq -c',$$

an obvious paradox. The theorem is proved. □

3.4.3 A discussion about the assumptions in Proposition 3.4.1

Let us show in this section that the conditions in Theorem 3.4.1 are realistic. Suppose the conditions of Theorem 3.4.1. Look at the drift $\mathfrak{L}\mathfrak{b}(\mathsf{x})$:

$$\begin{aligned} -p_S(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) - \mathfrak{b}(\mathsf{x} -^* \alpha^S)] + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{b}(\mathsf{x}) \vee \beta^B - \mathfrak{b}(\mathsf{x})] \\ -p_C(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})). \end{aligned}$$

The first term is bounded because of Proposition 2.3.1, as well as the second term. Look at the last term $\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x}))$:

$$\mathbb{Q}^{\mathsf{x}}[\beta^C = k] = \frac{pq^k}{\sum_{i\in\mathcal{S}(\mathsf{x})} pq^i} \mathbb{1}_{\{k\in\mathcal{S}(\mathsf{x})\}}, \tag{3.10}$$

and consequently, when $\mathfrak{b}^\circ(\mathsf{x}) > 0$,

$$\begin{aligned} & \mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{b}(\mathsf{x}), \alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})}](\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ = \; & \mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})} \mid \beta^C = \mathfrak{b}(\mathsf{x})]\tfrac{q^{\mathfrak{b}(\mathsf{x})}}{\sum_{k\in\mathcal{S}(\mathsf{x})} q^k}(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ \leq \; & \mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})} \mid \beta^C = \mathfrak{b}(\mathsf{x})]\tfrac{q^{\mathfrak{b}(\mathsf{x})}}{q^{\mathfrak{b}^\circ(\mathsf{x})} + q^{\mathfrak{b}(\mathsf{x})}}(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ = \; & \mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})} \mid \beta^C = \mathfrak{b}(\mathsf{x})]\tfrac{1}{1+q^{\mathfrak{b}(\mathsf{x})-\mathfrak{b}^\circ(\mathsf{x})}} q^{\mathfrak{b}(\mathsf{x})-\mathfrak{b}^\circ(\mathsf{x})}(\mathfrak{b}(\mathsf{x}) - \mathfrak{b}^\circ(\mathsf{x})) \\ \leq \; & \mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \mathsf{x}^{|\mathfrak{b}(\mathsf{x})} \mid \beta^C = \mathfrak{b}(\mathsf{x})] \times \tfrac{e^{-1}}{-\ln q} \leq \tfrac{e^{-1}}{-\ln q}. \end{aligned}$$

We see now that it is easy to choose $p_S(\mathsf{x}), p_B(\mathsf{x}), p_C(\mathsf{x})$ so that $\mathfrak{L}\mathfrak{b}(\mathsf{x}) > 0$ uniformly.

Consider next the three terms

$$\begin{aligned} & -p_S(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}]\mathfrak{d}(\mathsf{x}), p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) \wedge \beta^B - \mathfrak{d}(\mathsf{x})], \\ & -p_C(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g. \end{aligned}$$

The first term $p_S(\mathsf{x})\mathbb{Q}^{\mathsf{x}}[\alpha^S \geq \mathsf{x}^{|1}] \equiv 0$. For the second term, as β^B takes the form

$$\beta^B = \mathfrak{b}(\mathsf{x}) - (1-\epsilon)\beta^{B-} + \epsilon\beta^{B-},$$

the expectation

$$\begin{aligned}&\mathbb{E}^{\mathsf{x}}[\mathfrak{d}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) \wedge \beta^B] = \mathbb{E}^{\mathsf{x}}[(\mathfrak{d}(\mathsf{x}) - \beta^B)^+] \\ = \; &\mathbb{Q}^{\mathsf{x}}[\epsilon = 0]\mathbb{E}^{\mathsf{x}}[(\mathfrak{d}(\mathsf{x}) - \mathfrak{b}(\mathsf{x}) + \beta^{B-})^+] \leq \mathbb{Q}^{\mathsf{x}}[\epsilon = 0]\mathbb{E}^{\mathsf{x}}[(\beta^{B-} - h)^+],\end{aligned}$$

when $\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) \geq h$. When the constant h is large enough, $\mathbb{E}^{\mathsf{x}}[(\beta^{B-} - h)^+] = 0$. Finally, for the third term, as β^C follows the conditioned geometric distribution (3.10), we have, when $\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) > 0$, $g \geq 1$,

$$\begin{aligned}&\mathbb{Q}^{\mathsf{x}}[\beta^C = \mathfrak{d}(\mathsf{x}), \alpha^C \geq \Delta_{\beta^C}\mathsf{x}](\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g \\ = \; &\mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x} \mid \beta^C = \mathfrak{d}(\mathsf{x})]\frac{q^{\mathfrak{d}(\mathsf{x})}}{\sum_{k \in \mathcal{S}(\mathsf{x})} q^k}(\mathfrak{d}^\circ(\mathsf{x}) - \mathfrak{d}(\mathsf{x})) \wedge g \\ \geq \; &\mathbb{Q}^{\mathsf{x}}[\alpha^C \geq \Delta_{\mathfrak{d}(\mathsf{x})}\mathsf{x} \mid \beta^C = \mathfrak{d}(\mathsf{x})],\end{aligned}$$

which can be chosen uniformly bounded below.

Notice that no constraint has been imposed on $p_C(\mathsf{x})$ when $\mathfrak{b}(\mathsf{x}) > \mathfrak{d}(\mathsf{x})$. After the above analysis, it is easy to define the value of $p_C(\mathsf{x})$ to make $\mathfrak{L}_g(\mathfrak{b} - \mathfrak{d})(\mathsf{x}) \leq -c'$ for some $c' > 0$ when $\mathfrak{b}(\mathsf{x}) - \mathfrak{d}(\mathsf{x}) \geq h$.

3.5 The volume index in a bullish market

As in Section 3.4, we discuss the adaptation of our model to real market situations. A sufficient condition of the transience of $\mathbf{N}$ has been given in Theorem 3.3.1 in term of the volume index $\mathfrak{h}(\mathbf{N}_n)$ which tends to the infinity. However, in a real market situation, the bid process can rise continuously, but the volume should remain limited. In this section we search the parameterization of $\mathbf{N}$ which models such a market situation. We have the following theorem.

Theorem 3.5.1. *Suppose the assumptions in Theorem 3.4.1. Suppose the assumptions in Theorem 3.3.1, except the condition* $\mathfrak{L}\mathfrak{h}(\mathsf{x}) \geq \theta$ *which is replaced by : for some constant* $c' > 0, h' > 0$, $\mathfrak{L}\mathfrak{h}(\mathsf{x}) \leq -c'$ *whenever* $x^{|1} > h'$. *Then, the Markov chain* $\mathbf{N}$ *is transient, while the scope process* $(\mathfrak{b} - \mathfrak{d})(\mathbf{N}_n)$ *and the volume process* $\mathbf{N}_n^{|1}$ *turn back to small values, after whatever long period.*

Proof : This theorem can be proved in the same way as Theorem 3.4.1 has been proved. Notice only that, and the assumption of the theorem imply that, for some $h' > 0$, $\mathfrak{L}\mathfrak{h}(\mathsf{x}) \leq -c'$ whenever $x^{|1} > h'$, and consequently, the volume process $\mathbf{N}_n^{|1}$ can not remain long time above the level h', because of Lemma 2.3.1. □

The assumptions in Theorem 3.5.1 can be satisfied by reasonable parameters. In fact, it is enough to refine the conditions given in section 3.4.3.

Only the term

$$\mathfrak{L}\mathfrak{h}(\mathsf{x}) \quad = \quad -p_S(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^S] + p_B(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\alpha^B] - p_C(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C]$$

is to be considered. Notice that, as β^C takes values in $\mathcal{S}(\mathsf{x})$, $\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] \geq 1$ whenever $x^{|1} > 0$. We suppose that $p_C(\mathsf{x}), \alpha^C$ are so that

$$p_C(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^C] = a_{pc}$$

is a positive constant. We suppose that $a_b = \mathbb{E}^{\mathsf{x}}[\alpha^B]$ is a positive constant, as well as $\mathbb{E}^{\mathsf{x}}[\alpha^S]$. We will not modify the random variable $\alpha^S, \beta^{B-}, \beta^{B-}, \beta^C$ fixed in section 3.4.3. As for the probabilities $p_S(\mathsf{x}), p_B(\mathsf{x})$, we choose them so that

$$\begin{array}{l} 0 < p_S(\mathsf{x}) < 1 \quad \text{a constant,} \\ 0 < p_B(\mathsf{x}) \leq \frac{a_{pc}}{a_b}, \quad \text{for } \mathsf{x} \text{ such that } \mathsf{x}^{|1} > h'. \end{array}$$

Conclusion

We found that our model is enough flexible for letting the Markov Chain to be recurrent or transient. The recurrent case is well studied in the literature and we use the recurrent parameters set for the calibration in the next chapter 4.3. For the transient case, the introduction of the scope process is an original way to express the fact that all limit orders are not not too far to the bid price. With the fact that the volume process is bounded, the total volume H doesn't explode to infinity which is the financial reality. We proved in all cases, the existence of parameters and we derived in particular cases the set of the parameters which lead to the properties.

Chapitre 4

Estimations et calibrations du modèle

Contents

Introduction

Ce chapitre est consacré au traitement du problème de calibration et d'estimation de paramètres du modèle.

Dans le chapitre précédent, nous avons abordé le problème de classification de notre modèle et nous avons exhibé un ensemble de paramètres pour laquelle la chaine de Markov est récurrente, transiente. Dans ce chapitre, nous confrontons notre modèle avec les données du marché. Il s'agit de la validation du modèle.

Dans la littérature, afin de valider les modèles, ils étudient, de manière empirique les faits stylisés. Les modèles de carnet d'ordres qui étudient les problématiques d'estimation et de calibration, se ramènent à un problème d'estimation. En effet, ils sont modélisés à l'aide de processus de Poisson et ils estiment les intensités des processus de Poisson à l'aide des données et d'une méthode de maximum de vraisemblance. Il n'y a donc aucun degré supplémentaire dans leur modèle et il n'y a pas de calibration à faire.

Notre problème de calibration consiste à trouver les paramètres dans l'ensemble des probabilités $\mathbb{Q}$ dans Θ^{rec}. Pour réduire l'ensemble des lois d'arrivées, nous allons nous placer dans le cas d'une chaine de Markov récurrente. Il s'agit d'un choix artificiel mais nous justifions ce choix par la littérature existante [24] et [2].

Les faits stylisés et les critères de liquidité donnent des restrictions aux modèles stochastiques de carnet d'ordres. Les bons modèles devraient être capable de capturer simultanément la plupart d'entre eux avec peu de paramètres. En se plaçant dans Θ^{rec}, nous étudions les faits stylisés et les critères de liquidité en imposant des lois.

Dans la section 4.1, nous étudions le traitement et les statistiques descriptives des données. Les données du marché, provenant du future Bund, sont séparées en 2 sources : une possédant l'ensemble des transactions et une possédant les états du carnet. Cependant, les deux sources ne possèdent pas le mime horodatage et entre deux états successifs du carnet, plusieurs évènements peuvent se dérouler. Nous proposons une méthode de reconstruction. Cela donne, par la mime occasion, une information sur la qualité des données. Dans la section 4.2, nous étudions le problème d'estimation de paramètres et nous comparons les faits stylisés et des critères de liquidité entre le modèle simulé et les données du marché. Nous utilisons des méthodes d'estimation de maximum de vraisemblance. Nous donnons une paramétrisation concrète des problèmes de classification dans la section 4.3. Nous étudions les faits stylisés dans la section 4.4 (définit dans la sous-section 1.2.1) et les critères de liquidité dans la section 4.5 (définit dans la sous-section 1.2.2)

4.1 Présentation des données

Nous utilisons les données de niveau 2 du contrat du Future **Bund** négociées sur le marché de l'EUREX. Les données ont une précision de l'ordre de la milliseconde et ont été enregistrées entre mars 2013 et septembre 2013. Un contrat Future est un contrat autorisant l'achat d'un sous-jacent (pétroliers, actions ou devises) à une date future et à un prix déterminé à l'instant initial du contrat. Il permet aux traders de sécuriser le prix du sous-jacent qu'ils devront délivrer physiquement ou recevront à la date de livraison. Par conséquent àchaque instant, il peut exister un nombre infini de contrats, un pour chaque date de livraison. Sur un marché comme l'EUREX, les dates de livraison sont standardisées, tous les trois mois (mars, juin, septembre et décembre) et généralement peuvent être négociés en mime temps.

Dans nos données, nous avons 2 types de fichiers : "Quote Data raw" et "Trade Data Raw".

Le fichier "Trade Data Raw" est une liste de transaction décrit par un horodatage, le prix de la transaction, la taille de la transaction. L'horodatage de la transaction est l'instant de la réception et de la validation de l'ordre par le marché. Il y a un prix unique de transaction sur chaque ligne. Lorsqu'un ordre de marché est exécuté contre des ordres limites possédant différents niveaux de prix, l'exécution est reportée sur plusieurs transactions.

Trade Data Raw

Timestamp	LTP	LTV
06 :01 :18.767	145.96	1
06 :01 :18.908	145.97	1

FIGURE 4.1 – Représentation d'un fichier Trade Data Raw avec la colonne Timestamp correspondant à l'horodatage de la transaction, la colonne LTP correspondant au niveau de prix de la transaction et la colonne LTV correspondant à la taille de la transaction

Quote Data Raw

Timestamp	bid 0	Volume bid 0	Ask 0	Volume Ask 0	...	Ask 9	Volume Ask 9
06 :01 :20.517	145.96	48	145.97	41	...	146.06	54
06 :01 :20.766	145.96	53	145.97	46	...	146.06	54

FIGURE 4.2 – Représentation d'un fichier Quote Data Raw avec la colonne Timestamp correspondant à l'horodatage de la réception de l'état du carnet d'ordres, la colonne bid0 correspondant au meilleur niveau de prix à la vente et la colonne LTV correspondant au volume disponible au au niveau de prix bid0

Le fichier "Quote Data Raw" est une liste d'états du carnet d'ordre décrit par un horodatage, une liste fini (de taille K) de couple (prix, volume) qui sont les K meilleurs prix à l'achat et à la vente associées aux volumes correspondant. Les états du carnet d'ordre dans les données sont des représentations partielles des états du carnet d'ordre dans le marché. Pour des actifs financiers peu liquides et émergents, la liste des K meilleurs prix sont souvent non contigües. En regardant deux lignes consécutives, il est possible de remarquer des différences de volume sur plusieurs niveaux de prix. Cela implique qu'entre deux états consécutifs, il peut y avoir plus d'une arrivée d'évènements.

Il y a un décalage entre l'horodatage de la transaction et l'horodatage de la notification de la modification de l'état du carnet d'ordres par la transaction.

Nous expliquons les procédures pour traiter les données puis nous donnons quelques statistiques descriptives des données.

4.1.1 Traitement des données

Roulement (Rolling) Les futures possèdent plusieurs maturités simultanément. Nous traitons ce problème en gardant, pour chaque journée, la maturité enregistrant la plus grand somme de tailles de transaction, en éliminant les autres maturités. Nous concaténons les données de transaction en appliquant le traitement et nous faisons correspondre les données des états de carnet d'ordre.

Timestamp	bid 0	Volume bid 0	Ask 0	Volume Ask 0	...	Ask 9	Volume Ask 9
06 :01 :20.517	145.96	48	145.97	41	...	146.06	54
06 :01 :20.766	145.96	53	145.97	46	...	146.06	47

FIGURE 4.3 – Quote Data Raw

Fusion de transactions Lorsqu'un ordre marché traverse plusieurs niveaux de prix, nous rappelons que dans les données de transaction, cela crée plusieurs transactions. Si nous gardons les données originales de transaction, certains paramètres seront mal estimés. En effet, lorsque nous voulons estimer la variable aléatoire représentant la taille d'un ordre marché lorsqu'elle dépend de l'état, alors un ordre de marché ne pourra jamais dépasser le meilleur niveau de prix du carnet. Nous agrégeons toutes les transactions possédant le mime horodatage et le mime prix de transaction, en sommant la taille des transactions et en gardant le meilleur prix.

Hypothèses sur les données Dans le fichier de données "Quote Data Raw", plusieurs évènements peuvent arriver entre deux lignes consécutives représentant deux états du carnet d'ordres. Nous faisons l'hypothèse qu'au mime niveau de prix entre deux lignes consécutives, il ne peut y avoir qu'un seul évènement qui a produit le changement. Par exemple dans la représentation 4.3, au niveau de prix 146.06, la différence de volume est négative, nous considérons alors qu'il y a eu une annulation de taille 7 au niveau de prix 146.06. Au niveau de prix 146.06, la différence de volume est positive, nous considérons qu'il y a eu un ordre limite de taille 5.

Nous randomisons les évènements entre deux états afin d'éviter une possible dépendance. Nous construisons ainsi les données d'évènements.

Distinction entre une annulation aux meilleurs niveau de prix et une transaction Lorsque nous construisons les données d'évènements, dans le cas d'une différence de volume négative au niveau du bid, nous voulons savoir si l'évènement est une annulation ou une transaction. Pour décider, nous utilisons le fichier de données "Trade Data Raw". Nous prenons les horodatages que l'on note t du fichier de "Trade Data Raw" et nous cherchons dans une fenêtre $[t - \Delta t, t + \Delta t]$ dans le fichier d'évènements, s'il y a un évènement aux meilleurs niveaux de prix, possédant une différence de volume négative et la différence est égale à la taille de la transaction. Cette méthode possède plusieurs buts :

— Cela permet de différencier les annulations et les transactions.

— Cela permet de donner le signe des ordres marchés (une transaction n'a pas de signe). En effet si la transaction coïncide avec une différence de volume au niveau du bid (resp. de l'ask) alors l'ordre marché est une vente (resp. achat).

— Cela permet aussi de connaitre la granularité des données donc la qualité des données.

Nous faisons coïncider 72% des transactions avec les données d'évènements.

4.2 Estimations de paramètres

Pour la spécification de chaque variable aléatoire, nous invitons à lire α^S le paragraphe 2.3.3, α^C le paragraphe 2.3.5, β^C le paragraphe 2.3.4, α^B le paragraphe 2.3.1 et β^B le paragraphe 2.3.2. Nous nous référons au livre de Lejeune [46] pour l'estimation de paramètres.

Nous rappelons la fonction de vraisemblance d'un paramètre θ selon n observations $(x_1, \ldots, x_n)$ indépendants et identiquement distribuées d'une loi l_θ. Nous voulons estimer θ dans le cas discret :

$$L(x_1, \ldots, x_n, \theta) := \prod_{i=1}^{n} \mathbb{P}_\theta[X = x_i]$$

Pour trouver l'estimateur par la méthode du maximum de vraisemblance, nous cherchons le maximum de la fonction L par rapport à θ mais cela suppose d'imposer dans notre cas, une loi à priori. Si nous posons $X_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ et $S_n = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ alors dans le cas où la distribution est log-normale (μ, σ), l'espérance est $\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$ et la variance est $V[X] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1))$. il y a deux paramètres à estimer la moyenne μ et l'écart-type σ, nous trouvons :

$$\hat{\mu} = \log(X_n) - \frac{1}{2}\log\left(\frac{S_n}{X_n^2} + 1\right) \text{ et } \hat{\sigma} = \sqrt{\log\left(\frac{S_n}{X_n^2} + 1\right)},$$

Dans le cas binomiale (n, p) ($\mathbb{E}[X] = np$ et $V[X] = np(1-p)$), il y a un paramètre à estimer p, $\hat{p} = X_n$ avec le paramètre n donné.

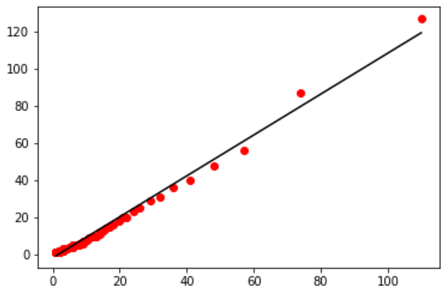
Dans le cas géométrique p ($\mathbb{E}[X] = \frac{1-p}{p}$ et $V[X] = \frac{1-p}{p^2}$), il y a un paramètre à estimer p, $\hat{p} = \frac{1}{X_n}$.

Dans un premier temps pour les cas indépendants de l'état courant des variables α^S et α^C, nous calculons les estimateurs des paramètres des variables aléatoires $\alpha^B, \beta^{B+}, \beta^{B-}, \alpha^S, \beta^{C-}, \alpha^C$ pour une loi log-normale, une loi binomiale et une loi géométrique. Nous pouvons calculer les intervalles de confiance asymptotique au seuil α de nos estimateurs à l'aide de $IC_\alpha = [\hat{\theta} - \frac{x_\alpha}{\sqrt{nI(\hat{\theta})}}, \hat{\theta} + \frac{x_\alpha}{\sqrt{nI(\hat{\theta})}}]$ avec x_α le $(100 - \alpha)$ percentile de la loi normale et $I(\hat{\theta})$ la matrice d'information de Fisher pour $\hat{\theta}$ (cf chapitre 7 section 3 de [46].

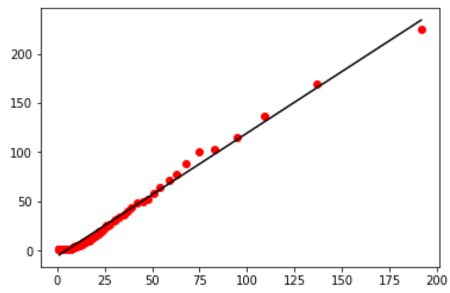
Afin de choisir une des lois, nous réalisons des tests d'adéquation du khi-deux.

	Log-normale $\hat{\mu}, \hat{\sigma}$	Binomiale $\hat{p}$	Géométrique $\hat{p}$
α^S	$2.44, 1.32\ (0.023, 0.043)$	$0.025\ (2 \times 10^{-5})$	$0.036\ (6 \times 10^{-4})$
α^B	$1.229, 1.552\ (0.008, 0.018)$	$0.008\ (1.3 \times 10^{-6})$	$0.088\ (4 \times 10^{-4})$
α^C	$1.041, 1.647\ (0.010, 0.024)$	$1.352\ (1.4 \times 10^{-6})$	$0.091\ (5 \times 10^{-4})$
β^{C-}	$0.835, 0.753\ (0.005, 0.005)$	$0.340\ (9 \times 10^{-6})$	$0.327\ (1.6 \times 10^{-3})$
β^{B-}	$0.623, 0.837\ (0.005, 0.005)$	$0.294\ (7 \times 10^{-6})$	$0.378\ (1.6 \times 10^{-3})$
β^{B+}	$0.001, 0.060\ (0.002, 1.9 \times 10^{-4})$	$0.001\ (3 \times 10^{-5})$	$0.997\ (0.002)$

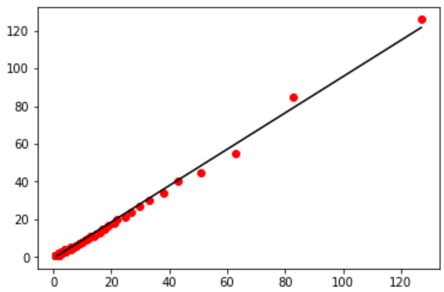
FIGURE 4.4 – Calcul des estimateurs des paramètres et du terme intervenant dans l'intervalle de confiance asymptotique ($\frac{x_\alpha}{\sqrt{nI(\hat{\theta})}}$) donné entre parenthèses



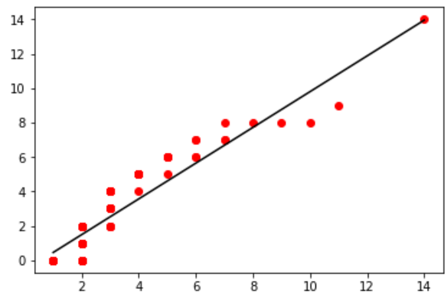
(a) pour la variable aléatoire α^B



(b) pour la variable aléatoire α^S



(c) pour la variable aléatoire α^S



(d) pour la variable aléatoire β^{B-}

FIGURE 4.5 – Diagramme Quantile-Quantile contre la loi log-normale de paramètres donnés dans 4.4

- Pour α^B, α^S, α^C, nous ne pouvons pas rejeter l'hypothèse nulle (égalité entre les données et la loi log-normale avec les paramètres estimés).
- Pour β^{B-} et β^{C-}, nous ne pouvons pas rejeter l'hypothèse nulle (égalité entre les données et les lois log-normale et géométrique avec les paramètres estimés).
- Pour β^{B+}, nous ne pouvons pas rejeter l'hypothèse nulle (égalité entre les données et la loi binomiale avec les paramètres estimés).

Les résultats des tests d'adéquation semblent cohérents car les variables α^B, α^S, α^C sont de même nature, ils représentent des tailles d'ordres. Il en est de même pour β^{B-} et β^{C-} qui correspondent aux placements d'ordres qui ne modifient pas le prix du bid. La variable aléatoire β^{B+} correspond aux placements d'ordres dans le spread. L'adéquation avec la loi binomiale, dont le support est fini, est en accord avec la bornitude du spread.

Pour estimer δ, nous devons introduire une fenêtre Δt pour considérer l'évènement $\{\delta = N\}$. Notre modèle est à temps discret et deux temps d'arrivée consécutifs des évènements sont espacées par Δt constant. Ainsi, nous calculons la fréquence $f_{\{\delta=A\}}$ d'avoir un évènement de type $A = \{S, C, B, N\}$ dans une fenêtre Δt :

$$f_{\{\delta=A\}} = \frac{\text{Occurrence de l'évènement de type A } \times \Delta t}{\text{T}}$$

avec T la durée de l'échantillon considérée.

Pour estimer les paramètres des variables aléatoires ϵ^B et ϵ^C, nous supposons que les variables ϵ^B et ϵ^C suivent des lois de Bernoulli. Nous donnons le terme intervenant dans l'intervalle de confiance pour les estimateurs. Nous calculons les fréquences (la proportion) en comptant le nombre d'occurrence de l'évènement

— d'avoir un ordre limite possédant un niveau de prix inférieur ou égal au prix du bid ($\epsilon^B = 0$),

— d'avoir un ordre limite possédant un niveau de prix supérieur au prix du bid ($\epsilon^B = 1$),

— d'avoir une annulation totale au niveau du bid, c'est à dire $\alpha^C = \mathsf{n}^{|\mathfrak{b}(\mathsf{n})}$ et $\beta^C = \mathfrak{b}(\mathsf{n})$ si l'état du carnet est n ($\epsilon^C = 1$)

— et d'avoir une annulation partielle, c'est à dire $\alpha^C \neq \mathsf{n}^{|\beta^C}$ ou $\beta^C \neq \mathfrak{b}(\mathsf{n})$ si l'état du carnet est n ($\epsilon^C = 0$)

Nous obtenons pour l'estimateur du paramètre de la variable aléatoire ϵ^B, $\hat{p}_{\epsilon^B} = 0.981(IC = 7.5 \times 10^{-4})$ et pour ϵ^C, $\hat{p}_{\epsilon^C} = 0.991(IC = 5 \times 10^{-4})$. Nous réalisons des tests d'adéquation du khi-deux. Pour ϵ^B, ϵ^C, α^C, nous ne pouvons pas rejeter l'hypothèse nulle (égalité entre les données et la loi de Bernoulli avec les paramètres estimés).

Pour estimer α^S dans le cas dépendant (voir le paragraphe 2.3.3), nous utilisons la moyenne temporelle empirique de $\mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}$, c'est à dire $\lim_{n\to\infty} \sum_{k=0}^{n} \mathsf{n}_k^{|\mathfrak{b}(\mathsf{N})-l}$. Si α^S est uniforme alors α^S est uniforme dans $\{1, \ldots, \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}\}$. Il n'y a pas de paramètre à estimer. En revanche, si α^S suit une loi binomiale alors α^S suit une loi binomiale de paramètres $(\mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}, p)$. Nous estimons p par une méthode de maximum de vraisemblance avec la moyenne temporelle empirique de $\mathsf{n}^{|\mathfrak{b}(\mathsf{n})-l}$.

Pour estimer α^C dans le cas dépendant (voir le paragraphe 2.3.5), nous utilisons la moyenne temporelle empirique de $\Delta_{\beta^C}\mathsf{n} - 1$, c'est à dire $\lim_{n\to\infty} \sum_{k=0}^{n} (\Delta_{\beta_k^C}\mathsf{n}_k - 1)$. Si α^C est uniforme alors α^C est uniforme dans $\{1, \ldots, \Delta_{\beta^C}\mathsf{n} - 1\}$. Il n'y a pas de paramètre à estimer. En revanche, si α^C suit une loi binomiale alors α^S suit une loi binomiale de paramètres $(\Delta_{\beta^C}\mathsf{n} - 1, p)$. Nous estimons p par une méthode de maximum de vraisemblance avec la moyenne temporelle empirique de $\Delta_{\beta^C}\mathsf{n} - 1$.

4.3 Calibration par rapport au problème de récurrence

D'après la section 3.2, nous exhibons une condition sur les paramètres pour connaitre la nature de la chaine de Markov **N**. Une condition suffisante est donnée par :

Soit $\mathsf{x} \in \mathcal{N}$, $D = \mathbb{E}^{\mathsf{x}}[\alpha^b] \ + \mathbb{E}^{\mathsf{x}}[\beta^b]$ et pour $\mathsf{x}^{|1}$ assez grand, si les paramètres vérifient cette inégalité :

$$p_b(\mathsf{x})D \leq p_c(\mathsf{x})\mathbb{E}^{\mathsf{x}}[\tilde{\alpha}^c],$$

alors **N** est récurrente. Nous cherchons si les données du marché donnent une chaine récurrente ou non.

Nous testons l'inégalité avec les paramètres et cherchons s'il existe un entier N pour caractériser la notion d'assez grand pour $\mathsf{x}^{|1}$. Afin d'étudier cette inégalité, nous représentons sur une figure, pour chaque endroit de l'espace $\mathcal{N}$, si la condition est vérifiée. Nous utilisons les caractéristiques principales de $\mathcal{N}$, c'est à dire le prix du bid $\mathfrak{b}(\mathsf{x})$ et le volume total $\mathsf{x}^{|1}$ pour représenter graphiquement des éléments de $\mathcal{N}$. $\mathfrak{b}(\mathsf{x})$ et $\mathsf{x}^{|1}$ suffisent car ce sont les caractéristiques misent en jeu pour la question de récurrence. Dans les données, nous n'avons pas accès à$\mathsf{x}^{|1}$, nous choisissons de considérer $\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}$ avec p=9. Sur chaque figure, nous représentons en abscisse le volume total $(\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p} - \mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min})/120$ et en ordonnée, le prix du bid $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}_{min}$ avec $\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min}$ la plus petite valeur de $\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}$ dans l'échantillon.

Nous étudions les deux cas pour α^C : indépendant et dépendant de l'état courant du carnet. Nous utilisons les lois et les paramètres obtenues de la section 4.2. Pour les variables β^{C-} et β^{B-}, nous utilisons par défaut la loi log-normale avec les paramètres estimés dans le tableau 4.4. Pour les autres variables, il n'y a pas d'ambiguïté voir résultat des tests d'adéquations 4.2.

α^C indépendant de l'état du carnet Lorsque α^C est indépendant de l'état du carnet, l'inégalité ne dépend pas de x.

Dans le cas où α^C, β^C suivent une loi log-normale de paramètres données dans le tableau4.4 alors la condition de récurrence est vérifiée nulle part (Figure 4.6).

α^C dépendant de l'état du carnet Nous considérons plusieurs cas : dans le cas où α^C suit une loi uniforme, nous retombons dans le cadre précédent. L'inégalité est vrai pour tout $\mathsf{x} \in \mathcal{N}$. La représentation est la mime que 4.6.

Dans le cas où α^C suit une loi binomiale, nous avons un cadre intéressant. Nous avons un ensemble de positions qui vérifie la condition et un autre ensemble qui ne la vérifie pas. Les deux ensembles sont disjoints et sont séparés par une frontière. La frontière est différente selon le choix de β^C (Figure 2a, 4.7a et 4.7b)

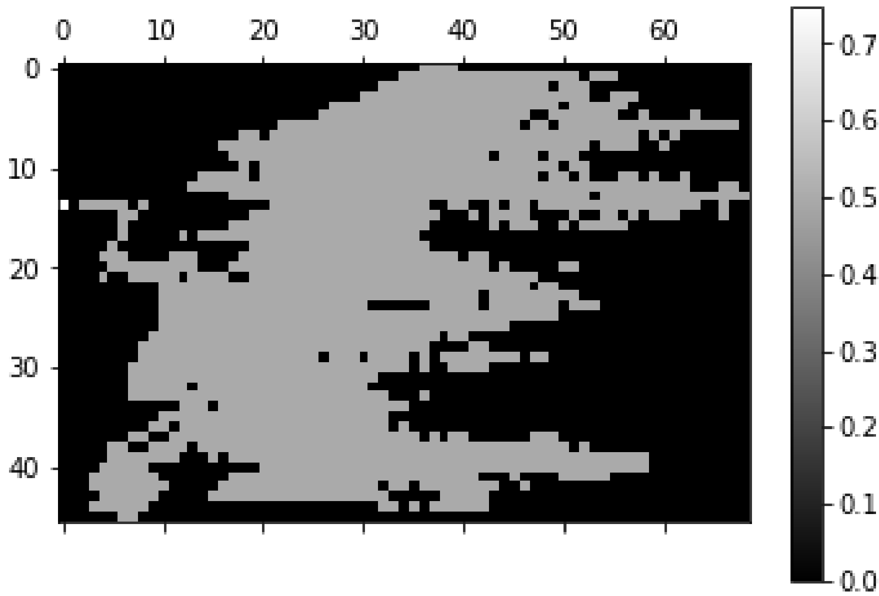
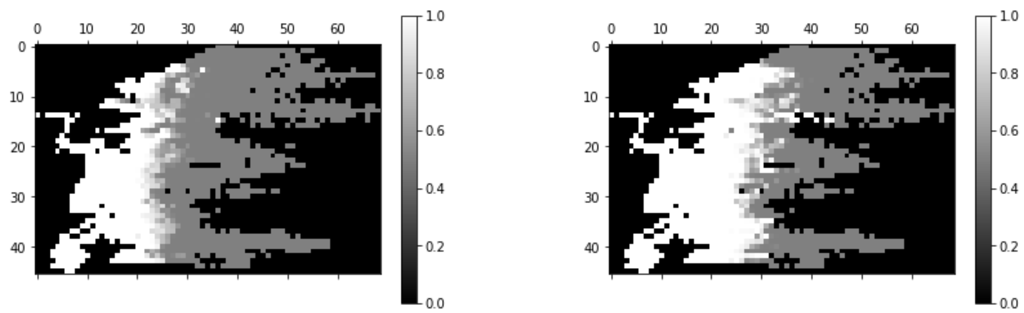


FIGURE 4.6 – Représentation des positions qui vérifient la condition de récurrence : les parties noires sont les positions non atteintes par les données du marché et la partie grise correspond aux positions vérifiant la condition. En abscisse, nous avons le volume total $(\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p} - \mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min})/120$ et en ordonnée, le prix du bid $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}_{min}$. α^C suit une loi log-normale et β^C suit une loi log-normale de paramètres données dans le tableau 4.4



(a) β^C suit une loi géométrique

(b) β^C suit une loi log-normale

FIGURE 4.7 – Représentation des positions qui vérifient la condition de récurrence : les parties noires sont les positions non atteintes par les données du marché, la partie blanche correspond aux positions ne vérifiant pas la condition, la partie grise correspond aux positions vérifiant la condition. En abscisse, nous avons le volume total $(\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p} - \mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min})/120$ et en ordonnée, le prix du bid $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}_{min}$

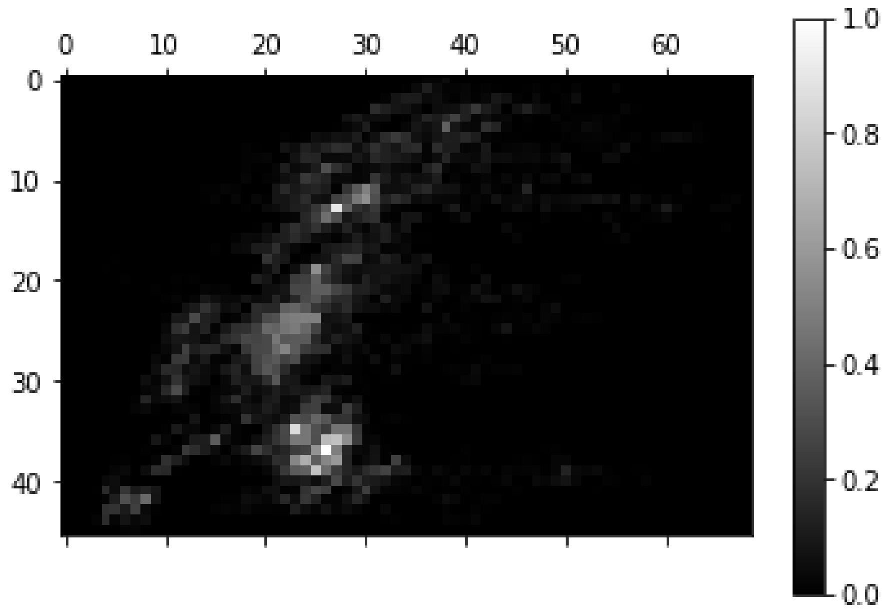


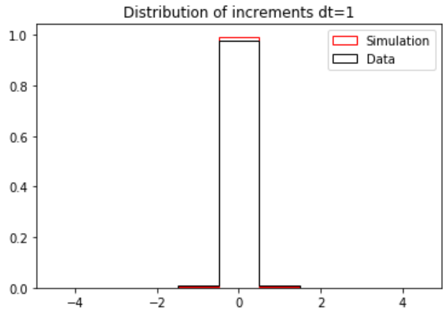
FIGURE 4.8 – Représentation de l'espace $\mathcal{N}$ des positions les plus visitées. Plus les zones sont blanches, plus elles sont visitées. En abscisse, nous avons le volume total $(\mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p} - \mathsf{x}^{|\mathfrak{b}(\mathsf{x})-p}{}_{min})/120$ et en ordonnée, le prix du bid $\mathfrak{b}(\mathsf{x}) - \mathfrak{b}_{min}$

La frontière entre les deux zones caractérise la notion de "assez grand" dans la condition de récurrence. La zone vérifiant la condition dans le cas de la loi géométrique est plus large (Figure 4.7a). La partie grise correspond à la zone où le drift est négatif, donc revient vers le "centre" de l'espace $\mathcal{N}$. Nous traçons une représentation de l'espace $\mathcal{N}$ pour caractériser le "centre" de l'espace $\mathcal{N}$ pour illustrer. Le "centre" de l'espace $\mathcal{N}$ a une forme de croissant correspondant à la partie proche de la frontière.

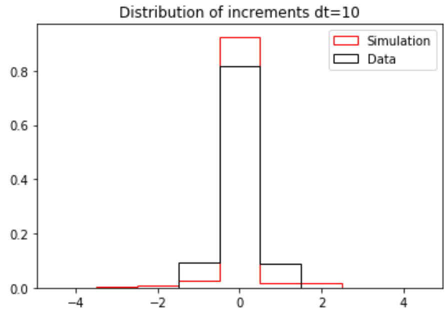
Conclusion Le cas α^C dépendant de l'état et suivant une loi binomiale semble plus correspondre à une certaine réalité notamment en décrivant le caractère récurrent en séparant en deux zones l'espace $\mathcal{N}$. Nous allons examiner de manière empirique, les inconvénients et les avantages du modèle simulé par rapport aux faits stylisés dans la section 4.4 et par rapport aux critères de liquidité de marché dans la section 4.5.

4.4 Faits stylisés

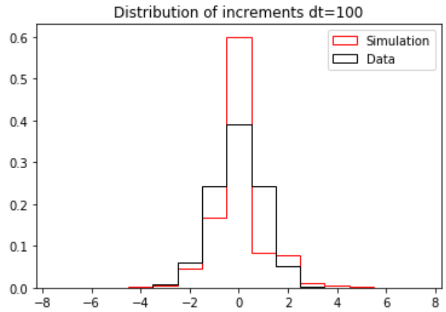
La section précédente nous indique l'importance de la dépendance de α^C par rapport à l'état. Nous considérons ce cas là. Nous utilisons les lois et les paramètres obtenues de la section 4.2. Pour les variables β^{C-} et β^{B-}, nous utilisons par défaut la loi log-normale avec les paramètres estimés dans le tableau 4.4. Pour les autres variables, il n'y a pas d'ambiguïté (voir les résultats des tests d'adéquations 4.2).



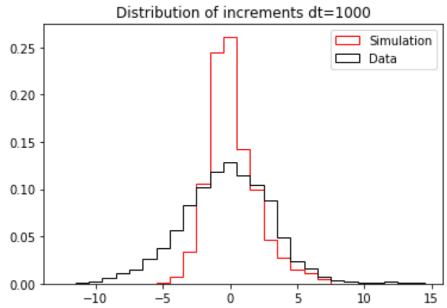
(a) $dt = 1$



(b) $dt = 10$



(c) $dt = 100$



(d) $dt = 1000$

FIGURE 4.9 – Distribution des incréments de prix pour différents Δt

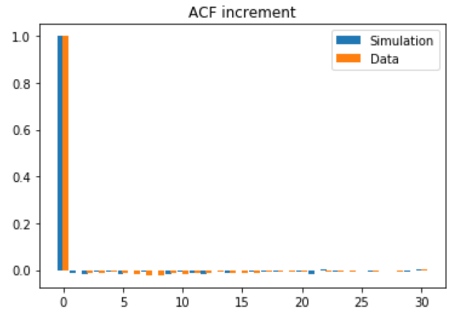
Nous nous référons à la sous section 1.2.1 du chapitre 1 pour une introduction sur les faits stylisés.

Afin d'étudier la distribution des incréments de prix, nous étudions la distribution empirique des incréments de prix. Pour définir la distribution empirique des incréments de prix, nous simulons une trajectoire de N et nous prenons la série des $R_n(\mathsf{N}_n, \Delta t) := (\mathfrak{b}(\mathsf{N}_{n+\Delta t}) - \mathfrak{b}(\mathsf{N}_n))$ avec $\Delta t \in \mathbb{N}^*$ une échelle de temps.

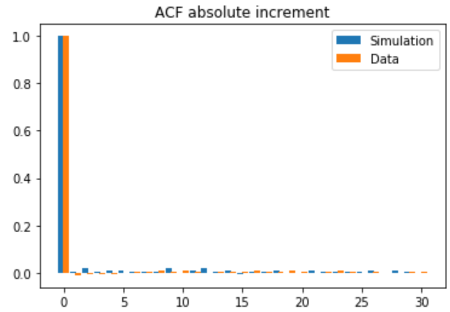
Distribution des incréments de prix Nous traçons l'histogramme de la série des (R_n) pour différentes valeurs de Δt avec $\Delta t = 1$ (figure 4.9a), $\Delta t = 10$ (figure 4.9b), $\Delta t = 100$ (figure 4.9c),$\Delta t = 1000$ (figure 4.9d). Nous calculons aussi la série des R_n pour les données de marché.

Pour les petites valeurs de Δt, la distribution empirique simulée correspond à la distribution des données du marché. Cependant plus Δt est grand, moins la distribution concorde avec les données du marché. La masse de la distribution simulée des R_n est concentrée au centre ce qui signifie que les données simulées produisent moins de variance pour les incréments. La non concordance est dû à un problème de choix de fenêtre de Δt. Nous observons aussi une asymétrie au niveau de la distribution simulée qui est dû à la sensibilité du choix des lois de $\beta^{C-}, \beta^{B-}, \beta^{B+}$ et des paramètres à estimer.

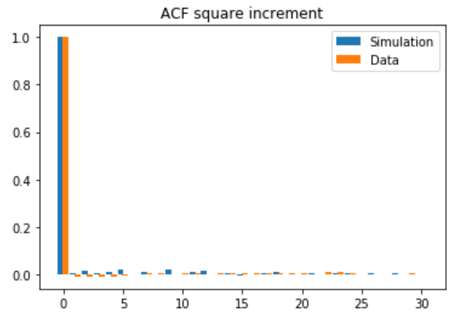
Autocorrélation des incréments de prix Nous calculons l'auto-corrélation empirique des incréments de prix à l'aide de la série des (R_n). Pour tracer l'autocorré-



(a) Fonction d'autocorrélation des incréments de prix



(b) Fonction d'autocorrélation de la valeur absolue des incréments de prix



(c) Fonction d'autocorrélation du carré des incréments de prix

FIGURE 4.10 – Représentation de différentes fonctions d'autocorrélation pour $dt = 1$

logramme, nous calculons pour $\tau \in \{0, K\}$:

$$C(\tau, \Delta t) = \frac{\sum_n R_n(\Delta t) R_{n+\tau}(\Delta t) - \sum_n R_n(\Delta t)^2}{V(\Delta t)}$$

avec $V(\Delta t)$ la variance de la série des $(R_n(\Delta t))$ Nous obtenons pour $\tau = 0$, $C(\tau, \Delta t) = 1$. Puis nous traçons donc la fonction d'auto-corrélation sur la figure 4.10a avec pour abscisse la valeur de τ. Nous n'avons pas de valeurs positives significatives pour l'auto-corrélation des incréments de prix du bid, ce qui suit l'intuition évoqué dans le paragraphe 1.2.1.1.

Regroupement de la volatilité Nous calculons l'autocorrélation empirique du carré des incréments de prix à l'aide de la série des (R_n). Pour tracer l'autocorrélogramme, on calcule pour $\tau \in \{0, K\}$:

$$C_2(\tau, \Delta t) = \frac{\sum_n R_n^2(\Delta t) R_{n+\tau}^2(\Delta t) - \sum_n R_n^2(\Delta t) R_n^2(\Delta t)}{V_2(\Delta t)}$$

avec $V_2(\Delta t)$ la variance de la série des $(R_n^2(dt))$ et

$$C_{abs}(\tau, \Delta t) = \frac{\sum_n |R_n(\Delta t)||R_{n+\tau}(\Delta t)| - \sum_n |R_n(dt)|^2}{V_{abs}(\Delta t)}$$

avec $V_{abs}(\Delta t)$ la variance de la série des $(|R_n(\Delta t)|)$ On a pour $\tau = 0$, $C_2(\tau, \Delta t) = 1$. Nous n'avons pas de valeurs positives significatives pour l'auto-corrélation du carré sur la figure 4.10c et de la valeur absolu sur la figure 4.10b des incréments de prix du bid.

4.5 Critères de liquidité de marché

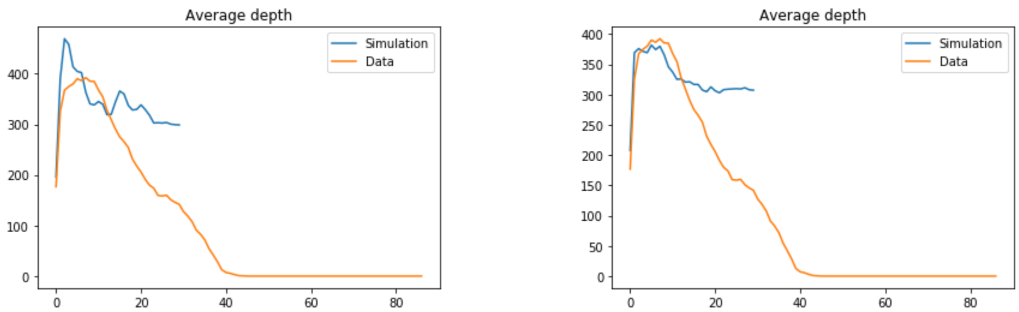
Nous utilisons les lois et les paramètres obtenues de la section 4.2. Pour les variables β^{C-} et β^{B-}, nous utilisons par défaut la loi log-normale avec les paramètres estimés dans le tableau 4.4. Pour les autres variables, il n'y a pas d'ambiguïté (voir les résultats des tests d'adéquations 4.2). Nous nous référons à la sous section 1.2.1 du chapitre 1 pour une introduction sur les faits stylisés.

Profil moyen Nous définissons le profil moyen comme la moyenne temporelle des volumes à chaque niveau de prix. Nous prenons comme référence le prix du bid :

$$\langle \Delta_{\mathfrak{b}(\mathsf{X})-p}\mathsf{X}\rangle = \lim_{n\to\infty} \sum_{k=0}^{n} \Delta_{\mathfrak{b}(\mathsf{X}_k)-p}\mathsf{X}_k.$$

Pour calculer cette quantité, nous simulons une trajectoire de la chaine de Markov $\mathbf{N}$ et nous gardons en mémoire $\Delta_{\mathfrak{b}(\mathsf{X}_k)-p}\mathsf{N}$ pour tout p niveau de prix et tout $k \in \mathbb{N}^*$ le nombre de pas de temps pris pour la simulation. Dans le cas de chaine de Markov récurrente positive, la valeur empirique converge bien vers le profile moyen théorique. Nous retrouvons bien le début de la forme du profil moyen avec une courbe croissante jusqu'à un maximum puis une décroissance. Sur les figures 4.11a et 4.11b, nous avons représenté en abscisse, les niveaux de prix en prenant comme référence le prix du bid, $\mathfrak{b}(\mathsf{X}_k) - p$, c'est à dire pour $p = 0$ nous avons le prix du bid et en ordonnée, nous représentons la moyenne temporelle du volume dans le carnet. Pour $p = 0$, nous avons Dans le cas où α^S dépend de l'état du carnet (Figure 4.11b), la courbe empirique s'accorde plus avec le profil moyen calculé par les données du marché en comparaison avec le profil moyen empirique avec α^S indépendant de l'état du carnet (Figure 4.11a). Cependant, la queue des profils ne décroit pas ce qui est dû au manque de données sur le carnet sur les volumes 10 niveaux en dessous du prix du bid. Pour avoir une meilleur concordance, Bouchaud [15] prend une distribution empirique de placement de prix à priori. Ils ne prennent pas de loi théorique, n'estime pas de paramètres et ne calibre pas par rapport aux données. Abergel [2] modélise et calibre à chaque niveau de prix un processus de Poisson.

Dans notre cas, nous avons modélisé le prix de placement d'ordre limite et le prix de placement d'annulation par une loi que nous calibrons avec les données de marché. Nous perdons de l'information en calibrant de cette manière mais nous



(a) α^S indépendant de l'état du carnet (b) α^S dépendant de l'état du carnet

FIGURE 4.11 – Représentation du profil moyen

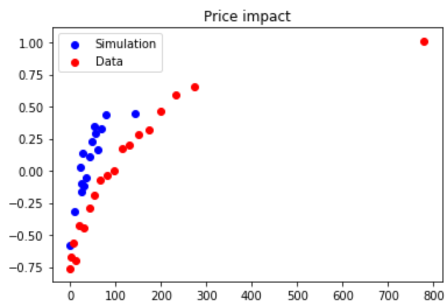
avons réduit le nombre de paramètres à estimer. De plus, nous obtenons une bonne concordance du profil moyen pour les valeurs proches du prix du bid.

Impact de prix Nous définissons l'impact de prix, dans notre cas :

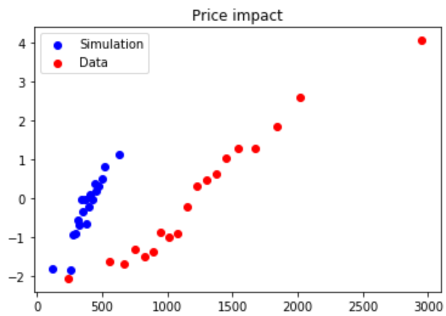
$$I_{ind}(\Delta t, q) := \mathbb{E}[R(\mathsf{N}_t, \Delta t)|\delta_t = \ \mathrm{S}\ , q_t = q]$$

Nous n'étudions dans notre cas, l'impact individuel de prix, c'est à dire l'espérance du changement de prix après Δt provoquer par un ordre marché de taille q. Dans notre cas, le prix considéré sera le prix du bid. Pour calculer cette quantité, nous simulons une trajectoire de la chaine de Markov $\mathbf{N}$ et gardons en mémoire l'ensemble des ordres marché, leur taille. à Δt fixé, nous récupérons pour chaque ordre marché l'incrément de prix $R(\mathsf{N}_t, \Delta t) := \mathfrak{b}(\mathsf{N}_{t+\Delta t}) - \mathfrak{b}(\mathsf{N}_t)$. Nous avons un problème de représentation dû à la distribution des tailles des ordres marchés qui possèdent une queue épaisse. Pour avoir une bonne représentation, nous prenons les quantiles de la distribution et nous calculons pour chaque quantile la moyenne du changement de prix. Nous obtenons dans le cas où α^S indépendant de l'état du carnet les courbes suivantes représentés sur les figures pour $\Delta t = 100$ 4.12a et pour $\Delta t = 1000$ 4.12b. La courbe empirique simulée ne concorde pas avec la courbe des données de marché.

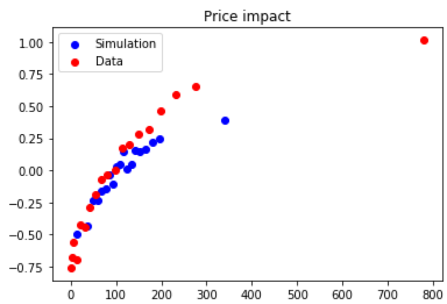
Dans le cas où α^S dépendant de l'état du carnet, nous représentons l'impact de prix sur les figures pour $\Delta t = 100$ 4.12c et pour $\Delta t = 1000$ 4.12d. Nous réussissons à avoir une meilleure concordance entre la courbe simulée et la courbe des données de marché. Cela peut s'expliquer notamment par la sélection de liquidité dû aux traders. Les traders choisissent d'envoyer des ordres de marché de grande taille lorsque la liquidité est élevée. En faisant dépendre α^S de l'état courant, nous caractérisons ce phénomène qui caractérise la concavité de la courbe de l'impact de prix. Nous retrouvons bien la concavité dans le cas α^S dépendant.



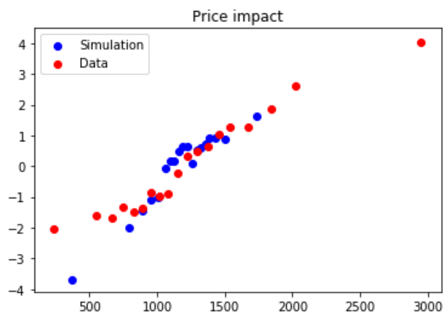
(a) α^S indépendant de l'état du carnet avec $\Delta t = 100$



(b) α^S indépendant de l'état du carnet avec $\Delta t = 1000$



(c) α^S dépendant de l'état du carnet avec $\Delta t = 100$



(d) α^S dépendant de l'état du carnet avec $\Delta t = 1000$

FIGURE 4.12 – Représentation de l'impact de prix

Conclusion

Nous avons donné une procédure pour traiter les données. Nous expliquons, l'intérêt de la dépendance de α^C par rapport à l'état courant n dans le cadre du problème de récurrence. α^C ne doit pas suivre une loi uniforme et une loi binomiale permet d'avoir une chaine récurrente. Le choix de β^{C-} n'est pas important dans ce problème. Nous n'obtenons pas de résultats satisfaisants pour les faits stylisés notamment pour la distribution des incréments de prix. Mais nous arrivons à un résultat intéressant pour les critères de liquidité et nous montrons l'intérêt de la dépendance de la variable aléatoire de α^S par rapport à l'état courant n.

Un problème plus général de calibration consisterait à calibrer selon les faits stylisés et les critères de liquidité.

Soit $(w_i)_{i \in 1,\dots,5} > 0$ fixé, notons l'ensemble Θ^{calib} les probabilités dans Θ^{rec} qui minimisent :

$$\begin{aligned}&w_1 d(R^M(p,\Delta t), R^Q(p,\Delta t)) + w_2 d(C_M(\tau,\Delta t) - C^Q(\tau,\Delta t)\\&w_3 d(C_2^M(\tau,\Delta t), C_2^Q(\tau,\Delta t))\\&w_4 d(M^M(p) - M^Q(p)) + w_5 d(I^M(dt,q), I^Q(dt,q))\end{aligned}$$

avec $d(X^M, X^Q)$ une distance entre la métrique donnée X^M par les données et la métrique donnée par le modèle.

Alors le problème de calibration consisterait à trouver les paramètres dans l'ensemble des probabilités $\mathbb{Q}$ dans Θ^{calib}. Le calcul explicite des métriques, dépendant des paramètres du modèle, est nécessaire dans ce problème de calibration.

Chapitre 5

Optimal liquidation in a limit order book

Contents

Introduction

This chapter deals with the optimal liquidation problem in a one-sided limit order book framework, we presented in 2.

In section 1.4, we describe the general optimal liquidation problem. In this framework, the (bid) price is derived through the depth, the state of the limit order book by the operator bid $\mathfrak{b}$. As our knowledge, this problem is not studied in the literature. Abergel et al [1] study a related problem in the market making context with a limit order book modeling.

In the case of limit order book framework, short-time scaled is relevant, permanent cost can be omitted. We choose the discrete time framework, so the strategy is the choice of the quantities $x_0, \ldots, x_K$. In the introduction of chapter 2, we showed the interest of the depth representation with the transaction gain function. As opposed to the optimal liquidation problem in a one-sided order book shape framework as Predoiu et al [56], we don't assume a reference price (semi-martingale).

In the section 5.1, we explain the optimal liquidation problem by defining the strategies, the gain function and the value function. In the section 5.2, we give the dynamic programming principle, we use for solving the one step model. The one step model is the optimal control with two instants. In order to give some intuition, we give an algorithm for the one step model and draw the strategy and the cost function with respect to the remaining quantity to sell. With the conjecture, we characterize the optimal strategy in the case of one step model in the section 5.3.

5.1 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a discrete filtration $\mathbb{F} = (\mathcal{F}_k)_{k\in\{0,\ldots,T\}}$ where $T \in \mathbb{N}^*$ is a finite horizon. We assume that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathbb{F}$. We consider an order-driven market, in which there is a **strategic trader** who has to sell $\mathsf{X} \geq 0$ shares before T. As we shall restrict her strategies to market orders, she will acts only on a one-sided Bid limit order book through sell market orders. Other investors may submit sell market orders, buy limit orders or may cancel limit orders.

5.1.1 Model settings

5.1.1.1 Order book mechanism

We assume that the dynamic of the limit order book will follow the same model as the one described and studied in the previous parts. We recall some notations.

1) $\mathcal{N}$ is the set of non negative and non increasing integer sequences $\mathsf{n} = (\mathsf{n}^{|1}, \mathsf{n}^{|2}, \ldots)$ such that there exists an $i \geq 0$ such that $\mathsf{n}^{|j} = 0$ from any $j \geq i$.

   The depth of the LOB n is an element of $\mathcal{N}$. Notice that n^1 represents the total volume of limit buy orders in the LOB.

2) The **sell operator** $\mathsf{T}^S : \mathcal{N} \times \mathbb{N} \to \mathcal{N}$ is defined by :

$$\mathsf{T}^S(\mathsf{n}, a) = (\mathsf{n} - a)^+ = ((\mathsf{n}^{|1} - a)^+, (\mathsf{n}^{|2} - a)^+, \ldots)$$

   It represents the order book after the arrival of sell order with size a.

3) The **buy operator** $\mathsf{T}^B : \mathcal{N} \times \mathbb{N} \times \mathbb{N}^* \to \mathcal{N}$ is defined by :

$$\mathsf{T}^B(\mathsf{n}, a, b) := \begin{cases} \mathsf{n}^{|i} + a & i \leq b \\ \mathsf{n}^{|i} & \text{otherwise} \end{cases}$$

It represents the order book after an arrival of buy order with a buy size of a and a buy price level of b.

4) The **cancel operator** $\mathsf{T}^C : \mathcal{N} \times \mathbb{N} \times \mathbb{N}^* \to \mathcal{N}$ is defined by :

$$\mathsf{T}^C(\mathsf{n}, a, b) := \begin{cases} \mathsf{n}^{|i} - a \vee \Delta_b \mathsf{n} = \mathsf{n}^i + (-\Delta_b \mathsf{n} \wedge -a) & i \leq b \\ \mathsf{n}^{|i} & \text{otherwise} \end{cases}$$

It represents the order book after an arrival of cancel order at a price level b and with size a.

With these operators, the dynamic of the limit order book may be described. The dynamic of the limit order book is the sequence of event arrival or the composition of corresponding events operators.

5) The **Bid Price** $\mathfrak{b} : \mathcal{N} \to \mathbb{N}^*$ is defined by

$$\mathfrak{b}(\mathsf{n}) = max\{i \in \mathbb{N}^* / \mathsf{n}^{|i} > 0\}$$

or

$$\mathfrak{b}(\mathsf{n}) = \sum_{i=1}^{+\infty} \mathbb{1}_{\{\mathsf{n}^{|i} > 0\}}$$

After an event arrival, we can compute the new Bid price. Its dynamic is an obvious consequence of the limit order book dynamic.

We define the **next Bid Price** as the new Bid price induced by sell order arrival $\mathfrak{b}^S : \mathcal{N} \times \mathbb{N} \to \mathbb{N}^*$ by

$$\mathfrak{b}^S(\mathsf{n}, a) := max\{i \in \mathbb{N}^* / \mathsf{n}^{|i} > a\} = \mathfrak{b}(\mathsf{T}^S(\mathsf{n}, a))$$

6) We define the operator $\Delta : \mathcal{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\Delta_p \mathsf{n} := \Delta(\mathsf{n}, p) = \mathsf{n}^{|p} - \mathsf{n}^{|p+1}, \qquad \text{for } \mathsf{n} \in \mathbb{N}$$

$\Delta_p \mathsf{n}$ represents the **volume of limit buy orders at the level price** p.

Randomness flow :We denote by
$(\delta_k)_{k \in \{0,\dots,T\}}$ the sequence of random variables corresponding to the events type.
$(\alpha_k^S)_{k \in \{0,\dots,T\}}$ the sequence of random variables corresponding to the size of sell orders.

$(\alpha_k^B)_{k\in\{0,\ldots,T\}}, (\beta_k^B)_{k\in\{0,\ldots,T\}}$ the sequence of random variables corresponding to the size and the price level of buy orders.
$(\alpha_k^C)_{k\in\{0,\ldots,T\}}, (\beta_k^C)_{k\in\{0,\ldots,T\}}$ the sequence of random variables corresponding to the size and the price level of cancel orders.

5.1.2 The control problem

5.1.2.1 Control strategies

For the strategic trader, a **control strategy** $Q = (Q_k)_{k\in\{0,\ldots,T\}}$ is a predictable and non decreasing process where, for all $k \in \{0, \ldots, T\}$, Q_k is a random variable valued in $\mathbb{N}$ and $Q_0 = 0$.
We denote by $\Delta Q_k := Q_{k+1} - Q_k$ the quantity of assets sold at time k, whereas Q_k is the number of shares sold by the strategic trader up to time k^-.

When the strategic trader knows the depth n, she may interact instantaneously with sell orders. The **controlled dynamics of the LOB** is given by the following equation :

$$\begin{aligned} \mathsf{N}_{k+1}^{|i} &= \check{\mathsf{N}}_{k+}^{|i} - \mathbb{1}_{\{\delta_k=S\}}(\alpha_k^S \wedge \check{\mathsf{N}}_{k+}^{|i}) \\ &\quad -\mathbb{1}_{\{\delta_k=C\}}\mathbb{1}_{\{\beta_k^C\leq i\}}(\alpha_k^C \wedge \Delta_{\beta_k^C}\check{\mathsf{N}}_{k+}^{|i}) \qquad \text{for all } (i,k) \in \mathbb{N}^* \times \{0,...,T-1\} \\ &\quad +\mathbb{1}_{\{\delta_k=B\}}\mathbb{1}_{\{\beta_k^B\leq i\}}\alpha_k^B, \end{aligned} \tag{5.1}$$

where we have defined the process representing the LOB just after the strategic trader sells :

$$\check{\mathsf{N}}_{k+}^{|i} = (\mathsf{N}_k^{|i} - \Delta Q_k)^+$$

We also consider that the strategic trader has the obligation to respect the **liquidation constraint**. The number of shares sold by the strategic trader has to be as close as possible to X at the horizon time T. However, the liquidity could fall and the strategic trader could not respect her inventory constraint. For $k \in \{1, T\}$, we impose that an admissible strategy Q should satisfy :

$$\begin{cases} Q_0 = 0 \\ 0 \leq \Delta Q_k \leq \mathsf{N}_{k-1}^{1} \\ \Delta Q_T = \mathsf{N}_T^{|1} \wedge (X - Q_T)(\text{ terminal liquidation contraint}) \end{cases}$$

5.1.2.2 State space and state process

The strategic trader sell orders induces the same mechanism as small investors sell orders. If, at time k, the state of depth is $\mathsf{N}_k = \mathsf{n} \in \mathcal{N}$, and the strategic trader decides to sell c_k. The new state induced by the strategic trader is $\mathsf{T}^S(\mathsf{n}, c_k)$. We can then define the state processes and their dynamics.

First we introduce the state space :

$$\mathcal{Y} := \{0, \dots, T\} \times \mathcal{N} \times \{0, \dots, X\}$$

We then define the state process as follows : $Y := (\tau, \mathsf{N}, Q)$ where τ represents the time and is such that $\tau_k = k$ for any $k \in \{0, \dots, T\}$, N is the LOB process associated to the strategy Q and its dynamic is given by equation (5.1).

$$\begin{aligned} \mathsf{N}^{|i}_{k+1} &= \check{\mathsf{N}}^{|i}_{k+} - \mathbb{1}_{\{\delta_k = S\}}(\alpha^S_k \wedge \check{\mathsf{N}}^{|i}_{k+}) \\ &\quad - \mathbb{1}_{\{\delta_k = C\}} \mathbb{1}_{\{\beta^C_k \le i\}}(\alpha^C_k \wedge \Delta_{\beta^C_k} \check{\mathsf{N}}^{|i}_{k+}) \qquad \text{for all } (i,k) \in \mathbb{N}^* \times \{0, ..., T-1\} \\ &\quad + \mathbb{1}_{\{\delta_k = B\}} \mathbb{1}_{\{\beta^B_k \le i\}} \alpha^B_k, \end{aligned} \tag{5.2}$$

where we have defined $\check{\mathsf{N}}^{|i}_{k+} = (\mathsf{N}^{|i}_k - \Delta Q_k)^+$. For $(k, \mathsf{n}) \in \{0, \dots, T\} \times \mathcal{N}$, we denote by $\mathsf{N}^{k,\mathsf{n},Q}$ the solution of equation (5.1) such that $\mathsf{N}^{k,\mathsf{n},Q}_k = \mathsf{n}$.

5.1.2.3 Admissible strategies

For $(k, \mathsf{n}, x) \in \mathcal{Y}$, we are now able to define the set of admissible strategies by

$$\begin{aligned} \mathcal{A}(k, \mathsf{n}, x) := \ & \{Q = (Q_j)_{j \in \{k, \dots, T+1\}} : Q \text{ is a } \mathbb{F} - \text{predictable and non decreasing process s.t.} \\ & Q_k = x, \ \forall j \in \{k, \dots, T-1\}, \ 0 \le \Delta Q_j \le (\mathsf{N}^{k,\mathsf{n},Q})^{|1}_j \\ & \text{and } \Delta Q_T = (\mathsf{N}^{k,\mathsf{n},Q})^{|1}_T \wedge (X - Q_T)\} \end{aligned} \tag{5.3}$$

Notice that, for $(k, \mathsf{n}, x) \in \{0, \dots, T\} \times \mathcal{N} \times [0, \dots, X]$, $\mathcal{A}(k, \mathsf{n}, x)$ is finite.

5.1.2.4 The gain function

Let $k \in \{0, \dots, T\}$ and $\mathsf{n} \in \mathcal{N}$ be the state of the LOB at time k. When the strategic trader submits a sell order of size $q \ge 0$ at time k, the strategic trader sells $q \wedge \mathsf{n}^{|\mathfrak{b}(\mathsf{n})}$ at price $\mathfrak{b}(\mathsf{n})$.
If $q < \mathfrak{b}(\mathsf{n})$, the gain is $q\mathfrak{b}(\mathsf{n})$. The strategic trader does not have direct impact on the Bid price in this case.
If $\mathfrak{b}(\mathsf{n}) - \mathfrak{b}^S(\mathsf{n}, q) = j > 0$, the strategic trader sells at price $\mathfrak{b}(\mathsf{n}) - p$ with p ranging from 0 to j. For each p ranging from 0 to $j-1$, she sells $\Delta_{\mathfrak{b}(\mathsf{n})-p}\mathsf{n}$ and for $p = z$, she sells $q - \mathsf{n}^{|\mathfrak{b}(\mathsf{n})-z}$.
Consequently, we are now able to define the gain function $\mathfrak{g}$ for a sell order of size q in an order book n by :

$$\mathfrak{g}(\mathsf{n}, q) := \sum_{i=1}^{\infty} i(q \wedge \mathsf{n}^{|i} - \mathsf{n}^{|i+1})^{+}$$

Notice that

$$\mathfrak{g}(\mathsf{n}, q) := \sum_{i=1}^{\infty} i(\mathsf{n}^{|i} - \mathsf{n}^{|i+1}) - \sum_{i=1}^{\infty} i(\mathsf{T}^{S}(\mathsf{n}, q)^{|i} - \mathsf{T}^{S}(\mathsf{n}, q)^{|i+1})$$

An integration by parts formula gives $\sum_{i=1}^{\infty} i(\mathsf{n}^{|i} - \mathsf{n}^{|i+1}) = \sum_{i=1}^{\infty} \mathsf{n}^{|i}$.
Therefore, if we define $\mathfrak{s}(\mathsf{n}) := \sum_{k=0}^{\infty} \mathsf{n}^{|i}$, we may rewrite $\mathfrak{g}$ in the following way :

$$\mathfrak{g}(\mathsf{n}, q) = \mathfrak{s}(\mathsf{n}) - \mathfrak{s}(\mathsf{T}^{S}(\mathsf{n}, q))$$

5.1.2.5 The value function

The strategic trader aims at maximizing the expected value of the total gain obtained at the final time T. We consider the following value function v defined on $\mathcal{Y}$ by,

$$v(k, \mathsf{n}, x) := \sup_{Q \in \mathcal{A}(k,\mathsf{n},x)} J^{Q}(k, \mathsf{n}, x),$$

where we have set

$$J^{Q}(k, \mathsf{n}, x) := \mathbb{E}_{k,\mathsf{n},q}\Big[\sum_{j=k}^{T-1} \mathfrak{g}(\mathsf{N}_{j}^{k,\mathsf{n},Q}, \varDelta Q_{s}) + \mathfrak{g}\left(\mathsf{N}_{T}^{k,\mathsf{n},Q}, \varDelta Q_{T}\right)\Big]$$

Recall that $\varDelta Q_T = (X - Q_T) \wedge \left(\mathsf{N}_{T}^{k,\mathsf{n},Q}\right)^{|1}$.
Since $\mathcal{A}(k, \mathsf{n}, x)$ is finite, the supremum is reached and there exists an optimal strategy. Hence we have

$$v(k, \mathsf{n}, x) := \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} J^{Q}(k, \mathsf{n}, x)$$

Notice that we have the following boundary conditions :

$$v(T, \mathsf{n}, x) = \mathfrak{g}(\mathsf{n}, x \wedge \mathsf{n}^{|1}) \qquad \text{and} \qquad v(k, \mathsf{n}, X) = 0$$

5.2 Analytical properties

5.2.1 Dynamic programming principle

Theorem 5.2.1 (Dynamic programming principle)**.**
Let $(k, \mathsf{n}, x) \in \mathcal{Y}$. For any stopping time τ taking values in $\{k+1, \ldots, T\}$, we have :

$$v(k, \mathsf{n}, x) = \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} \{\mathbb{E}_{t,\mathsf{n},x}[\sum_{j=k}^{\tau-1} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \varDelta Q_j) + v(\tau, \mathsf{N}_\tau^{k,\mathsf{n},Q}, Q_\tau)]\}$$

Proof : Let $(k, \mathsf{n}, x) \in \mathcal{Y}$, from the definition of the value function, it follows that :

$$v(k, \mathsf{n}, x) = \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} \mathbb{E}_{k,\mathsf{n},q}[\sum_{j=k}^{T} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \varDelta Q_j)]$$

For any stopping time τ taking values in $\{k+1, \dots, T\}$, we split the sum inside the expectation and get :

$$v(k, \mathsf{n}, x) = \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} \mathbb{E}^{\mathsf{P}}_{k,\mathsf{n},q}[\sum_{j=k}^{\tau-1} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \varDelta Q_j) + \sum_{j=\tau}^{T} \mathfrak{g}(\mathsf{N}_j^{k,\mathsf{n},Q}, \varDelta Q_j)]$$

We consider an optimal strategy $Q^* \in \mathcal{A}(\tau, \mathsf{N}^{k,\mathsf{n},Q}, Q_\tau)$, and conclude. □

Notice that if we apply the dynamic programming principle for $\tau := k+1$ then, for $(k, \mathsf{n}, x) \in \mathcal{Y}$, we have :

$$v(k, \mathsf{n}, x) = \max_{Q \in \mathcal{A}(k,\mathsf{n},x)} \{\mathfrak{g}(\mathsf{n}, \varDelta Q_k) + v(k+1, \mathsf{N}_{k+1}^{k,\mathsf{n},Q}, Q_{k+1})]\}$$

5.2.2 One step model

In this section, we study the one period optimal control problem. We consider that the strategic trader has x remaining quantity to sell in two transactions, one at the beginning of the period and one at the end of the period. We illustrate the possible events during this period in the figure 5.1.

Figure 5.1 is a representation of controlled path in a one step model. At time K-1, the strategic trader owns x shares to sell and the market is at the state n. The strategic trader choose to sell $q \in \{0, \dots, x\}$. The new state of the market is $\mathsf{T}^S(\mathsf{n}, q)$ and the remaining share is $x - q$. We emphasize that the trader can react just after looking the state n. The market reacts and evolves according the model. At time K, the strategic trader owns $x - q$ shares and sell $x - q$ in order to respect the liquidation constraint.

In order to give some intuition, we compute the value function and an optimal strategy for different LOB profiles of LOB such as a block order shape as in Predoiu [56] or such as randomly generated LOB profiles.

By using the dynamic programming principle and for $k = K - 1$, we obtain :

$$v(K-1, \mathsf{n}, x) = \max_{q \in [0, \dots, x \wedge \mathsf{n}^{|1}]} \{\mathfrak{g}(\mathsf{n}, q) + \mathbb{E}[g(\mathsf{N}_{k+1}, x-q)|\mathsf{N}_k = \mathsf{T}^S(\mathsf{n}, q)]\}$$

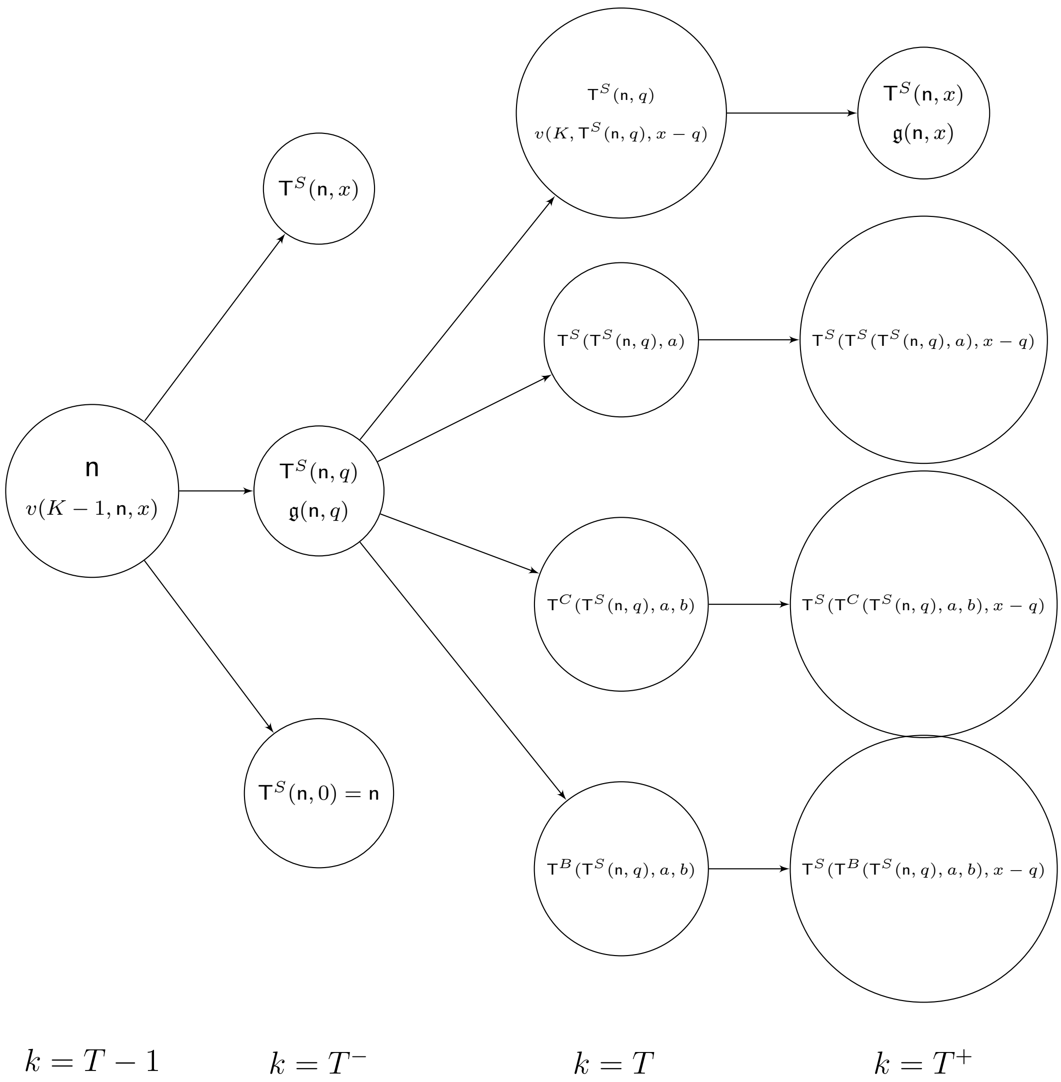


FIGURE 5.1 – Representation of controlled path in a one step model

And we define J

$$J(K-1, \mathsf{n}, x, q) := \mathfrak{g}(\mathsf{n}, q) + \mathbb{E}[g(\mathsf{N}_{k+1}, x-q)|\mathsf{N}_k = \mathsf{T}^S(\mathsf{n}, q)]$$

5.2.2.1 Algorithm

We start with initializing the LOB n, the number of shares to liquidate x, the law parameters $\mathbf{p} := (\mathbf{p}^E, \mathbf{p}^S, \mathbf{p}^{PC}, \mathbf{p}^{VC}, \mathbf{p}^{PB}, \mathbf{p}^{VB})$ for the events arrivals. In order to optimize the algorithm, we compute the probability vectors $\mathbb{P}^\gamma$ of $\gamma = \{\delta, \alpha^S, \alpha^C, \beta^C, \alpha^B, \beta^B\}$ defined by

$$\mathbb{P}^\gamma := (\mathbb{P}[\gamma = i])_{i \in support\{\gamma\}}$$

We now describe the algorithm for computing the value function and finding an optimal strategy for the one period model.

— For each iteration $q \in [0, \dots, x]$, we follow the following steps :

  — Step 1 : Compute the new depth $\mathsf{m} := \mathsf{T}^S(\mathsf{n}, q)$, the Bid Price of the new depth $\mathfrak{b}^S := \mathfrak{b}^S(\mathsf{n}, q)$

  — Step 2 : Compute the gain $J_0 := \mathfrak{g}(\mathsf{n}, q)$ and the gain $J_x := \mathfrak{g}(\mathsf{m}, x-q)$

  — Step 3 : Compute J_S, J_C, J_B :

$$J_S := \sum_{a \in supp\{\alpha^S\}} \mathfrak{g}(\mathsf{T}^S(\mathsf{m}, a), x-q)\mathbb{P}[\alpha^S = i]$$

$$J_C := \sum_{a,b \in supp\{\alpha^C\} \times supp\{\beta^C\}} \mathfrak{g}(\mathsf{T}^C(\mathsf{m}, a, \mathfrak{b}^S - \beta^C), x-q)\mathbb{P}[\alpha^C = a]\mathbb{P}[\beta^C = b]$$

$$J_B := \sum_{a,b \in supp\{\alpha^B\} \times supp\{\beta^B\}} \mathfrak{g}(\mathsf{T}^B(\mathsf{m}, a, \mathfrak{b}^S - \beta^B + u), x-q)\mathbb{P}[\alpha^B = a]\mathbb{P}[\beta^B = b]$$

  — Step 4 : Compute $J[q]$:

$$J[q] := J_0 + \mathbb{P}[\delta = N]J_x + \mathbb{P}[\delta = S]J_S + \mathbb{P}[\delta = C]J_C + \mathbb{P}[\delta = B]J_B$$

— Step 5 : Compute q^* and v :

$$q^* := argmax J[q]$$

$$v := max J[q]$$

We plot in figure 5.2 a representation of different depth profiles. The horizontal axis represent the price scale and the vertical axis represent the cumulative volume. All curves are represented by piece-wise linear curve. The green curve correspond to

Data:
- the depth n
- x : remaining share (integer)
- E : events (vector size 4)

for $k \in [\![0, x]\!]$ **do**
  Compute $newdepth = executionMarket(depth, k)$;
  Compute $J0 = gain(depth, k)$;
  Compute $JN = gain(newdepth, x - k)$;
  for $i \in [\![1, n_s]\!]$ **do**
    Compute
    $JS = JS + prsellsize[i] \times gain(executionMarket(newdepth, i), x - k)$
  end
  for $i, j \in [\![1, n_b]\!] \times [\![1, m_b]\!]$ **do**
    Compute $JB = JB + prbuysize[i] \times prbuyprice[j] \times gain(executionLimit(newdepth, i, bestPrice(newdepth) - j + buyprice[2]), x - k)$)
  end
  for $i, j \in [\![1, n_c]\!] \times [\![1, m_c]\!]$ **do**
    Compute $JC = JC + prcancelsize[i] \times prcancelprice[j] \times gain(executionCancel(newdepth, i, bestPrice(newdepth) - j), x - k)$
  end
  Compute $J[k] = J0 + JN + JS + JB + JC$
end
Result: J : (vector size x+1)

Algorithm 2: Algorithm for computing the value function and finding an optimal strategy for the one period model

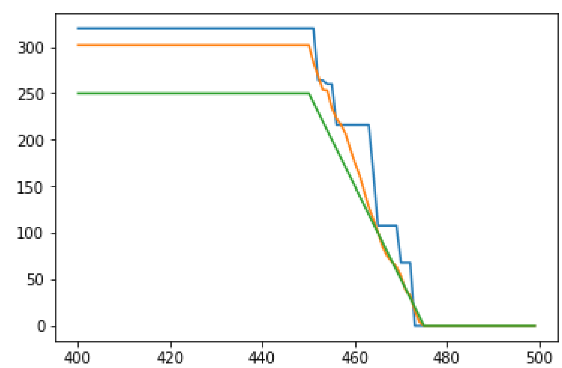
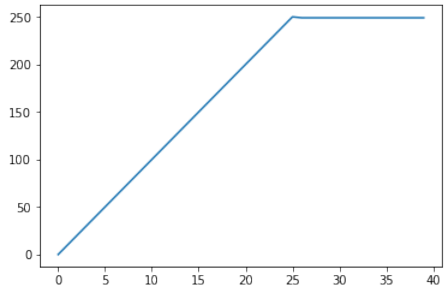
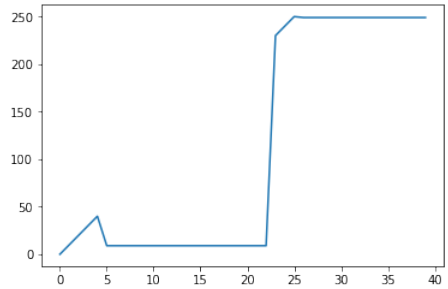


FIGURE 5.2 – Depth profiles



(a) Recurence



(b) Transience

FIGURE 5.3 – Strategy for linear n

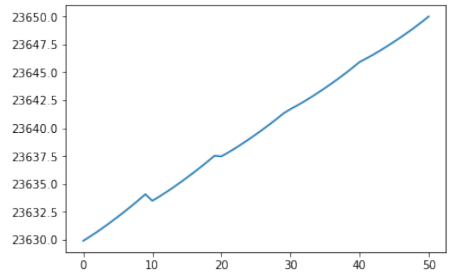
the block shape order book. The block shape order is define by $\mathsf{n} := (\mathsf{n}^{|k})_{k\in\mathbb{N}^*}$ such as :

$$\mathsf{n}^{|k} = 10\left[(475-k)^+ - (450-k)^+\right] \text{ and for } k \in \{450,\dots,475\}\ \Delta_k\mathsf{n} = 10 \quad (5.4)$$

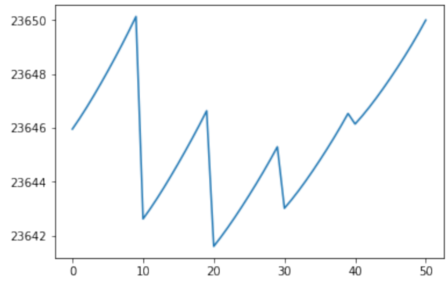
The orange and the blue curves correspond to randomly generated order book. We generate the values $\Delta_k\mathsf{n}$ for $k \in \{450,\dots,475\}$ and $\Delta_k\mathsf{n} = 0$ outside $\{450,\dots,475\}$. For each k, we generate a realization from uniform law with parameters $(a = 0, b = 20)$. For each k, we generate a realization from binomial law with parameters $(n = 20, p = 0.5)$.

We represent the optimal share $q/10$ at time $K-1$ against the remaining share x to liquidate in the case of the block shape order book in the case of recurrence parameters in figure 5.3a and transient parameters in figure 5.3b. We refer to the chapter 4 for the discussion of parameters which lead to the recurrence property or the transience property. In the recurrent case, the optimal share q is proportional to x until $q = \mathsf{n}^{|1}$. in the transient case, the optimal share q is proportional to x from $q = 0$ to $q = q_0$ and from $q = q_1$ to $q = \mathsf{n}^{|1}$. Between q_0 and q_1, the optimal share $q = \mathsf{n}^{|\mathfrak{b}(\mathsf{n})} - 1$.

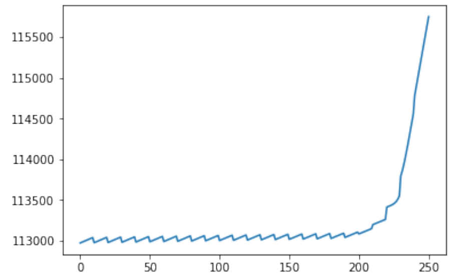
In the figure 5.4, we compute the function J. At the x-axis, we represent the remaining share x to liquidate in the case of the block shape order book and in the other axis, the function J. In figures 5.4a, 5.4b, 5.4c and 5.4d, we notice the same shape for J. For all x and parameters, J is a sawtooth function. We conjecture that J is increasing on each interval $q = \{\mathsf{n}^{|k+1} \wedge x, \dots, \mathsf{n}^{|k} - 1 \wedge x\}$ for all $k \in \{\mathfrak{b}(\mathsf{n})+1, \dots, 1\}$.



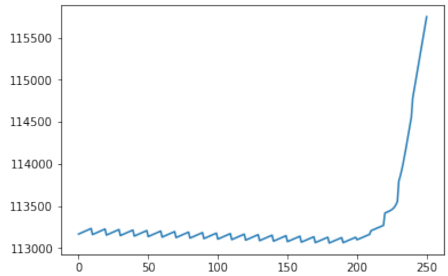
(a) Recurrence J for $x = 50$



(b) Transience J for $x = 50$



(c) Recurrence J for $x = 250$



(d) Transience J for $x = 250$

FIGURE 5.4 – Computation of J for $x = \{50, 250\}$

5.3 Characterization of the optimal strategy for one step model

5.3.1 Main result

We want to simplify the expression of J. By noticing that

$$\mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q) = \mathfrak{g}(\mathsf{n}, x) - \mathfrak{g}(\mathsf{n}, q) \tag{5.5}$$

and by computing the drift for the function $\mathfrak{g}$ with $\mathsf{m} = \mathsf{T}^S(\mathsf{n}, q)$, we have

$$\mathfrak{L}\mathfrak{g}(\mathsf{m}, x - q) = \mathbb{E}[\mathfrak{g}(\mathsf{N}_{k+1}, x - q) - \mathfrak{g}(\mathsf{N}_k, x - q)|\mathsf{N}_k = \mathsf{m}]. \tag{5.6}$$

Thus,

$$\begin{aligned} J(K-1, \mathsf{n}, x, q) \quad &= \mathfrak{g}(\mathsf{n}, q) + \mathbb{E}[\mathfrak{g}(\mathsf{N}_{k+1}, x - q)|\mathsf{N}_k = \mathsf{T}^S(\mathsf{n}, q)] \\ &= \mathfrak{g}(\mathsf{n}, q) + \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q) + \mathfrak{L}\mathfrak{g}(\mathsf{m}, x - q) \\ &= \mathfrak{g}(\mathsf{n}, x) + \mathfrak{L}\mathfrak{g}(\mathsf{m}, x - q) \end{aligned} \tag{5.7}$$

We define $\mathfrak{L}\mathfrak{g}(\mathsf{m}, x - q) := J^S(\mathsf{n}, x, q) + J^B(\mathsf{n}, x, q) + J^C(\mathsf{n}, x, q)$ with

— $J^S(\mathsf{n}, x, q) := p^S\mathbb{E}[\mathfrak{g}(\mathsf{T}^S(\mathsf{T}^S(\mathsf{n}, q), \alpha^S), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)]$

— $J^B(\mathsf{n}, x, q) := p^B\mathbb{E}[\mathfrak{g}(\mathsf{T}^B(\mathsf{T}^S(\mathsf{n}, q), \alpha^B, \beta^B), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)]$

— $J^C(\mathsf{n}, x, q) := p^C\mathbb{E}[\mathfrak{g}(\mathsf{T}^C(\mathsf{T}^S(\mathsf{n}, q), \alpha^C, \beta^C), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)]$

By following the conjecture,

Definition 5.3.1. *We define $\mathcal{A}_{\mathfrak{b}}$ the set of sell size which leads to the same next bid price :*

$$\text{For } \mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})], \mathcal{A}_{\mathfrak{b}}(\mathsf{n}, x) := \{c \in [0, x], \mathfrak{b}^S(\mathsf{n}, c) = \mathfrak{b}\}$$

The main result of this part is the following :

Theorem 5.3.1. *For all $\mathsf{n} \in \mathcal{N}$, for all $q \in \mathcal{A}_{\mathfrak{b}}$ with $\mathfrak{b}^S(\mathsf{n}, q)$, the function $J(K-1, \mathsf{n}, x, .)$ is non decreasing in q. Then, for all $\mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})]$, for all $q \in \mathcal{A}_{\mathfrak{b}}$,*

$$\arg\max_{q \in \mathcal{A}_{\mathfrak{b}}} J(K-1, \mathsf{n}, x, q) = \mathsf{n}^B - 1$$

This theorem means that in this model, the strategic trader only need to know the next bid price to define his sell size at time $K-1$. For proving this theorem, we need to compute the functions J^S, J^B, J^C. In order to compute these functions, we need to understand compositions of transition operators such as $\mathsf{T}^S\mathsf{T}^S$, $\mathsf{T}^C\mathsf{T}^S$ and $\mathsf{T}^B\mathsf{T}^S$. Since for all $\mathsf{n} \in \mathcal{N}$, $(a, b) \in \mathbb{N}$ we have

$$\mathsf{T}^S(\mathsf{T}^S(\mathsf{n}, a), b) = \mathsf{T}^S(\mathsf{n}, a + b), \tag{5.8}$$

the function J^S is the easiest to compute.

5.3.2 Computations of J^S, J^B, J^C functions

In order to compute the functions J^S, J^B, J^C, we need to define some notations such as

— D^α the support of the random variable α,

— $F^\alpha(y) := \sum_{a \in \mathsf{D}^\alpha} \mathbb{1}_{\{a \le y\}} \mathbb{P}[\alpha = a]$ the cumulative distribution of α,

— $D^\alpha(y) := \sum_{a \in D^\alpha} a \mathbb{1}_{\{a \le y\}} \mathbb{P}[\alpha = a]$,

— $H^\alpha(y) := yF^\alpha(y) - D^\alpha(y)$

— $\hat{F}^\alpha(y) := 1 - F^\alpha(y)$

— $\hat{D}^\alpha(y) = \bar{\alpha} - D^\alpha(y)$

— $\hat{H}^\alpha(y) := y\hat{F}^\alpha(y) - \hat{D}^\alpha(y)$

— H is positive and non decreasing and $\hat{H}$ is non decreasing

We define for a function $q \to f(q)$ with $q \in \mathbb{N}$, the increment operator for f $\Delta_q f$ such as $\Delta_q f(q) := f(q+1) - f(q)$.

We give a lemma for the decomposition of $\mathfrak{g}$.

Lemma 5.3.1. *For $\mathsf{n} \in \mathcal{N}$, $x \in \mathbb{N}$, we have*

$$\mathfrak{g}(\mathsf{n}, x) = x\mathfrak{b}^S(\mathsf{n}, x) + \hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x))$$

.

We use these lemma by separating the price part and the volume part. We emphasize the importance of the dependence of $\mathfrak{b}^S(\mathsf{n},x)$. For $\mathfrak{b} \in \mathbb{N}$ fixed and for all $x \in \mathcal{A}_{\mathfrak{b}}$, we notice the same shape for $\mathfrak{g}$ as the J function.

— For $\mathsf{n} \in \mathcal{N}$ fixed, the function $b \to \hat{\mathfrak{s}}(\mathsf{n},b)$ is decreasing in $\mathbb{N}$.

— For $\mathsf{n} \in \mathcal{N}$ fixed, the function $x \to \mathfrak{b}^S(\mathsf{n},x)$ is decreasing in $\mathbb{N}$.

Then, For $\mathsf{n} \in \mathcal{N}$ fixed, the function $x \to \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))$ is increasing in $\mathbb{N}$.

Lemma 5.3.2 (Computation of J^S)**.** *For all* $\mathsf{n} \in \mathcal{N}$*, for all* $q \in \{0,\dots,x\}$*, we have*

$J^S(\mathsf{n},x,q) := p_S\mathbb{E}[\mathfrak{g}(\mathsf{T}^S(\mathsf{T}^S(\mathsf{n},q),\alpha^S),x-q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n},q),x-q)] = A(\mathsf{n},x) - A(\mathsf{n},q)$

with $A(\mathsf{n},q) := \sum_{i\leq \mathfrak{b}^S(\mathsf{n},q)} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1 - q) + \mathbb{E}[\alpha^S]\right]$*. Moreover, for* $\mathfrak{b} \in [0,\mathfrak{b}(\mathsf{n})]$*, for all* $q \in \mathcal{A}_{\mathfrak{b}}$*,* $J^S(\mathsf{n},x,q)$ *is non decreasing in* q*.*

Proof : Firstly, we want to compute $\mathbb{E}[\mathfrak{g}(\mathsf{T}^S(\mathsf{n},\alpha^S),x) - \mathfrak{g}(\mathsf{n},x)]$. Then, we replace $\mathsf{n} := \mathsf{T}^S(\mathsf{n},q)$ and $x := x-q$ for computing the term J^S.

Since we have 5.3.1, we focus on the two functions $\mathfrak{b}^S$ and $\hat{\mathfrak{s}}$. By definition of the function $\mathfrak{b}^S$, for $\mathsf{n} \in \mathcal{N}$, we have $\mathfrak{b}^S(\mathsf{n},\alpha^S) = \sum_{i\geq 1} 1\!\!1_{\{\alpha^S \leq \mathsf{n}^{|i}-1\}}$ and $\mathfrak{b}^S(\mathsf{T}^S(\mathsf{n},\alpha^S),x) = \mathfrak{b}^S(\mathsf{n},\alpha^S + x)$. We have :

$$\mathbb{E}[\mathfrak{b}^S(\mathsf{T}^S(\mathsf{n},\alpha^S),x) - \mathfrak{b}^S(\mathsf{n},x)] = -\sum_{i=1}^{\mathfrak{b}^S(\mathsf{n},x)} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) \tag{5.9}$$

By definition of the function $\hat{\mathfrak{s}}$, we have :

$$\begin{aligned} \hat{\mathfrak{s}}(\mathsf{T}^S(\mathsf{n},a),\mathfrak{b}^S(\mathsf{T}^S(\mathsf{n},a),x)) &= \sum_{i>\mathfrak{b}^S(\mathsf{n},a+x)} (\mathsf{n}^{|i} - a) 1\!\!1_{\{i\leq \mathfrak{b}^S(\mathsf{n},a)\}} \\ &= \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},a+x)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},a)) \\ &+ a(\mathfrak{b}^S(\mathsf{n},a+x) - \mathfrak{b}^S(\mathsf{n},a)) \end{aligned} \tag{5.10}$$

We compute the expectation of $\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},\alpha^S + x))$ and the $\alpha^S\mathfrak{b}^S(\mathsf{n},\alpha^S + x)$. Since the set $\{i \in \mathbb{N},\, i > \mathfrak{b}^S(\mathsf{n},x+\alpha^S)\}$ is equivalent to the set $\{i \in \mathbb{N},\, \alpha^S \geq \mathsf{n}^{|i} - x\}$ and for $i > \mathfrak{b}^S(\mathsf{n}, x)$,$\hat{F}^{\alpha^S}(\mathsf{n}^{|i} - x - 1) = 1$, then

$$\begin{aligned}
\mathbb{E}[\hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x + \alpha^S))] &= \mathbb{E}[\sum_{i > \mathfrak{b}^S(\mathsf{n}, x+\alpha^S)} \mathsf{n}^{|i}] \\
&= \mathbb{E}[\sum_{i \geq 1} \mathbb{1}_{\{\alpha^S \geq \mathsf{n}^{|i} - x\}} \mathsf{n}^{|i}] \\
&= \sum_{i \geq 1} \mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - x - 1) \\
&= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - x - 1) + \sum_{i > \mathfrak{b}^S(\mathsf{n}, x)} \mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - x - 1) \\
&= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - x - 1) + \hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x))
\end{aligned} \tag{5.11}$$

We compute

$$\begin{aligned}
\mathbb{E}[\alpha^S \mathfrak{b}^S(\mathsf{n}, x + \alpha^S)] &= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} D^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) \\
&= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \mathbb{E}[\alpha^S] - \hat{D}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x)
\end{aligned} \tag{5.12}$$

Then for the function $\hat{\mathfrak{s}}$, by using the equations 5.11 and 5.12 we obtain :

$$\begin{aligned}
\mathbb{E}[\hat{\mathfrak{s}}(\mathsf{T}^S(\mathsf{n}, \alpha^S) \quad &, \quad \mathfrak{b}^S(\mathsf{T}^S(\mathsf{n}, a), x)) - \hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x))] \\
= & \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \left[\mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) - \hat{D}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) + \mathbb{E}[\alpha^S]\right] \\
- & \sum_{i \leq \mathfrak{b}(\mathsf{n})} \left[\mathsf{n}^{|i} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1) - \hat{D}^{\alpha^S}(\mathsf{n}^{|i} - 1) + \mathbb{E}[\alpha^S]\right]
\end{aligned} \tag{5.13}$$

Then, we conclude for $\mathfrak{g}$ with equations 5.9 and 5.13 :

$$\begin{aligned}
\mathbb{E}[\mathfrak{g}(\mathsf{T}^S(\mathsf{n}, \alpha^S), x) - \mathfrak{g}(\mathsf{n}, x)] &= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) + \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) + \mathbb{E}[\alpha^S]\right] \\
&- \sum_{i \leq \mathfrak{b}(\mathsf{n})} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1) + \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1) + \mathbb{E}[\alpha^S]\right]
\end{aligned} \tag{5.14}$$

By setting $\mathsf{n} := \mathsf{T}^S(\mathsf{n}, q)$ and $x := x - q$ in equation 5.14, we compute J^S,

$$\begin{aligned}
J^S(\mathsf{n}, x, q) &= E[\mathfrak{g}(\mathsf{T}^S(\mathsf{T}^S(\mathsf{n}, q), \alpha^S), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)] \\
&= \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, x)} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) + \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - x) + \mathbb{E}[\alpha^S]\right] \\
&- \sum_{i \leq \mathfrak{b}^S(\mathsf{n}, q)} \left[\hat{H}^{\alpha^S}(\mathsf{n}^{|i} - 1 - q) + \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - 1 - q) + \mathbb{E}[\alpha^S]\right]
\end{aligned} \tag{5.15}$$

We compute for for $\mathfrak{b} \in \mathbb{N}$ and $q, q+1 \in \mathcal{A}_{\mathfrak{b}}(\mathsf{n}, x)$,

$$\Delta_q J^S(\mathsf{n}, x, q) \quad = \quad \sum_{i \le \mathfrak{b}^S(\mathsf{n},q)} \hat{F}^{\alpha^S}(\mathsf{n}^{|i} - q - 1) \ge 0 \tag{5.16}$$

Then J^S is increasing in q for all $q \in \mathcal{A}_{\mathfrak{b}}(\mathsf{n}, x)$. The lemma is proved. □

If we define $J^C(\mathsf{n}, x, q) := p_c \mathbb{E}[\mathfrak{g}(\mathsf{T}^C(\mathsf{T}^S(\mathsf{n}, q), \alpha^C, \beta^C), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)]$,

Lemma 5.3.3 (Computation of J^C)**.** *for all* $\mathsf{n} \in \mathcal{N}$*, for all* $q \in \{0, \ldots, x\}$*, we have*

$$\frac{J^C(\mathsf{n}, x, q)}{p_c} = A^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q)) + \mathbb{Q}[\beta^{C-} = 0] B^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q), q)$$

Moreover, for $\mathfrak{b} \in [0, \mathfrak{b}(\mathsf{n})]$*, for all* $q \in \mathcal{A}_{\mathfrak{b}}$*,* $J^C(\mathsf{n}, x, q)$ *is non decreasing in* q*.*

Proof : If there is a cancel event arrival and the cancel price level is lower than the $\mathfrak{b}^S(\mathsf{n}, x)$, the new bid price doesn't change. It means that, for $\mathfrak{b}(\mathsf{n}) - b < \mathfrak{b}^S(\mathsf{n}, x)$, $\mathfrak{b}^S(\mathsf{T}^C(\mathsf{n}, a, \mathfrak{b}(\mathsf{n}) - b), x) - \mathfrak{b}^S(\mathsf{n}, x) = 0$.

Otherwise, we have for $y_b := \mathfrak{b}(\mathsf{n}) - b$,

$$\mathfrak{b}^S(\mathsf{T}^C(\mathsf{n}, a, y_b), x) - \mathfrak{b}^S(\mathsf{n}, x) = \left(\mathfrak{b}^S(\mathsf{n}, x + a) - \mathfrak{b}^S(\mathsf{n}, x)\right) \mathbb{1}_{\{\mathfrak{b}(\mathsf{n}) - \mathfrak{b}^S(\mathsf{n},x) \ge b\}} \mathbb{1}_{\{a \le \Delta_{y_b} \mathsf{n}\}} \tag{5.17}$$

We have $\alpha^C(\Delta_{\beta^c}\mathsf{n}) = \mathbb{1}_{\{\epsilon^c = 1\}} \Delta_{\beta^c}\mathsf{n} + \mathbb{1}_{\{\epsilon^c = 0\}} \xi^c_{\Delta_{\beta^c}\mathsf{n}}$ and $\beta^C(\mathsf{n}) = \mathfrak{b}(\mathsf{n}) - \beta^{C-}$. By using equation 5.17, we compute,

$$\mathbb{E}[\mathfrak{b}^S(\mathsf{T}^C(\mathsf{n}, \alpha^C, \beta^C), x) - \mathfrak{b}^S(\mathsf{n}, x)] \quad = \quad - \sum_{i \le \mathfrak{b}^S(\mathsf{n},x)} \mathbb{E}[\mathbb{1}_{\{\beta^{C-} \le \Delta \mathfrak{b}^S(\mathsf{n},0,x)\}} \hat{F}^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \tag{5.18}$$

Since the expectation of $\mathfrak{b}^S$ is computed, we analyze $\hat{\mathfrak{s}}$

$$\hat{\mathfrak{s}}(\mathsf{m}, \mathfrak{b}^S(\mathsf{m}, x)) - \hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x)) = \left(\hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x + a)) + a(\mathfrak{b}^S(\mathsf{n}, x + a) - y_b)\right) \mathbb{1}_{\{y_b \ge \mathfrak{b}^S(\mathsf{n},x)\}} \mathbb{1}_{\{a \le \Delta_{y_b} \mathsf{n}\}} \tag{5.19}$$

$$\mathbb{E}[(\hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x + \alpha^C(\Delta_{\beta^c}\mathsf{n}))) - \hat{\mathfrak{s}}(\mathsf{n}, \mathfrak{b}^S(\mathsf{n}, x))] = \sum_{i \le \mathfrak{b}^S(\mathsf{n},x)} \mathsf{n}^{|i} \mathbb{E}[\mathbb{1}_{\{\beta^{C-} \le \Delta \mathfrak{b}^S(\mathsf{n},0,x)\}} \hat{F}^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \tag{5.20}$$

$$\mathbb{E}[\alpha^C(\Delta_{\beta^c}\mathsf{n})(\mathfrak{b}^S(\mathsf{n}, x+\alpha^C(\Delta_{\beta^c}\mathsf{n})) - \mathfrak{b}(\mathsf{n}) + \beta^{C-})] = \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},0,x)\}}\alpha^C(\Delta_{\beta^c}\mathsf{n})(\beta^{C-} - \mathfrak{b}(\mathsf{n}))] + \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},0,x)\}}(D^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \tag{5.21}$$

We conclude then for $\mathfrak{g}$:

$$\begin{aligned}\mathbb{E}[\mathfrak{g}(\mathsf{T}^C(\mathsf{n},\alpha^C,\beta^C),x) - \mathfrak{g}(\mathsf{n},x)] &= \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} (\mathsf{n}^{|i} - x)\mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},0,x)\}}\hat{F}^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},0,x)\}}D^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},0,x)\}}\alpha^C(\Delta_{\beta^c}\mathsf{n})(\beta^{C-} - \mathfrak{b}(\mathsf{n}))] \end{aligned} \tag{5.22}$$

We compute for $\mathsf{n} := \mathsf{T}^S(\mathsf{n}, q)$ and $x := x-q$ and we recall that $\Delta_{\mathfrak{b}^S(\mathsf{n},q)-\beta^{C-}}\mathsf{T}^S(\mathsf{n}, q) = \Delta_{\mathfrak{b}^S(\mathsf{n},q)-\beta^{C-}}\mathsf{n}\mathbb{1}_{\{\beta^{C-}>0\}} + (\mathsf{n}^{\mathfrak{b}^S(\mathsf{n},q)} - q)\mathbb{1}_{\{\beta^{C-}=0\}}$

$$\begin{aligned}\frac{J^C(\mathsf{n},x,q)}{p_C} &= \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} (\mathsf{n}^{|i} - x)\mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}\hat{F}^{\alpha^C(\Delta_{\beta^c}\mathsf{T}^S(\mathsf{n},q))}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}D^{\alpha^C(\Delta_{\beta^c}\mathsf{T}^S(\mathsf{n},q))}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \mathbb{E}[\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}\alpha^C(\Delta_{\beta^c}\mathsf{T}^S(\mathsf{n},q))(\beta^{C-} - \mathfrak{b}^S(\mathsf{n},q))] \\ &= A^c(\mathsf{n},x,\mathfrak{b}^S(\mathsf{n},q)) + B^c(\mathsf{n},x,\mathfrak{b}^S(\mathsf{n},q),q)\mathbb{Q}[\beta^{C-} = 0] \end{aligned} \tag{5.23}$$

with

$$\begin{aligned}A^c(\mathsf{n},x,\mathfrak{b}^S(\mathsf{n},q)) &= \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} (\mathsf{n}^{|i} - x)\mathbb{E}[\mathbb{1}_{\{\beta^{C-}>0\}}\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}\hat{F}^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} \mathbb{E}[\mathbb{1}_{\{\beta^{C-}>0\}}\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}D^{\alpha^C(\Delta_{\beta^c}\mathsf{n})}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \mathbb{E}[\mathbb{1}_{\{\beta^{C-}>0\}}\mathbb{1}_{\{\beta^{C-}\leq \Delta\mathfrak{b}^S(\mathsf{n},q,x)\}}\alpha^C(\Delta_{\beta^c}\mathsf{n})(\beta^{C-} - \mathfrak{b}^S(\mathsf{n},q))] \end{aligned} \tag{5.24}$$

and

$$\begin{aligned}B^c(\mathsf{n},x,\mathfrak{b}^S(\mathsf{n},q),q) &= \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} (\mathsf{n}^{|i} - x)\hat{F}^{\alpha^C(\mathsf{n}^{\mathfrak{b}^S(\mathsf{n},q)}-q)}(\mathsf{n}^{|i} - 1 - x)] \\ &+ \sum_{i\leq \mathfrak{b}^S(\mathsf{n},x)} D^{\alpha^C(\mathsf{n}^{\mathfrak{b}^S(\mathsf{n},q)}-q)}(\mathsf{n}^{|i} - 1 - x)] \\ &- \mathbb{E}[\alpha^C(\mathsf{n}^{\mathfrak{b}^S(\mathsf{n},q)} - q)(\mathfrak{b}^S(\mathsf{n},q))] \end{aligned} \tag{5.25}$$

It is easy to notice for $\mathfrak{b} \in \mathbb{N}$ fixed and $q, q+1 \in Ar_{\mathfrak{b}}(\mathsf{n}, x)$, $\Delta_q A^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q)) = 0$

We assume that $\mathbb{E}[\alpha^C(N)/N] = m > 1$ with m constant in N. This assumption is true in the case of the specification of α^C 2.3.5 when α^C follows a uniform law with parameters $(1, \Delta_b\mathsf{n})$ or α^C follow a binomial law with parameters $(\Delta_b\mathsf{n}, p)$.

We compute $\Delta_q B^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q), q)$ and if we define $\Delta f_q^{\alpha^C}(j) := f^{\alpha^C}_{\mathsf{n}^{|\mathfrak{b}^S(\mathsf{n},q)}-q-1}(j) - f^{\alpha^C}_{\mathsf{n}^{|\mathfrak{b}^S(\mathsf{n},q)}-q}(j)$

$$\begin{aligned}\Delta_q B^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q), q) &= \sum_{i\le \mathfrak{b}^S(\mathsf{n},x)} \sum_{j=1}^{\mathsf{n}^{|i}-x-1} j\Delta f_q^{\alpha^C}(j) + \sum_{i\le \mathfrak{b}^S(\mathsf{n},x)} (\mathsf{n}^{|i} - x)\Delta \hat{F}_q^{\alpha^C}(\mathsf{n}^{|i} - x - 1) \\ &+ m\mathfrak{b}^S(\mathsf{n}, q) \qquad (5.26)\end{aligned}$$

For example, if the random variable α^C is state dependent (we refer to to the specification of α^C 2.3.5 or the section for parameter estimation 4.2) and follows a uniform law with parameters $(1, \Delta_{\mathfrak{b}}\mathsf{n})$, we have

$$f_q^{\alpha^C}(j) = \frac{1}{\mathsf{n}^{|b} - q} \text{ and } \Delta f_q^{\alpha^C}(j) = \frac{1}{(\mathsf{n}^{|b} - q - 1)(\mathsf{n}^{|b} - q)} > 0$$

$$\hat{F}_q^{\alpha^C}(j) = 1 - \frac{j}{\mathsf{n}^{|b} - q} \text{ and } \Delta \hat{F}_q^{\alpha^C}(j) = \frac{-j}{(\mathsf{n}^{|b} - q - 1)(\mathsf{n}^{|b} - q)} < 0$$

Then, we compute $\Delta_q B^c$,

$$\Delta_q B^c(\mathsf{n}, x, \mathfrak{b}^S(\mathsf{n}, q), q) = -\frac{1}{2} \sum_{i\le \mathfrak{b}^S(\mathsf{n},x)} \frac{(\mathsf{n}^{|i} - x - 1)(\mathsf{n}^{|i} - x)}{(\mathsf{n}^{|b} - q - 1)(\mathsf{n}^{|b} - q)} + m\mathfrak{b}^S(\mathsf{n}, q) > 0$$

For $q \in \mathcal{A}_{\mathfrak{b}}(\mathsf{n})$, $\Delta_q B^c > 0$. The lemma is proved. □

Lemma 5.3.4 (Computation of J^B)**.** *For all* $\mathsf{n} \in \mathcal{N}$, *for all* $q \in \{0, \ldots, x\}$, *we have*

$$\frac{J^B(\mathsf{n}, x, q)}{p_B} = \mathbb{E}[\mathfrak{g}(\mathsf{T}^B(\mathsf{T}^S(\mathsf{n}, q), \alpha^B, \beta^B), x - q) - \mathfrak{g}(\mathsf{T}^S(\mathsf{n}, q), x - q)] = A^B(\mathsf{n}, x, x - q)$$

Moreover $J^B(\mathsf{n}, x, q)$ *is non decreasing in* q.

Proof : If there is a buy event arrival and the buy price level y_b is lower than the $\mathfrak{b}^S(\mathsf{n}, x)$, the new bid price doesn't change. Since the volume x of a sell event and the volume a of a buy event are opposite sign, we need to distinguish the two case. When the volume a is greater than x (absolute value), the new bid price is the buy price level. When the volume x is greater than a, the new bid price is the maximum between the buy price level and the new bid price after the difference of x and a. Then :

$$\mathfrak{b}^S(\mathsf{T}^B(\mathsf{n}, a, y_b), x) - \mathfrak{b}^S(\mathsf{n}, x) = \begin{cases} y_b - \mathfrak{b}^S(\mathsf{n}, x) & \text{if } a > x \text{ and } \mathfrak{b}^S(\mathsf{n}, x) \le y_b \\ \mathfrak{b}^S(\mathsf{n}, x - a) - \mathfrak{b}^S(\mathsf{n}, x) & \text{if } a \le x \text{ and } \mathfrak{b}^S(\mathsf{n}, x - a) > y_b \\ y_b - \mathfrak{b}^S(\mathsf{n}, x) & \text{if } a \le x \text{ and } \mathfrak{b}^S(\mathsf{n}, x - a) \le y_b \end{cases}$$

We have $y_b = \mathfrak{b}(\mathsf{n}) - \mathbb{1}_{\{\epsilon^B=0\}}\beta^{B-} + \mathbb{1}_{\{\epsilon^B=1\}}\beta^{B+}$.

We recall that $\mathfrak{I}(\mathsf{n},x) = \mathfrak{b}(\mathsf{n}) - \mathfrak{b}^S(\mathsf{n},x)$. By definition, β^{B+} is positive and β^{B-} is non negative.

$$\mathfrak{b}^S(\mathsf{T}^B(\mathsf{n},a,y_b),x) - \mathfrak{b}^S(\mathsf{n},x) = \begin{cases} [\beta^{B+} + \mathfrak{I}(\mathsf{n},x)]\mathbb{1}_{\{\epsilon^B=1\}} \\ [-\beta^{B-} + \mathfrak{I}(\mathsf{n},x)]\mathbb{1}_{\{a>x\}}\mathbb{1}_{\{\beta^{B-}\leq\mathfrak{I}(\mathsf{n},x)\}}\mathbb{1}_{\{\epsilon^B=0\}} \\ [\mathfrak{b}^S(\mathsf{n},x-a) - \mathfrak{b}^S(\mathsf{n},x)]\mathbb{1}_{\{a\leq x\}}\mathbb{1}_{\{\beta^{B-}>\mathfrak{I}(\mathsf{n},x-a)\}}\mathbb{1}_{\{\epsilon^B=0\}} \\ [-\beta^{B-} + \mathfrak{I}(\mathsf{n},x)]\mathbb{1}_{\{a\leq x\}}\mathbb{1}_{\{\mathbb{1}_{\{\beta^{B-}\leq\mathfrak{I}(\mathsf{n},x-a)\}}\}}\mathbb{1}_{\{\epsilon^B=0\}} \end{cases}$$

Then for the expectation of $\mathfrak{b}^S$ with $\beta^B := y_b$ and $\mathfrak{b}^S(\mathsf{n},x-\alpha^B) - \mathfrak{b}^S(\mathsf{n},x) = \mathfrak{I}(\mathsf{n},x) - \mathfrak{I}(\mathsf{n},x-\alpha^B)$, we have

$$\begin{aligned} \mathbb{E}[(\mathfrak{b}^S(\mathsf{T}^B(\mathsf{n},\alpha^B,\beta^B),x) - \mathfrak{b}^S(\mathsf{n},x))] &= \mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}(\beta^{B+} + \mathfrak{I}(\mathsf{n},x))] \\ &+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}(-\beta^{B-} + \mathfrak{I}(\mathsf{n},x))] \\ &- \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x))] \\ &- \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \end{aligned} \tag{5.27}$$

With $\mathsf{m} := \mathsf{T}^B(\mathsf{n},a,y_b)$, we define $\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x)) := \hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))$. Thus,

$$\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x)) = \begin{cases} \hat{\mathfrak{s}}(\mathsf{n},y_b) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) & \text{if } a>x \text{ and } \mathfrak{b}^S(\mathsf{n},x)\leq y_b \\ \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x-a)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) + a(y_b - \mathfrak{b}^S(\mathsf{n},x-a)) & \text{if } a\leq x \text{ and } \mathfrak{b}^S(\mathsf{n},x-a) > y_b \\ \hat{\mathfrak{s}}(\mathsf{n},y_b) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) & \text{if } a\leq x \text{ and } \mathfrak{b}^S(\mathsf{n},x-a)\leq y_b \end{cases}$$

We have $y_b = \mathfrak{b}(\mathsf{n}) - \mathbb{1}_{\{\epsilon^B=0\}}\beta^{B-} + \mathbb{1}_{\{\epsilon^B=1\}}\beta^{B+}$ and $\hat{\mathfrak{s}}(\mathsf{n},b) = 0$ for $b \geq \mathfrak{b}(\mathsf{n})$

$$\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x)) = \begin{cases} [-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))]\mathbb{1}_{\{\epsilon^B=1\}} \\ [\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n}) - \beta^{B-}) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))]\mathbb{1}_{\{a>x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x)\geq\beta^{B-}\}}\mathbb{1}_{\{\epsilon^B=0\}} \\ [\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x-a)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) + a(-\beta^{B-} + \mathfrak{I}(\mathsf{n},x-a))]\mathbb{1}_{\{a\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-a)<\beta^{B-}\}}\mathbb{1}_{\{\epsilon^B=0\}} \\ [\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n}) - \beta^{B-}) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))]\mathbb{1}_{\{a\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-a)\geq\beta^{B-}\}}\mathbb{1}_{\{\epsilon^B=0\}} \end{cases}$$

Then for the $\hat{\mathfrak{s}}$:

$$\begin{aligned} \mathbb{E}[\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x))] &= -\mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\ &+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n}) - \beta^{B-}) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\ &+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\alpha^B\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\ &+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)<\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x-\alpha^B)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\ &+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n}) - \beta^{B-}) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \end{aligned} \tag{5.28}$$

$$\begin{aligned}
\mathbb{E}[\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x))] &= -\mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n})-\beta^{B-})-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\alpha^B\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)<\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x-\alpha^B))-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n})-\beta^{B-})-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))]
\end{aligned} \tag{5.29}$$

We focus on the term $\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n})-\beta^{B-})$:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n})-\beta^{B-})] &= \sum_{b=0}^{\mathfrak{I}(\mathsf{n},x)}\sum_{i>\mathfrak{b}(\mathsf{n})-b}^{\mathfrak{b}(\mathsf{n})}\mathsf{n}^{|i}\mathbb{P}[\beta^{B-}=b] \\
&= \sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}(F^{\alpha^B}(\mathfrak{I}(\mathsf{n},x))-F^{\alpha^B}(\mathfrak{b}(\mathsf{n})-i))
\end{aligned} \tag{5.30}$$

and then

$$\mathbb{E}[\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)\geq\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}(\mathsf{n})-\beta^{B-})] = \sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}(F^{\alpha^B}(\mathfrak{I}(\mathsf{n},x-\alpha^B))-F^{\alpha^B}(\mathfrak{b}(\mathsf{n})-i)) \tag{5.31}$$

$$\begin{aligned}
&\mathbb{E}[\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x))] = -\mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
+\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\alpha^B\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\
+\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\mathbb{1}_{\{\mathfrak{I}(\mathsf{n},x-\alpha^B)<\beta^{B-}\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x-\alpha^B))-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
-\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}F^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i)] \\
-\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}F^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i)]
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&\mathbb{E}[\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x))] = -\mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
+\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\alpha^B\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\
-\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}\hat{F}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B)] \\
-\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}F^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i)] \\
+\ &\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\leq x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}(\hat{F}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))-F^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i))]
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
\mathbb{E}[\Delta\hat{\mathfrak{s}}(\mathsf{m},\mathfrak{b}^S(\mathsf{m},x))] &= \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\alpha^B\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\
-\ & \mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] - \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
+\ & \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B> x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}\hat{F}^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i)] \\
+\ & \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}^{\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}F^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B)] \\
+\ & \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}\hat{F}^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i))]
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
\mathbb{E}[(\mathfrak{b}^S(\mathsf{T}^B(\mathsf{n},\alpha^B,\beta^B),x)-\mathfrak{b}^S(\mathsf{n},x))] &= \mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}(\beta^{B+}+\mathfrak{I}(\mathsf{n},x))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}(-\beta^{B-}+\mathfrak{I}(\mathsf{n},x))] \\
&- \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x))] \\
&- \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))]
\end{aligned} \tag{5.35}$$

We obtain for $\mathfrak{g}$:

$$\begin{aligned}
\mathbb{E}[(\mathfrak{g}(\mathsf{T}^B(\mathsf{n},\alpha^B,\beta^B),x) &- \mathfrak{g}(\mathsf{n},x))] = \mathbb{E}[\mathbb{1}_{\{\epsilon^B=1\}}x(\beta^{B+}+\mathfrak{I}(\mathsf{n},x))-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\alpha^B\le x\}}\mathbb{1}_{\{\epsilon^B=0\}}x(-\beta^{B-}+\mathfrak{I}(\mathsf{n},x))-\hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x))] \\
&- \hat{F}^{\alpha^B}(x)x\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x))\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}] \\
&+ \mathbb{E}[\mathbb{1}_{\{\alpha^B\le x\}}\mathbb{1}_{\{\epsilon^B=0\}}(\alpha^B-x)\hat{H}^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B>x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}\mathsf{n}^{|i}\hat{F}^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i)] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}^{\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}F^{\beta^{B-}}(\mathfrak{I}(\mathsf{n},x-\alpha^B)] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}\mathsf{n}^{|i}\hat{F}^{\beta^{B-}}(\mathfrak{b}(\mathsf{n})-i))]
\end{aligned}$$

We compute for $\mathsf{n} := \mathsf{T}^S(\mathsf{n},q)$ and $x := x-q$. We denote $\Delta\mathfrak{b}^S(\mathsf{n},q,x) := \mathfrak{b}^S(\mathsf{n},q)-\mathfrak{b}^S(\mathsf{n},x)$. We have

$$\hat{\mathfrak{s}}(\mathsf{T}^S(\mathsf{n},q),\mathfrak{b}^S(\mathsf{T}^S(\mathsf{n},q),x-q)) = \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},q)) - q\Delta\mathfrak{b}^S(\mathsf{n},q,x)$$

$$\begin{aligned}
\frac{J^B(\mathsf{n},x,q)}{p_B} &= \mathbb{E}[(q-x)[\mathbb{1}_{\{\epsilon^B=0\}}\beta^{B-} - \mathbb{1}_{\{\epsilon^B=1\}}\beta^{B+}] + x\Delta\mathfrak{b}^S(\mathsf{n},q,x) + \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},x)) - \hat{\mathfrak{s}}(\mathsf{n},\mathfrak{b}^S(\mathsf{n},q))] \\
&- \hat{F}^{\alpha^B}(x-q)(x-q)\hat{H}^{\beta^{B-}}(\Delta\mathfrak{b}^S(\mathsf{n},q,x))\mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}] \\
&+ \mathbb{E}[\mathbb{1}_{\{\alpha^B\le x-q\}}\mathbb{1}_{\{\epsilon^B=0\}}(\alpha^B - x + q)\hat{H}^{\beta^{B-}}(\Delta\mathfrak{b}^S(\mathsf{n},q,x-\alpha^B))] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B> x-q\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}^{\mathfrak{b}^S(\mathsf{n},q)}(\mathsf{n}^{|i}-q)\hat{F}^{\beta^{B-}}(\mathfrak{b}^S(\mathsf{n},q)-i)] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x-q\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x)}^{\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}(\mathsf{n}^{|i}-q)F^{\beta^{B-}}(\Delta\mathfrak{b}^S(\mathsf{n},q,x-\alpha^B)] \\
&+ \mathbb{E}[\mathbb{1}_{\{\epsilon^B=0\}}\mathbb{1}_{\{\alpha^B\le x-q\}}\sum_{i>\mathfrak{b}^S(\mathsf{n},x-\alpha^B)}^{\mathfrak{b}^S(\mathsf{n},q)}(\mathsf{n}^{|i}-q)\hat{F}^{\beta^{B-}}(\mathfrak{b}^S(\mathsf{n},q)-i))]
\end{aligned}$$

We compute $\Delta_q J^B(\mathsf{n},x,q)$ for $q\in\mathcal{A}_\mathfrak{b}$ in the same way as $\Delta_q J^C(\mathsf{n},x,q)$. We find $\Delta_q J^B(\mathsf{n},x,q) > 0$. The lemma is proved. □

Conclusion

The main result of this chapter is the strategy is reduced to choose the price impact instead of the quantity, in the case of the one step model. In order to find this result, we just use the dynamical programming principle and the different relations between the operators of $\mathcal{N}$.

Deuxième partie

Optimal execution in a one-sided order book with stochastic volume process

Chapitre 6

Optimal execution in a one-sided limit order book with stochastic volume process

Contents

This part is independent of the first part of the thesis.

Introduction

This chapter deals with the optimal liquidation problem in a one-sided limit order shape framework, we presented in chapter 2.

In section 1.4, we describe the general optimal liquidation problem and we refer to an introduction of the limit order book shape model in subsection 1.4.2. In our model, we are in a continuous-time framework, the strategy can be decomposed with a jump part, and a singular part and a continuous part. It leads to a singular control problem. We refer to the Pham's book [55] for general stochastic control theory for continuous diffusion. Since we allow to a general diffusion with a jump part, we use the Oksendal and Sulem's book [54].

We explain the optimal liquidation problem by defining the strategies, the gain function and the value function in the section 6.1. In the section 6.2, we give the dynamic programming principle and the Hamilton- Jacobi-Bellman equations for the value function. We characterize the value function as the unique continuous viscosity solution of the Hamilton-Jacobi-Bellman equations in the section 6.3. We give specific examples of our model and present numerical results obtained by a numerical approach in the section 6.4.

6.1 Model

The cost of a financial transaction depends on the two following quantities : the liquidity in the market, which is summarized through the limit order book and the volume of the trade, this is the quantity of shares to be traded. In the usual setup of mathematical finance, liquidity is assumed to be infinite and the mean price paid for the transaction does not depend on the volume. In this case, the quantity of shares traded does not impact the shape of the limit order book. In contrast, when large traders participate in the market this assumption no longer holds. To liquidate large positions or to perform big purchases, large traders incur costs that depend on the volume of the transaction and the current shape of the limit order book. A large trader who wants to liquidate or purchase a large quantity of shares over a fixed period of time would like to split the big trade into smaller trades, and benefit from the execution timing to minimize the costs. This problem is known as the *optimal order execution trade.*

To model the mean price of a transaction there are two aspects to take into account :

- The quantity of shares to be traded (ΔX_t)
- The liquidity in the market (f_t), modeled through the Limit Order Book. Liquidity in a LOB has three components : *Depth*, *Resilience*, and *Tightness.* Since we work on a one-sided LOB we disregard this last aspect and will model LOB only through its density. We will, have

$$S_t^{mean} = F(\Delta X_t, f_t)$$

where ΔX_t is the quantity to trade and $f_t : \Omega \mapsto \mathcal{F}(\mathbb{R}_+)$ is the density of the order book. In general, to model f_t is complicated because it involves infinite dimensional processes. We take the following approach : assume that there is a fix form f from the start. Then we model

$$f_t(\omega)(a) = f(a + Y_t(\omega))$$

where Y_t models the volume up to time t, not only including the investor but also other agents in the market.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness. We assume that this filtered probability space supports a standard $\mathbb{P}$-Brownian motion W and M a random Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with mean measure $\gamma_t dtm(dz)$ where $\gamma : [0,T] \to (0, \bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$.

A financial agent wants to buy $\overline{X}$ shares of an illiquid asset over the time interval $[0,T]$. Without loss of generality we will assume that all quantities are discounted.

Let $(A_t)_{t\geq 0}$ be the reference price of the assets, which we assume to be a continuous $\mathbb{P}$-martingale. In our model, we assume that, in the absence of trading, the number of available shares at time t in the price interval $[A_t, A_t + x)$ is $F(x)$. F is a non-decreasing and left-continuous function associated to an infinite measure μ on $[0, +\infty)$ in the following way :

$$F(x) := \mu([0,x)), \quad \text{for all } x \geq 0. \tag{6.1}$$

Assumption 6.1.1. *We will make some assumption on the function F*

i. There exists $b > 0$, $\alpha > 0$ and $a > 0$ such that

$$F(x) \geq bx^{\alpha}, \quad \textit{for all } x \geq a. \tag{6.2}$$

ii. There exists $K > 0$ and $q > 0$ such that, for any $x, x' \geq 0$,

$$\textit{if } \mid F(x) - F(x') \mid < q \textit{ then } \mid x - x' \mid \leq K. \tag{6.3}$$

Examples :

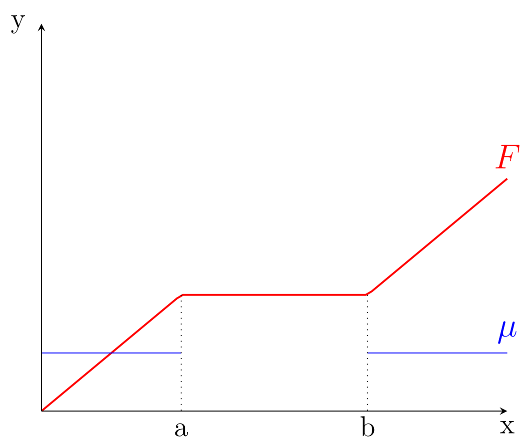


FIGURE 6.1 – Modified Block Order Book

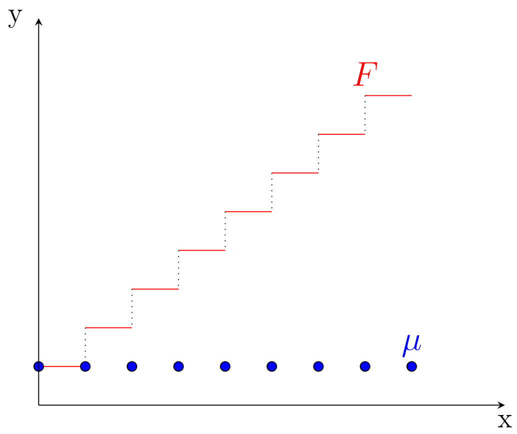


FIGURE 6.2 – Discrete Order Book

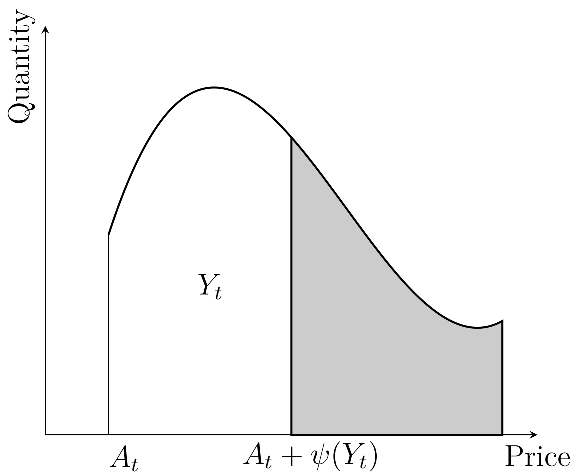
Investor's strategies. The strategies of the agent are given by non-decreasing right-continuous adapted processes $(X_t)_{0\leq t\leq T}$ with $X_T = \overline{X}$. We assume that $X_{0-} = 0$ and denote by $\Delta X_t = X_t - X_{t-}$ the jump at time t.

The dynamics of the volume effect process (Y_t)**.** We assume that the strategy of our investor has impact on the price. When the financial agent follows strategy X, we assume that at time t, the ask price is no longer the reference price but is given by $A_t + D_t$ where $D_t := \psi(Y_t)$, with Y_t representing the dynamics of the volume effect process

$$dY_t = dX_t - h(Y_{t^-})dt + \sigma(Y_{t^-})dW_t + \int_{\mathbb{R}} Y_{t^-}q(Y_{t^-}, z)\bar{M}(dt, dz); Y_{0-} = y. \qquad (6.4)$$

and the left-continuous function ψ given by follows

$$\psi(y) := \sup\{a \geq 0 | F(a) < y\}, \text{ for } y > 0 \text{ and } \psi(0) := 0. \qquad (6.5)$$



Assumption 6.1.2. *We assume that h, σ, q are such that,*

i) For all $n \in \mathbb{N}$ and $y \geq 0$, we have

$$\sup_{t\in[0,T];\ X\in\mathcal{A}} \mathbb{E}\Big[(Y_t^{y,X})^n\Big] < +\infty, \tag{6.6}$$

where $\mathcal{A}$ is the set of non-decreasing right-continuous adapted processes $(X_t)_{0\leq t\leq T}$ with $X_T = \overline{X}$ and $X_{0-} = 0$.

ii) There exists $\beta > 0$ such that

$$\lim_{y\to+\infty} \sigma(y)^2 y^{\beta-2} < +\infty \tag{6.7}$$

We denote

$$\check{Y}_{t^-} := Y_{t^-} + \varDelta_M Y_t$$

where $\varDelta_M Y_t$ is the jump of the measure M at time t.

Remarks :

1.) The volume effect process Y incorporates both the impact from the investor's trading through the term dX_t and from other agents in the market.

2.) When the investor follows the strategy $(X_t)_{0\leq t\leq T}$, for any given $a \geq 0$, the number of assets available between prices A_t and A_t+a becomes $(F(a)-Y_{t^-})_+$.

3.) Equation (6.4) specifies how the order book is affected by the transactions of the trader. Suppose that there is a large transaction at time t with the investor buying $\varDelta X_t$ shares, then by equation (6.4), $Y_t = \check{Y}_{t^-} + \varDelta X_t$. Right after this transaction the ask price jumps from $A_t + \psi(\check{Y}_{t^-})$ to $A_t + \psi(\check{Y}_{t^-} + \varDelta X_t)$.

Strategy cost. We now can write the cost of the strategy $X = (X_t)_{0\leq t\leq T}$ as

$$\begin{aligned} C(X) := & \int_0^T (A_t + \check{D}_{t^-}) dX_t^c + \sum_{0\leq t\leq T} [A_t \varDelta X_t + (\varPhi(Y_t) - \varPhi(\check{Y}_{t-}))], \\ = & \int_0^T \psi(\check{Y}_{t-}) dX_t^c + \sum_{0\leq t\leq T} (\varPhi(Y_t) - \varPhi(\check{Y}_{t-})) + \int_0^T A_t dX_t. \end{aligned}$$

where X^c denotes the continuous part of X, $\check{D}_{t^-} := \psi(\check{Y}_{t^-})$ and $\varPhi(y) \triangleq \int_0^{\psi(y)} \xi dF(\xi)$. Notice that it follows from the monotonicity and the left-continuity of F, that for all $y > 0$ we have $F(\psi(y)) = y$ and then

$$\varPhi(y) = \int_0^y \psi(\zeta)\, d\zeta, \quad y > 0. \tag{6.8}$$

Notice that the integration by parts formula implies that

$$\mathbb{E}\left[\int_0^T A_t dX_t\right] = \mathbb{E}\left[A_T\overline{X} - A_0X_{0-} + \int_0^T X_{t-}dA_s\right].$$

Under suitable conditions on A, since X is nonnegative and bounded by $\overline{X}$ and $X_{0-} = 0$, this can be simplified to obtain

$$\mathbb{E}\left[\int_0^T A_t dX_t\right] = \overline{X}A_0.$$

The expected cost for the investor following strategy X becomes

$$\mathbb{E}\left[\int_0^T \psi(\check{Y}_{t-})dX_t^c + \sum_{0\le t\le T}(\Phi(Y_t) - \Phi(\check{Y}_{t-})) \; + \; \bar{X}A_0\right]. \tag{6.9}$$

Control Problem We analyze (6.9) as a control problem in which the controls are given by the strategies X and Y is a state variable whose dynamics are given by (6.4). As usual we introduce a time variable t and make the problem dynamic by introducing the following value function for $0 \le t < T$ and $x \in [0, \overline{X}]$ and $y \ge 0$.

$$v(t,x,y) = \inf_{X\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T \psi(\check{Y}_{s-}^{t,y,X})dX_s^c + \sum_{t\le s\le T}(\Phi(Y_s^{t,y,X}) - \Phi(\check{Y}_{s-}^{t,y,X}))\right], \tag{6.10}$$

where $Y_s^{t,y,X}$ for $t \le s \le T$ denotes the solution of (6.4) with $Y_{t-}^{t,y,X} = y$ and the set of admissible controls $\mathcal{A}(t,x)$ is given by

$$\mathcal{A}(t,x) \triangleq \{X : X \nearrow;\; X_{t-} = x;\; X_T = \overline{X}\}. \tag{6.11}$$

The value function at terminal time T is given by

$$v(T,x,y) = \Phi(y + \overline{X} - x) - \Phi(y). \tag{6.12}$$

Notice that

$$v(t,\overline{X},y) = 0. \tag{6.13}$$

6.2 Analytical properties of the value function

Let $(t,x,y) \in \mathcal{S} := [0,T) \times [0,\overline{X}] \times [0,+\infty)$ be the state variable at time t. If the investor immediately buys $\overline{X} - x$, the associated cost would be equal to

$\Phi(y+\overline{X}-x)-\Phi(y)$ therefore, we have

$$v(t,x,y) \leq \Phi(y+\overline{X}-x)-\Phi(y) \text{ on } [0,T)\times[0,\overline{X}]\times[0,+\infty).$$

6.2.1 Continuity

In this section we will denote for any $t,y\geq 0$ and $X\in\mathcal{A}(t,x)$

$$C(t,x,y,X) := \int_t^T \psi(\check{Y}_{u^-}^{t,y,X})dX_u^c + \sum_{t\leq u\leq T}(\Phi(Y_u^{t,y,X})-\Phi(\check{Y}_{u^-}^{t,y,X})). \tag{6.14}$$

To prove continuity of the value function we need the following lemmas.

Lemma 6.2.1. *For any non-decreasing right-continuous adapted process X and any initial time t, the flow $y\to Y_\cdot^{t,y,X}$ is continuous.*

Proof : See Theorems V.37 and V.38 of [58] □

Lemma 6.2.2. *For any $t\in[0,T]$ and $y\geq 0$, the value function $x\to v(t,x,y)$ is decreasing.*

Proof : Let $t\in[0,T]$ and $y\geq 0$. Suppose that $0\leq x<x'\leq\overline{X}$. Let $\epsilon>0$ be arbitrary and assume that $X\in\mathcal{A}(t,x)$ satisfies

$$v(t,x,y)+\epsilon \geq \mathbb{E}[C(t,x,y,X)].$$

Let

$$\tau \triangleq \inf\{s\geq t : X_s - x \geq \overline{X}-x'\}.$$

We define the strategy $X'\in\mathcal{A}(t,x')$ as follows. $X'_s = x'+(X_s-x)$ for $s<\tau$ and $X_s=\overline{X}$ for $s>\tau$. We have that

$$C(t,x',y,X') \leq C(t,x,y,X).$$

Taking expectations on both sides we deduce that

$$v(t,x',y) \leq \mathbb{E}[C(t,x',y,X')] \leq \mathbb{E}[C(t,x,y,X)] \leq v(t,x,y)+\epsilon.$$

Since $\epsilon>0$ was arbitrary the conclusion follows. □

Lemma 6.2.3. *Let $0\leq s\leq t\leq T$, $x\in[0,\overline{X}]$, $y\geq 0$ and $X\in\mathcal{A}(t,x)$. For any random variable ξ such that for any $\eta>0$, $\mathbb{P}\left(\xi>\sqrt{\eta}\right)\leq\sqrt{\eta}$ and ξ admits moments of any order, we have*

$$\mathbb{E}\Big[\Phi\left(Y_t^{s,y,X}+\xi\right)-\Phi\left(Y_t^{s,y,X}\right)\Big] \leq \rho_y(\eta). \tag{6.15}$$

where ρ_y is a continuous function defined on $\mathbb{R}^+$ such that $\lim_{\zeta\to 0}\rho_y(\zeta)=0$.

Proof : Let ξ be a random variable such that for any $\eta>0$, $\mathbb{P}\left(\xi>\sqrt{\eta}\right)\le\sqrt{\eta}$ and ξ admits moments of any order. Let $\eta>0$, we have

$$\begin{aligned}
\varDelta &:= \mathbb{E}\Big[\varPhi\left(Y_t^{s,y,X}+\xi\right)-\varPhi\left(Y_t^{s,y,X}\right)\Big]\\
&= \mathbb{E}\Big[\int_0^{\xi}\psi\left(Y_t^{s,y,X}+\zeta\right)d\zeta\Big]\\
&\le \mathbb{E}\Big[\int_0^{\sqrt{\eta}}\psi\left(Y_t^{s,y,X}+\zeta\right)d\zeta\,\mathbb{1}_{\{\xi\le\sqrt{\eta}\}}\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{\xi}\psi\left(Y_t^{s,y,X}+\zeta\right)d\zeta\,\mathbb{1}_{\{\xi>\sqrt{\eta}\}}\Big].
\end{aligned}\tag{6.16}$$

We will find an upper bound for the first term of inequality (6.16). First, we notice that assumption (6.2) implies that we have

$$\text{For all } y\ge F(a),\quad \psi(y)\le b^{-\frac{1}{\alpha}}y^{\frac{1}{\alpha}}\tag{6.17}$$

As ψ is non decreasing, we have

$$\begin{aligned}
\varDelta_1 &:= \mathbb{E}\Big[\int_0^{\sqrt{\eta}}\psi\left(Y_t^{s,y,X}+\zeta\right)d\zeta\,\mathbb{1}_{\{\xi\le\sqrt{\eta}\}}\Big]\\
&\le \mathbb{E}\Big[\int_0^{\sqrt{\eta}}\left(\frac{F(a)+\zeta}{b}\right)^{\frac{1}{\alpha}}\mathbb{1}_{\{Y_t^{s,y,X}\le F(a)\}}d\zeta\Big]\\
&\quad+\mathbb{E}\Big[\int_0^{\sqrt{\eta}}\left(\frac{Y_t^{s,y,X}+\zeta}{b}\right)^{\frac{1}{\alpha}}\mathbb{1}_{\{Y_t^{s,y,X}>F(a)\}}d\zeta\Big]\\
&= \frac{\alpha}{b^{\frac{1}{\alpha}}(1+\alpha)}\mathbb{E}\Big[\left(F(a)\right)^{\frac{\alpha+1}{\alpha}}\mathbb{1}_{\{Y_t^{s,y,X}\le F(a)\}}\left((1+\frac{\sqrt{\eta}}{F(a)})^{\frac{\alpha+1}{\alpha}}-1\right)\Big]\\
&\quad\frac{\alpha}{b^{\frac{1}{\alpha}}(1+\alpha)}\mathbb{E}\Big[\left(Y_t^{s,y,X}\right)^{\frac{\alpha+1}{\alpha}}\mathbb{1}_{\{Y_t^{s,y,X}>F(a)\}}\left((1+\frac{\sqrt{\eta}}{Y_t^{s,y,X}})^{\frac{\alpha+1}{\alpha}}-1\right)\Big]\\
&\le C^y\left((1+\frac{\sqrt{\eta}}{F(a)})^{\frac{\alpha+1}{\alpha}}-1\right),
\end{aligned}$$

where we have set

$$C^y:=\frac{\alpha}{b^{\frac{1}{\alpha}}(1+\alpha)}\left(F(a)^{\frac{\alpha+1}{\alpha}}+\mathbb{E}\Big[\left(Y_t^{s,y,X}\right)^{\frac{\alpha+1}{\alpha}}\Big]\right).$$

Now we turn to evaluating the second term of inequality (6.2.3). As ψ is non decreasing, we deduce from inequality (6.17) and Cauchy-Schwartz inequality that

$$
\begin{aligned}
\Delta_2 &:= \mathbb{E}\Big[\int_0^\xi \psi\left(Y_t^{s,y,X}+\zeta\right) d\zeta \mathbb{1}_{\{\xi>\sqrt{\eta}\}}\Big] \\
&\leq \mathbb{E}\Big[\xi\left(\frac{Y_t^{s,y,X}+\xi}{b}\right)^{\frac{1}{\alpha}} \mathbb{1}_{\{\xi>\sqrt{\eta}\}}\Big] \\
&\leq \left(\mathbb{E}\Big[\xi^2\left(\frac{Y_t^{s,y,X}+\xi}{b}\right)^{\frac{2}{\alpha}}\Big]\mathbb{P}\left(\xi>\sqrt{\eta}\right)\right)^{\frac{1}{2}}
\end{aligned}
$$

As ξ admits moments of any order and $\mathbb{P}\left(\xi > \sqrt{\eta}\right) \leq \sqrt{\eta}$, we conclude the proof thanks to assumption (6.6). ☐

Theorem 6.2.1. *The value function v defined in* (6.10) *is a continuous function on* $[0,T]\times[0,\overline{X}]\times[0,\infty)$.

Proof : We shall study separately the continuity in each of the variables.
(1.) Continuity in x uniformly with respect to t :
We fix $t\in[0,T]$, $y\geq 0$ and $0\leq x' < x \leq \overline{X}$. Let $\varepsilon>0$, there exists $X\in\mathcal{A}(t,x)$ which satisfies

$$v(t,x,y)+\varepsilon \geq \mathbb{E}[C(t,x,y,X)].$$

By adding a jump of size $x-x'$ at time T to X we obtain $\widehat{X}\in\mathcal{A}(t,x')$. We have that

$$C(t,x',y,\widehat{X}) = C(t,x,y,X) + \Phi(Y_T^{t,y,X}+(x-x')) - \Phi(Y_T^{t,y,X}).$$

Taking expectations on both sides of this equation we deduce that

$$\mathbb{E}[C(t,x',y,\widehat{X})] \leq v(t,x,y)+\varepsilon+\mathbb{E}[(\Phi(Y_T^{t,y,X}+(x-x'))-\Phi(Y_T^{t,y,X}))].$$

On the other hand since, by Lemma 6.2.2, the value function is decreasing in x, we conclude that

$$v(t,x,y)\leq v(t,x',y)\leq v(t,x,y)+\varepsilon+\mathbb{E}[(\Phi(Y_T^{t,y,X}+(x-x'))-\Phi(Y_T^{t,y,X}))].$$

We deduce from Lemma (6.2.3) that

$$0\leq v(t,x',y)-v(t,x,y)\leq \varepsilon+\rho_y(x-x').$$

Therefore, we have obtained the continuity of the value function v in x uniformly in t.

(2.) Continuity in y uniformly with respect to t and x :
Fix $y\geq 0$, $y'\geq 0$ and $\epsilon>0$. Let $t\in[0,T]$ and $x\in[0,\overline{X}]$. There exists $X\in\mathcal{A}(t,x)$

which satisfies

$$v(t,x,y)+\epsilon \geq \mathbb{E}[C(t,x,y,X)].$$

Let $\widehat{X} \in \mathcal{A}(t,x)$ such that

$$d\widehat{X}^c_s = 0 \qquad \text{and} \qquad \Delta\widehat{X}_s = \Delta X_s \qquad \text{for all } t \leq s < T. \tag{6.18}$$

We define the following stopping time :

$$\tau := \inf\{s \geq t : \; Y^{t,y',\widehat{X}}_s \leq Y^{t,y,X}_s\}. \tag{6.19}$$

We define the following random variable :

$$\xi_\tau := \int_t^\tau dX^c_u - \Big[Y^{t,y,X}_\tau - Y^{t,y',\widehat{X}}_\tau\Big], \tag{6.20}$$

Notice that when $y' \leq y$, $\tau = t$ and $\xi_\tau \leq 0$. Now we construct the following strategy $\widetilde{X}$ belonging to $\mathcal{A}(t,x)$:

$$d\widetilde{X}_s := \begin{cases} d\widehat{X}_s, & \text{for } t \leq s < \tau \\ \Big(\Delta X_s + (Y^{t,y,X}_s - Y^{t,y',\widehat{X}}_s)\Big) \wedge \Big(\overline{X} - \widehat{X}_{\tau^-}\Big), & \text{for } s = \tau \\ dZ_s, & \text{for } \tau < s \leq T, \end{cases}$$

where Z is the non decreasing process such that $Z_\tau = 0$,

$$\text{on } \{\xi_\tau \geq 0\}, \quad dZ_s := \begin{cases} dX_s, & \text{for } \tau < s < T, \\ \Delta X_s + \xi_\tau, & \text{for } s = T \end{cases}$$

and,

$$\text{on } \{\xi_\tau < 0\}, \quad dZ_s = \begin{cases} dX_s, & \text{for } \tau < s < \theta, \\ \overline{X} - \Big(\widehat{X}_\tau + X_{\theta^-} - X_\tau\Big) & \text{for } s = \theta < T \\ 0, & \text{for } \theta < s \leq T, \end{cases}$$

where θ is defined by

$$\theta := \inf\{u \geq \tau : \; \widehat{X}_\tau + X_u - X_\tau \geq \overline{X}\}.$$

We then obtain the following inequality

$$\begin{aligned} v(t,x,y') - v(t,x,y) &\leq \varepsilon + \mathbb{E}[C(t,x,y',\widetilde{X}) - C(t,x,y,X)] \\ &= \varepsilon + R_1 + R_2 + R_3 + R_4 + R_5, \end{aligned}$$

where we have set

$$R_1 = \mathbb{E}\int_t^T \psi(\check{Y}_{u^-}^{t,y',\widetilde{X}})d\widetilde{X}_u^c - \int_t^T \psi\left(\check{Y}_{u^-}^{t,y,X}\right)dX_u^c \leq 0$$

$$R_2 = \mathbb{E}\sum_{t\leq u<\tau}\Big[\Phi(Y_u^{t,y',\widetilde{X}}) - \Phi(\check{Y}_{u^-}^{t,y',\widetilde{X}})\Big] - \Big[\Phi(Y_u^{t,y,X}) - \Phi(\check{Y}_{u^-}^{t,y,X})\Big]$$

$$R_3 = \mathbb{E}\Big[\Phi(Y_\tau^{t,y',\widetilde{X}}) - \Phi(\check{Y}_{\tau^-}^{t,y',\widetilde{X}})\Big] - \Big[\Phi(Y_\tau^{t,y,X}) - \Phi(\check{Y}_{\tau^-}^{t,y,X})\Big]$$

$$R_4 = \mathbb{E}\mathbb{1}_{\{\xi_\tau\geq 0\}}\Big[\Phi(Y_T^{t,y',\widetilde{X}} + \xi_\tau) - [\Phi(Y_T^{t,y,X})\Big]$$

$$R_5 \leq \mathbb{E}\mathbb{1}_{\{\xi_\tau< 0\}}\Big[\Phi(\check{Y}_{\theta^-}^{t,y',\widetilde{X}} + \Delta\widetilde{X}_\theta) - \Phi(\check{Y}_{\theta^-}^{t,y,X} + \Delta X_\theta)\Big] \leq 0.$$

We begin with finding an upper bound for R_2. As we have

$$\sum_{t\leq u<\tau}\Delta X_u \leq \overline{X},$$

there exists $\delta > 0$ such that

$$\sum_{t\leq u<\tau;\ \Delta X_u<\delta}\Delta X_u < \epsilon.$$

We split the sum in R_2, between a sum of big jumps of X and small jumps of X. More precisely we can write that $R_2 = S_1 + S_2$ where

$$S_1 := \mathbb{E}\sum_{t\leq u<\tau;\ \Delta X_u\geq\delta}\Big[\Phi(Y_u^{t,y',\widetilde{X}}) - \Phi(\check{Y}_{u^-}^{t,y',\widetilde{X}})\Big] - \Big[\Phi(Y_u^{t,y,X}) - \Phi(\check{Y}_{u^-}^{t,y,X})\Big]$$
$$S_2 := \mathbb{E}\sum_{t\leq u<\tau;\ \Delta X_u<\delta}\Big[\Phi(Y_u^{t,y',\widetilde{X}}) - \Phi(\check{Y}_{u^-}^{t,y',\widetilde{X}})\Big] - \Big[\Phi(Y_u^{t,y,X}) - \Phi(\check{Y}_{u^-}^{t,y,X})\Big]$$

First, notice that Φ is non decreasing and that $\check{Y}_{u^-}^{t,y,X} \leq \check{Y}_{u^-}^{t,y',\widetilde{X}} \leq \check{Y}_{u^-}^{t,y',X}$ for $u < \tau$, therefore we have

$$S_1 \leq \mathbb{E}\sum_{t\leq u<\tau;\ \Delta X_u\geq\delta}\Phi(\check{Y}_{u^-}^{t,y',X} + \Delta X_u) - \Phi(\check{Y}_{u^-}^{t,y,X} + \Delta X_u).$$

We denote by $u_i \in [t,\tau)$ the random times at which $\Delta X_u \geq \delta$. We have a finite number of u_i, lower than $\lceil\frac{\overline{X}}{\delta}\rceil$.
For $u \in [t,\tau)$, we set $\gamma_u := \check{Y}_{u^-}^{t,y',X} - \check{Y}_{u^-}^{t,y,X}$ and notice that $\gamma_u \geq 0$ a.s.Hence, we

obtain

$$\begin{aligned} S_1 &\leq \mathbb{E}\Big[\sum_{t\leq u<\tau;\ \Delta X_u\geq\delta}\int_0^{\gamma_u}\psi\left(\check{Y}_{u^-}^{t,y,X}+\zeta\right)d\zeta\Big] \\ &\leq \sum_{i=1}^{\lceil\frac{\overline{X}}{\delta}\rceil}\mathbb{E}\Big[\int_0^{\gamma_{u_i}}\psi\left(\check{Y}_{u_i^-}^{t,y,X}+\zeta\right)d\zeta\Big] \end{aligned}$$

Moreover, for any $u\in[t,\tau)$, it follows from Itô's formula that,

$$\begin{aligned} \mathbb{E}[\gamma_u] &= y'-y+\mathbb{E}\Big[\int_0^{\tau}h(Y_{u^-}^{t,y,X})-h(Y_{u^-}^{t,y',X})\,du\Big] \\ &\leq y'-y. \end{aligned}$$

Thanks to Lemma (6.2.3), we can then assert that

$$S_1\leq\left\lceil\frac{\overline{X}}{\delta}\right\rceil\rho_y(y'-y). \tag{6.21}$$

Once again, it follows from the inequality $\check{Y}_{u^-}^{t,y',\widetilde{X}}\leq\check{Y}_{u^-}^{t,y',X}$ for any $u<\tau$, that

$$S_2 \;\leq\; \mathbb{E}\Big[\sum_{t\leq u<\tau;\ \Delta X_u<\delta}\int_0^{\Delta X_u}\psi\left(\check{Y}_{u^-}^{t,y',X}+\zeta\right)-\psi\left(\check{Y}_{u^-}^{t,y,X}+\zeta\right)d\zeta\Big]$$

From the uniform continuity of the flow and the assumption (6.3), we know that for ε small enough, we have

$$S_2 \;\leq\; \mathbb{E}\Big[\sum_{t\leq u<\tau;\ \Delta X_u<\delta}K\Delta X_u\Big]\leq K\varepsilon.$$

We obviously have $R_3\leq 0$ because Φ is non decreasing and that $\check{Y}_{\tau^-}^{t,y,X}\leq\check{Y}_{\tau^-}^{t,y',\widetilde{X}}\leq\check{Y}_{\tau^-}^{t,y',X}$ for $u<\tau$.

We turn to finding an upper bound for R_4. Notice that we have $R_4=0$ when $y'\leq y$ therefore we assume here that $y'>y$. From Itô's formula we deduce that there exists a martingale M such that :

$$Y_{\tau}^{t,y,X}-Y_{\tau}^{t,y',\widehat{X}} \;=\; y-y'+\int_t^{\tau}dX_u^c+\int_t^{\tau}h(Y_{u^-}^{t,y',\widehat{X}})-h(Y_{u^-}^{t,y,X})\,du+M_{\tau},$$

thus, as h is non increasing, we have $\mathbb{E}\Big[\xi_{\tau}\Big]\leq y'-y$. We may apply Lemma (6.2.3) to get

$$R_4 = \mathbb{E}\Big[\int_0^{\xi_\tau^+} \psi(Y_T^{t,y',\widetilde{X}} + \zeta)\, d\zeta\Big] \\ \leq \rho_{y'}(y' - y)$$

We conclude this proof by recalling that $\zeta \to \rho_\zeta(\eta)$ is continuous for any $\eta \geq 0$.

(3.) Continuity in t : Let $x \in [0, \overline{X}]$, $y \geq 0$ and $0 \leq s < t$. We first observe that given a strategy $X \in \mathcal{A}(t,x)$ as in (6.11), we can construct a strategy $\widehat{X} \in \mathcal{A}(s,x)$ by not doing anything on $[s,t)$ and then, we follow the strategy $\widehat{X} = X'$, where $dX'_u = \delta X_u$ on $[t,T]$, up to time τ where the stopping time τ is defined by

$$\tau := \inf\{u \geq t : \ \check{Y}_{u^-}^{s,y,\widehat{X}} \leq y\}.$$

then, we set

$$d\widehat{X}_u = \begin{cases} \Big(\Delta X_\tau + (y - \check{Y}_{\tau^-}^{s,y,\widehat{X}})\Big) \wedge (\overline{X} - x), & \text{for } u = \tau \\ dX_u, & \text{for } \tau < u < \theta \\ \overline{X} - \widehat{X}_{u^-}, & \text{for } u = \theta \\ 0, & \text{for } \theta < u \leq T, \end{cases}$$

where we have set

$$\theta := \inf\{u \geq t : \ \widehat{X}_{u^-} + \Delta X_u \geq \overline{X}\}.$$

Considering an ε-optimal strategy $X \in \mathcal{A}(t,x)$ and following the lines of the previous step of the proof, we can show that

$$\begin{aligned} v(s,x,y) - v(t,x,y) &\leq C(s,x,y,\widehat{X}) - C(t,x,y,X) + \varepsilon \\ &\leq \mathbb{E}\Big[C(t,x,Y_{t^-}^{s,y,\widehat{X}},\widehat{X}) - C(t,x,y,X)\Big] + \varepsilon \\ &\leq f(\varepsilon) \quad \text{where } f \text{ is s.t. } \lim_0 f = 0 \end{aligned}$$

Now, we consider a strategy $X \in \mathcal{A}(s,x)$ such that $v(s,x,y) \geq C(s,x,y,X) - \varepsilon$. We define the strategy $\widetilde{X} \in \mathcal{A}(t,x)$ by

$$d\widetilde{X}_u = \begin{cases} \Big(\Delta X_t + (\check{Y}_{t^-}^{s,y,X} - y)^+\Big) \wedge (\overline{X} - x), & \text{for } u = t \\ dX_u, & \text{for } t < u < \theta \\ \overline{X} - \widetilde{X}_{u^-}, & \text{for } u = \theta \text{ and } \theta \leq T \\ \Delta X_T + X_t - X_{s^-} - (\check{Y}_{t^-}^{s,y,X} - y)^+ (= \overline{X} - \widetilde{X}_{T^-}), & \text{for } u = T \text{ and } \theta = +\infty, \end{cases}$$

where we have set $\theta = 0$ on $\{\Delta X_t + (\check{Y}^{s,y,X}_{t^-} - y)^+ \geq \overline{X} - x\}$ and

$$\theta := \inf\{u \geq t : \ \widetilde{X}_{u^-} + \Delta X_u \geq \overline{X}\} \quad \text{on } \{\Delta X_t + (\check{Y}^{s,y,X}_{t^-} - y)^+ < \overline{X} - x\},$$

with the convention $\inf \emptyset = +\infty$. Following again the lines of the previous step of the proof, we can show that

$$\begin{aligned} v(s,x,y) - v(t,x,y) &\geq C(s,x,y,X) - C(t,x,y,\widetilde{X}) - \varepsilon \\ &\geq \mathbb{E}\Big[C(t, X_{t^-}, Y^{s,y,X}_{t^-}, \widehat{X}) - C(t,x,y,\widetilde{X})\Big] + \varepsilon \\ &\geq g(\varepsilon) \quad \text{where } f \text{ is s.t. } \lim_0 g = 0 \end{aligned}$$

(3.) Conclusion : We can combine the previous considerations to get join continuity because the arguments are uniform in the other variables. □

6.2.2 Dynamic programming principle

We may state the standard Dynamic programming principle associated to our control problem

Dynamic Programming Principle (DPP)
Let $(t,x,y) \in [0,T) \times [0,\overline{X}] \times \mathbb{R}^+$. For any stopping time τ taking values in (t,T), we have

$$\begin{aligned} v(t,x,y) = \inf_{X \in \mathcal{A}(t,x)} \mathbb{E}\Big[\int_t^\tau \psi(\check{Y}^{t,y,X}_{s-}) dX^c_s + \sum_{t \leq s \leq \tau} (\Phi(Y^{t,y,X}_s) - \Phi(\check{Y}^{t,y,X}_{s-})) \\ + v(\tau, X_\tau, Y^{t,y,X}_\tau)\Big]. \end{aligned} \tag{6.22}$$

6.2.3 Heuristic HJB

If the value function is smooth, we would have by Itô's lemma that for $r > t$

$$\begin{aligned} v(r, X_{r-}, E^{t,y,X}_{r-}) = v(t,x,y) + \int_t^r \left(\frac{\partial v}{\partial t} + \tilde{\mathcal{L}}^\pi\right)(u, X_{u^-}, Y^{t,y,X}_{u^-})\, du \\ + \sum_{t \leq u < r} \Delta v(u, X_u, Y^{t,y,X}_u) + \text{local martingale}, \end{aligned}$$

where

$$\tilde{\mathcal{L}}^\pi v \triangleq \pi \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\frac{\partial^2 v}{\partial y^2}\sigma^2 - h\frac{\partial v}{\partial y} + \gamma_t \int_{\mathbb{R}} (v(t,x,y - q(y,z)) - v(t,x,y))\, m(dz);$$

and

$$dX^c_u = \pi_u \, du.$$

We should have then that

$$0 = \inf_{X \in \mathcal{A}(t,x)} \mathbb{E}\bigg[\int_t^{\tau} \left(\frac{\partial v}{\partial t} + \mathcal{L}^{\pi} \right) (u, X_{u^-}, Y_{u^-}^{t,y,X})\, du \tag{6.23}$$
$$+ \sum_{t \le u < \tau} (\Phi(E_u^{t,y,X}) - \Phi(E_{u^-}^{t,y,X}) + \Delta v(u, X_u, Y_u^{t,y,X}))\bigg].$$

where

$$\mathcal{L}^{\pi} v \triangleq \tilde{\mathcal{L}}^{\pi} + \psi\pi.$$

Along an optimal strategy X^* it should be the case that at all times

$$\Phi(E_u^{t,y,X^*}) - \Phi(E_{u^-}^{t,y,X^*}) + \Delta v(u, X_u^*, Y_u^{t,y,X^*}) = 0.$$

Taking $\tau = t + h$ with $h > 0$ in (6.23), dividing by h and taking the limit as h goes to 0, we conjecture that the value function should satisfy

$$0 \le \frac{\partial v}{\partial t} + \inf_{\pi \ge 0} \mathcal{L}^{\pi} v,$$

as long as the infimum is finite and equal to

$$\mathcal{L} \triangleq \mathcal{L}^0, \tag{6.24}$$

which occurs iff

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \psi \ge 0.$$

These considerations would lead us to propose the following HJB quasi-variational inequality for v

$$\max \left(-\frac{\partial v}{\partial t} - \mathcal{L}v, -\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} - \psi \right) = 0. \tag{6.25}$$

where $\mathcal{L}$ is given by (6.24).

6.3 Viscosity Characterization of the value function

Theorem 6.3.1. *The value function v is the unique continuous viscosity solution on $\mathcal{S}$ to the variational inequality :*

$$\max \left(-\frac{\partial v}{\partial t} - \mathcal{L}v,\ -\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} - \psi \right) = 0, \tag{6.26}$$

satisfying the following growth condition :

$$0 \le v(t,x,y) \le \Phi(y+\overline{X}-x) - \Phi(y) \text{ on } [0,T)\times[0,\overline{X}]\times[0,+\infty), \quad (6.27)$$

and with boundary data $v(t,\overline{X},y) = 0$ *and* $v(T,x,y) = \Phi(y+\overline{X}-x) - \Phi(y)$,

Proof : We divide the proof in three steps : we first show that v is a subsolution of equation 6.26 then that v is a supersolution of equation 6.26 and finally we establish a comparison theorem which will lead to the unicity of the solution of equation 6.26.
Proof of subsolution property.
Consider any $z_0 := (t_0,x_0,y_0) \in \mathcal{S}$ and $\varphi \in C^2(\mathcal{S})$ s.t. $v-\varphi \le 0$ on $\mathcal{S}$ and $v(z_0) = \varphi(z_0)$. We define $\varepsilon > 0$ such that $t_0+\varepsilon < T$. Let us consider an admissible control $\hat{X} \in \mathcal{A}(t_0,x_0)$ where we decide to buy $0 \le \eta < \overline{X} - x_0$ assets at time t_0 and then to do nothing until the time $t_0+\varepsilon$. From the dynamic programming principle **(DPP)**, we have

$$\begin{aligned} \varphi(z_0) = v(z_0) &\le \Phi(y_0+\eta) - \Phi(y_0) + \mathbb{E}\left[v(t_0+\varepsilon, x_0+\eta, Y^{t_0,y_0,\hat{X}}_{t_0+\varepsilon})\right] \\ &\le \Phi(y_0+\eta) - \Phi(y_0) + \mathbb{E}\left[\varphi(t_0+\varepsilon, x_0+\eta, Y^{t_0,y_0,\hat{X}}_{t_0+\varepsilon})\right]. \quad (6.28) \end{aligned}$$

Applying Itô's formula to the process $\varphi(t,x_0+\eta,Y^{t_0,y_0,\hat{X}}_t)$ between t_0 and $t_0+\varepsilon$, and taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}\left[\varphi(t_0+\varepsilon, x_0+\eta, Y^{t_0,y_0,\hat{X}}_{t_0+\varepsilon})\right] &= \varphi(t_0,x_0+\eta,y_0+\eta) \\ &+\mathbb{E}\left[\int_{t_0}^{t_0+\varepsilon} (\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi)(t,x_0+\eta,Y^{t_0,y_0,\hat{X}}_{t^-})dt\right] \quad (6.29) \end{aligned}$$

Combining relations (6.28) and (6.29), we have

$$\begin{aligned} \mathbb{E}\left[\int_{t_0}^{t_0+\varepsilon} (\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi)(t,x_0+\eta,Y^{t_0,y_0,\hat{X}}_{t^-})dt\right] &\ge \varphi(t_0,x_0,y_0) - \varphi(t_0,x_0+\eta,y_0+\eta) \\ &- [\Phi(y_0+\eta) - \Phi(y_0)] \quad (6.30) \end{aligned}$$

⋆ Take first $\eta = 0$. By dividing the above inequality by ε and letting ε going to 0, we conclude that

$$\left[\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\right](z_0) \ge 0. \quad (6.31)$$

⋆ Take now $\eta > 0$ in (6.30). By sending ε to 0, we get

$$0 \le \varphi(t_0,x_0+\eta,y_0+\eta) - \varphi(t_0,x_0,y_0) + \Phi(y_0+\eta) - \Phi(y_0).$$

It follows from equation (6.8) that

$$\begin{aligned} 0 &\le \int_0^\eta \left[\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}\right](t_0,x_0+s,y_0+s)\,ds+\int_{\psi(y_0)}^{\psi(y_0+\eta)}\xi\,dF(\xi)\\ &= \int_0^\eta \left[\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}+\psi\right](t_0,x_0+s,y_0+s)\,ds \end{aligned}$$

Dividing by η and letting $\eta\to 0$, we obtain

$$0 \le \left[\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}+\psi\right](z_0). \tag{6.32}$$

This proves the required subsolution property

$$\max\left(-\frac{\partial\varphi}{\partial t}-\mathcal{L}\varphi,\ -\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}-\psi,\right)(z_0)\le 0, \tag{6.33}$$

Proof of the supersolution property.

We prove the supersolution property by contradiction. Suppose that the claim is not true. Therefore, there exists $z_0\in\mathcal{S}$, a C^2 function φ with $(\varphi-v)(z_0)=0$ and $\varphi\le v$ on $\mathcal{S}$, and $\eta>0$ such that

$$-\frac{\partial\varphi}{\partial t}(z)-\mathcal{L}\varphi(z_0)<-\eta \quad\text{and}\quad -\left[\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}+\psi\right](z_0)<-\eta \tag{6.34}$$

From the regularity of φ, we deduce that there exists $\varepsilon>0$ such that $t_0+\varepsilon<T$ and, for all $z\in\overline{B}_\varepsilon(z_0)$, where $B_\varepsilon(z_0):=((t_0-\varepsilon)_+,t_0+\varepsilon)\times((x_0-\varepsilon)_+,(x_0+\varepsilon)\wedge\overline{X})\times((y_0-\varepsilon)_+,y_0+\varepsilon)\subset\mathcal{S}$, we have

$$-\frac{\partial\varphi}{\partial t}(z)-\mathcal{L}\varphi(z)\ <-\eta, \tag{6.35}$$

$$-\left[\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}+\psi\right](z)\ <-\eta, \tag{6.36}$$

Notice that inequality (6.36) derives from the monotony and the left-continuity of ψ. For any admissible control $X\in\mathcal{A}(t_0,x_0)$, consider the exit time $\tau_\varepsilon=\inf\{t\ge t_0,(t,X_t,Y_t^{t_0,y_0,X})\notin\overline{B}_\varepsilon(z_0)\}$. We notice that $\tau_\varepsilon<T$. Applying Itô's formula to the

process $\varphi(t, X_t, Y_t^{t_0,y_0,X})$ between t_0 and τ_ε^-, we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(\tau_\varepsilon^-, X_{\tau_\varepsilon^-}, Y_{\tau_\varepsilon^-}^{t_0,y_0,X})\right] &= \varphi(z_0) + \mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon}\left[\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\right](t, X_{t-}, \check{Y}_{t-}^{t_0,y_0,X})dt\right] \\ &+\mathbb{E}\left[\int_0^{\tau_\varepsilon}\left[\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\right](t, X_{t^-}, \check{Y}_{t^-}^{t_0,y_0,X})\, dX_t^c\right] \\ &+\mathbb{E}\left[\sum_{t_0\le t<\tau_\varepsilon}[\varphi(t, X_t, Y_t^{t_0,y_0,X}) - \varphi(t, X_{t^-}, \check{Y}_{t^-}^{t_0,y_0,X})]\right]\end{aligned} \quad (6.37)$$

From (6.36), and noting that $\Delta X_t := X_t - X_{t^-} = Y_t^{t_0,y_0,X} - \check{Y}_{t^-}^{t_0,y_0,X}$ for all $t_0 \le t < \tau_\varepsilon$, we have

$$\begin{aligned}\varphi(t, X_t, Y_t^{t_0,y_0,X}) - \varphi(t, X_{t^-}, \check{Y}_{t^-}^{t_0,y_0,X}) &= \int_0^{\Delta X_t}\left[\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\right](t, X_{t^-} + \zeta, \check{Y}_{t^-}^{t_0,y_0,X} + \zeta)\, d\zeta \\ &\ge \eta\Delta X_t - \int_0^{\Delta X_t}\psi(\check{Y}_{t^-}^{t_0,y_0,X} + \zeta)\, d\zeta \\ &= \eta\Delta X_t - \left[\Phi(Y_t^{t_0,y_0,X}) - \Phi(\check{Y}_{t^-}^{t_0,y_0,X})\right]. \quad (6.38)\end{aligned}$$

Hence, plugging last inequality, (6.35) and (6.36) in (6.37), we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(\tau_\varepsilon^-, X_{\tau_\varepsilon^-}, \check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X})\right] &\ge \varphi(z_0) + \eta\mathbb{E}\left[\tau_\varepsilon - t_0 + \int_{t_0}^{\tau_\varepsilon^-} dX_t\right] \\ &-\mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon}\psi(\check{Y}_{t^-}^{t_0,y_0,X})dX_t^c\right] \\ &-\mathbb{E}\left[\sum_{t_0\le t<\tau_\varepsilon}[\Phi(Y_t^{t_0,y_0,X}) - \Phi(\check{Y}_{t^-}^{t_0,y_0,X})]\right] \quad (6.39)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}v(z_0) &= \varphi(z_0) \\ &\le \mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon}\psi(\check{Y}_{t^-}^{t_0,y_0,X})dX_t^c + \sum_{t_0\le t<\tau_\varepsilon}[\Phi(Y_t^{t_0,y_0,X}) - \Phi(Y_{t^-}^{t_0,y_0,X})] + \varphi(\tau_\varepsilon^-, X_{\tau_\varepsilon^-}, \check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X})\right] \\ &\quad -\eta\left(\mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon^-} dt + \int_{t_0}^{\tau_\varepsilon^-} dX_t\right]\right)\end{aligned} \quad (6.40)$$

On the set $\{\check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X} \in [y_0 - \varepsilon, y_0 + \varepsilon]\}$, we have $\check{Z}_{\tau_\varepsilon^-}^{z_0,X} := (\tau_\varepsilon, \check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X}, X_{\tau_\varepsilon^-}) \in \overline{B}_\varepsilon(z_0)$, then $Z_{\tau_\varepsilon}^{z_0,X}$ belongs to the boundary $\delta\overline{B}_\varepsilon(z_0)$ or is out of $\overline{B}_\varepsilon(z_0)$. Hence, there exists a random variable $\gamma \in [0, 1]$ such that, on $\{\check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X} \in [y_0 - \varepsilon, y_0 + \varepsilon]\}$, we have

$$Z_{\tau_\varepsilon}^{(\gamma)} := \left(\tau_\varepsilon, \check{Y}_{\tau_\varepsilon^-}^{t_0,y_0,X} + \gamma\Delta X_{\tau_\varepsilon}, X_{\tau_\varepsilon^-} + \gamma\Delta X_{\tau_\varepsilon}\right) \in \delta\overline{B}_\varepsilon(z_0).$$

On the one hand, it follows from the **DPP**, that, on $\{\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} \in [y_0-\varepsilon, y_0+\varepsilon]\}$, we have

$$v(Z^{(\gamma}_{\tau_\varepsilon}) \le v(Z^{z_0,X}_{\tau_\varepsilon}) + \Phi(Y^{t_0,y_0,X}_{\tau_\varepsilon}) - \Phi(\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} + \gamma \Delta X_{\tau_\varepsilon}).$$

On the other hand, on $\{\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} \in [y_0-\varepsilon, y_0+\varepsilon]\}$, we have

$$\begin{aligned}
\varphi(\check{Z}^{z_0,X}_{\tau_\varepsilon^-}) &= \varphi(Z^{(\gamma)}_{\tau_\varepsilon}) - \int_0^{\gamma\Delta X_{\tau_\varepsilon}} \left[\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\right](t, X_{t^-}+\zeta, \check{Y}^{t_0,y_0,X}_{t^-}+\zeta)\, d\zeta \\
&\le \varphi(Z^{(\gamma)}_{\tau_\varepsilon}) - \gamma\eta\Delta X_{\tau_\varepsilon} + \int_0^{\gamma\Delta X_{\tau_\varepsilon}} \psi(\check{Y}^{t_0,y_0,X}_{t^-}+\zeta)\, d\zeta \\
&\le v(Z^{(\gamma)}_{\tau_\varepsilon}) - \gamma\eta\Delta X_{\tau_\varepsilon} + \Phi(\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} + \gamma\Delta X_{\tau_\varepsilon}) - \Phi(\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-}) \\
&\le v(Z^{z_0,X}_{\tau_\varepsilon}) - \gamma\eta\Delta X_{\tau_\varepsilon} + \Phi(Y^{t_0,y_0,X}_{\tau_\varepsilon}) - \Phi(\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-})
\end{aligned} \tag{6.41}$$

Plugging equation (6.41) in (6.40), we obtain

$$\begin{aligned}
v(z_0) \le\; & \mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon} \psi(\check{Y}^{t_0,y_0,X}_{t^-}) dX^c_t + \sum_{t_0 \le t \le \tau_\varepsilon} [\Phi(Y^{t_0,y_0,X}_t) - \Phi(Y^{t_0,y_0,X}_{t^-})] + v(Z^{z_0,X}_{\tau_\varepsilon})\right] \\
& -\eta\left(\mathbb{E}\left[\int_{t_0}^{\tau_\varepsilon^-} dt + \int_{t_0}^{\tau_\varepsilon^-} dX_t + \gamma\Delta X_{\tau_\varepsilon} \mathbb{1}_{\{\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} \in [y_0-\varepsilon, y_0+\varepsilon]\}}\right]\right)
\end{aligned} \tag{6.42}$$

From the dynamic programming principle **(DPP)**, it follows that (6.40) implies that

$$0 \le -\inf_X \mathbb{E}\Big[\tau_\varepsilon - t_0 + \int_{t_0}^{\tau_\varepsilon^-} dX_t + \gamma\Delta X_{\tau_\varepsilon} \mathbb{1}_{\{\check{Y}^{t_0,y_0,X}_{\tau_\varepsilon^-} \in [y_0-\varepsilon, y_0+\varepsilon]\}}\Big].$$

Therefore there exists a sequence of admissible control $(X^n)_{n\ge 0}$ such that

$$\mathbb{E}\Big[\tau^n_\varepsilon - t_0 + X^n_{\tau^{n-}_\varepsilon} - x_0 + \gamma\Delta X^n_{\tau^n_\varepsilon} \mathbb{1}_{\{\check{Y}^{t_0,y_0,X^n}_{\tau^{n-}_\varepsilon} \in [y_0-\varepsilon, y_0+\varepsilon]\}}\Big] \le \frac{1}{n},$$

where $\tau^n_\varepsilon = \inf\{t \ge t_0, (t, X^n_t, Y^{t_0,y_0,X^n}_t) \notin \overline{B}_\varepsilon(z_0)\}$. As $\tau^n_\varepsilon - t_0 \ge 0$, $X^n_{\tau^{n-}_\varepsilon} - x_0 \ge 0$ and $\gamma\Delta X^n_{\tau^n_\varepsilon} \in [0, \varepsilon]$, on the one hand, we have

$$\begin{aligned}
\frac{1}{n} &\ge \varepsilon\mathcal{P}\left(y_0 - \varepsilon < \check{Y}^{t_0,y_0,X^n}_{\tau^{n-}_\varepsilon} < y_0 + \varepsilon - \gamma\Delta X^n_{\tau^n_\varepsilon}\right) \\
&\ge \varepsilon\mathcal{P}\left(y_0 - \varepsilon < \check{Y}^{t_0,y_0,X^n}_{\tau^{n-}_\varepsilon} < y_0\right),
\end{aligned}$$

and, on the other hand, we have

$$\lim_{n\to+\infty} \tau^n_\varepsilon = t_0 \quad \text{and} \quad \lim_{n\to+\infty} X^n_{\tau^{n-}_\varepsilon} = x_0.$$

Therefore, $\lim_{n\to+\infty} \check{Y}^{t_0,y_0,X^n}_{\tau^{n-}_\varepsilon} = y_0$ and $\mathbb{P}\left(y_0 - \varepsilon < \check{Y}^{t_0,y_0,X^n}_{\tau^{n-}_\varepsilon} < y_0\right)$ tends to 1 when

n goes to $+\infty$. It leads to a contradiction, then we obtain the required viscosity supersolution property :

$$\max\left(-\frac{\partial\varphi}{\partial t} - \mathcal{L}\varphi,\ -\frac{\partial\varphi}{\partial x} - \frac{\partial\varphi}{\partial y} - \psi\right) \geq 0, \tag{6.43}$$

□

Lemma 6.3.1 (Comparison Principle)**.** *If v is a continuous viscosity subsolution of (6.26) and w is a continuous viscosity supersolution of (6.26), such that v and w satisfy the growth condition 6.27 and*

$$v(t,\overline{X},y) \leq w(t,\overline{X},y) \quad \textit{and} \quad v(T,x,y) \leq w(T,x,y),$$

then $v \leq w$.

Proof of the comparison principle.

Construction of a strict subsolution. Let $\alpha > 0$ such that

$$\lim_{y\to+\infty} \sigma(y)^2 y^{\alpha-2} < +\infty \qquad \text{and} \qquad \lim_{y\to+\infty} \frac{\Phi(y,\overline{X}-x) - \Phi(y)}{y^\alpha} = 0.$$

On $[0,T]\times[0,\overline{X}]\times\mathbb{R}^+$, we set

$$\varphi(t,x,y) = -e^{-ct}\left((-a_1x + a_2)y^\alpha - b_1x + b_2\right),$$

where a_1, a_2, b_1, b_2 and c are positive constants such that $a_2 > a_1\overline{X}$ and $b_2 > b_1\overline{X}$. We set $A(x) = -a_1x + a_2 > 0$ on $[0,\overline{X}]$. On the one hand, we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi &= e^{-ct}\left(c\left(A(x)y^\alpha - b_1x + b_2\right) - \frac{A(x)\sigma^2}{2}\alpha(\alpha-1)\sigma(y)^2y^{\alpha-2}\right) \\
&\quad + e^{-ct}\left(+\alpha h(y)A(x)y^{\alpha-1} + \gamma_t A(x)\int_{\mathbb{R}} y^\alpha - (y-q(y,z))^\alpha\, dz\right) \\
&\geq A(x)e^{-ct}\left(cy^\alpha - \frac{\sigma^2}{2}\alpha(\alpha-1)\sigma(y)^2y^{\alpha-2}\right) + ce^{-cT}(-b_1\overline{X} + b_2) \\
&\geq ce^{-cT}(-b_1\overline{X} + b_2).
\end{aligned}$$

The last inequality comes from the linear growth assumption on σ and is true for c big enough. On the other hand, we have

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} = e^{-ct}\left(a_1y^\alpha + b_1 - \alpha A(x)y^{\alpha-1}\right) \geq \varepsilon,$$

for b_1 big enough compared to a_1 and a_2.
Therefore, φ is a continuous function defined on $[0,T]\times[0,\overline{X}]\times\mathbb{R}^+$ such that for

all $m > 0$

$$\lim_{y\to+\infty} \frac{1}{m}\varphi(t,x,y) + \Phi(y,\overline{X}-x) - \Phi(y) = -\infty, \quad \forall (t,x) \in [0,T]\times[0,\overline{X}].$$

and $v_m := v + \frac{1}{m}\varphi$ is a strict subsolution of equation (6.26) in the sense that

$$\max\left(-\frac{\partial v_m}{\partial t} - \mathcal{L}v_m,\ -\frac{\partial v_m}{\partial x} - \frac{\partial v_m}{\partial y} - \psi\right) \le -\frac{\varepsilon}{m} < 0,$$

Toward Ishii's lemma :

Let $m \ge 1$. We need to show that $\varrho := \sup_{(t,x)} v_m - w \le 0$, with $v_m := v + \frac{1}{m}\varphi$. Suppose on the contrary that $\varrho > 0$. Since,

$$\lim_{y\to\infty} v_m - w = -\infty,\ \lim_{x\to\overline{X}} v_m - w \le 0 \text{ and } \lim_{t\to T} v_m - w \le 0, \tag{6.44}$$

it is clear that this supremum is attained at some point $(t_0,x_0,y_0) \in [0,T)\times[0,\overline{X})\times\mathbb{R}^+$ i.e. $\varrho = v_m(t_0,x_0,y_0) - w(t_0,x_0,y_0)$, with $0 \le t_0 < T$, $0 \le x_0 < \overline{X}$ and $0 \le y_0$.

Let $\bar{y} \ge y_0$, for $k \ge 1$, define $\Phi_i(t,x,y,z) = v_m(t,x,y) - w(t,x,z) - \phi_k(y,z)$, where $\phi_k(y,z) := k/2 \mid y - z \mid^2$. Let $\varrho_k = \sup_{[0,T]\times[0,\overline{X}]\times[0,\bar{y}]^2} \Phi_k(t,x,y,z)$, which is attained at some point $(\hat{t}_k,\hat{x}_k,\hat{y}_k,\hat{z}_k) \in [0,T]\times[0,\overline{X}]\times[0,\bar{y}]^2$. By taking a subsequence, we can also assume that there exists a point $(\hat{t}_0,\hat{x}_0,\hat{y}_0,\hat{z}_0)$ to which $(\hat{t}_k,\hat{x}_k,\hat{y}_k,\hat{z}_k)$ converges as $k\to\infty$. enough, we can then assume that $\hat{t}_k < T$, and $\hat{x}_k > 0$.

In order to show that $\lim_k \hat{x}_k = \lim_k \hat{y}_k = \hat{x}_0$, consider the following inequality :

$$\Phi_k(\hat{t}_0,\hat{x}_0,\hat{y}_0,\hat{y}_0) \le \Phi_k(\hat{t}_k,\hat{x}_k,\hat{y}_k,\hat{z}_k).$$

In particular, we have

$$\frac{k}{2} \mid \hat{y}_k - \hat{z}_k \mid^2 \ \le\ -v_m(\hat{t}_0,\hat{x}_0,\hat{y}_0) + w(\hat{t}_0,\hat{x}_0,\hat{y}_0) + v_m(\hat{t}_k,\hat{x}_k,\hat{y}_k) - w(\hat{t}_k,\hat{x}_k,\hat{z}_k).$$

As v_m and w are continuous on the compact set $[0,T]\times[0,\overline{X}]\times[0,\bar{y}]$, there exists $C > 0$ such that

$$\mid \hat{y}_k - \hat{z}_k \mid^2 \ \le\ \frac{C}{k}.$$

Letting k go to $+\infty$, we find $\hat{y}_0 = \hat{z}_0$. Finally, we show that ϱ_k tends to ϱ when k goes to $+\infty$. To do so, note that $\varrho \le \varrho_k$ since $\Phi_k(\hat{t}_k,\hat{x}_k,\hat{y}_k,\hat{z}_k) \ge \Phi_k(t_0,x_0,y_0,y_0) = v_m(t_0,x_0,y_0) - w(t_0,x_0,y_0) = \varrho$. Moreover, we have

$$\varrho_k \le v_m(\hat{t}_k,\hat{x}_k,\hat{y}_k) - w(\hat{t}_k,\hat{x}_k,\hat{z}_k) - \frac{i}{2} \mid \hat{y}_k - \hat{z}_k \mid^2 \le v_m(\hat{t}_k,\hat{x}_k,\hat{y}_k) - w(\hat{t}_k,\hat{x}_k,\hat{y}_k)$$

which converges to $v_m(\hat{t}_0,\hat{x}_0,\hat{y}_0) - w(\hat{t}_0,\hat{x}_0,\hat{y}_0) \le \varrho$ since v_m and w are continuous.

Since this limit is less or equal to ϱ, we conclude that $\varrho_k \to \varrho$ and $\frac{k}{2} \mid \hat{y}_k - \hat{z}_k \mid^2 \to 0$ when $k \to \infty$. Moreover, we have $v_m(\hat{t}_0, \hat{x}_0, \hat{y}_0) - w(\hat{t}_0, \hat{x}_0, \hat{y}_0) = \varrho$.

Ishii's lemma :

We can now apply Theorem 3.2 of [25] at the point $(\hat{t}_k, \hat{x}_k, \hat{y}_k, \hat{z}_k)$. There exist M and $M' \in \mathbb{R}$ such that

$$\begin{pmatrix} -k- \mid\mid A \mid\mid & 0 \\ 0 & -k- \mid\mid A \mid\mid \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \leq A + \frac{1}{k} A^2 \tag{6.45}$$

with $A = D^2\phi_k(\hat{y}_k, \hat{z}_k) = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$, where $\phi_k(y,z) := k/2 \mid y - z \mid^2$ and ,from the relation between the notion of superjets and our definition of viscosity supersolutions, we have

$$\varepsilon/m \leq \min\left(\mathcal{K}[\frac{\partial \phi_k}{\partial y}(\hat{y}_k, \hat{z}_k), M](\hat{y}_k) + \gamma_{\hat{t}_k} I[v_m](\hat{t}_k, \hat{x}_k, \hat{y}_k);\ \frac{\partial \phi_k}{\partial y}(\hat{y}_k, \hat{z}_k) + \psi(\hat{y}_k)\right) \tag{6.46}$$

$$0 \geq \min\left(\mathcal{K}[-\frac{\partial \phi_k}{\partial z}(\hat{y}_k, \hat{z}_k), -M'](\hat{z}_k) + \gamma_{\hat{t}_k} I[w](\hat{t}_k, \hat{x}_k, \hat{z}_k);\ -\frac{\partial \phi_k}{\partial z}(\hat{y}_k, \hat{z}_k) + \psi(\hat{z}_k)\right) \tag{6.47}$$

in which we have set

$$\mathcal{K}[p,M](y) = \frac{\sigma^2(y)}{2} M - h(y)p \text{ and } I[g](t,x,y) = \int_{\mathbb{R}} (g(t,x,y-q(y,\zeta)) - g(t,x,y)) m(d\zeta).$$

From the second inequality, we have two cases :

i. $-\frac{\partial \phi_k}{\partial z}(\hat{y}_k, \hat{z}_k) + \psi(\hat{z}_k) \leq 0.$

ii. $\mathcal{K}[-\frac{\partial \phi_k}{\partial z}, -M'](\hat{y}_k, \hat{z}_k) + \gamma_{\hat{t}_k} I[w](\hat{t}_k, \hat{x}_k, \hat{z}_k) \leq 0.$

Subtracting inequalities 6.46 and 6.47, we obtain, in the first case

$$\begin{aligned} \frac{\varepsilon}{m} &\leq \frac{\partial \phi_k}{\partial y}(\hat{y}_k, \hat{z}_k) + \frac{\partial \phi_k}{\partial z}(\hat{y}_k, \hat{z}_k) + \psi(\hat{y}_k) - \psi(\hat{z}_k) \\ &\leq \psi(\hat{y}_k) - \psi(\hat{z}_k). \end{aligned}$$

We get a contradiction by letting k going to $+\infty$

In the second case, we have

$$\begin{aligned}
\frac{\varepsilon}{m} &\leq \mathcal{K}[\frac{\partial \phi_k}{\partial y}(\hat{y}_k, \hat{z}_k), M](\hat{y}_k) + \gamma_{\hat{t}_k} I[v_m](\hat{t}_k, \hat{x}_k, \hat{y}_k) \\
&\quad - \left(\mathcal{K}[-\frac{\partial \phi_k}{\partial z}(\hat{y}_k, \hat{z}_k), -M'](\hat{z}_k) + \gamma_{\hat{t}_k} I[w](\hat{t}_k, \hat{x}_k, \hat{z}_k)\right) \\
&\leq \frac{1}{2}(M\sigma(\hat{y}_k)^2 + M'\sigma(\hat{z}_k)^2) - k(h(\hat{y}_k) - h(\hat{z}_k))(\hat{y}_k - \hat{z}_k) \\
&\quad + \gamma_{\hat{t}_k}\left(I[v_m](\hat{t}_k, \hat{x}_k, \hat{y}_k) - I[w](\hat{t}_k, \hat{x}_k, \hat{z}_k)\right)
\end{aligned}$$

From the Lipschitz continuity of h, the continuity of γ and v_m and w, we obtain

$$\begin{aligned}
\frac{\varepsilon}{m} &\leq \frac{1}{2}\lim_{k\to\infty}\left\langle \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}(\sigma(\hat{y}_k), \sigma(\hat{z}_k)), (\sigma(\hat{y}_k), \sigma(\hat{z}_k))\right\rangle \\
&\quad + \gamma_{\hat{t}_0}\left(I[v_m](\hat{t}_0, \hat{x}_0, \hat{y}_0) - I[w](\hat{t}_0, \hat{x}_0, \hat{z}_0)\right) \\
&\leq \frac{1}{2}\lim_{k\to\infty}\left\langle \left(A + \frac{1}{k}A^2\right)(\sigma(\hat{y}_k), \sigma(\hat{z}_k)), (\sigma(\hat{y}_k), \sigma(\hat{z}_k))\right\rangle \\
&\quad + \gamma_{\hat{t}_0} I[\Phi](\hat{t}_0, \hat{x}_0, \hat{y}_0), \\
&\leq \frac{1}{2}\lim_{k\to\infty}\left\langle \left(A + \frac{1}{k}A^2\right)(\sigma(\hat{y}_k), \sigma(\hat{z}_k)), (\sigma(\hat{y}_k), \sigma(\hat{z}_k))\right\rangle \\
&= \lim_{k\to\infty}\frac{3}{2}k(\sigma(\hat{y}_k) - \sigma(\hat{z}_k))^2, \\
&= 0.
\end{aligned}$$

The last equality follows from the Lipschitz continuity of σ and the property :

$$\lim_{k\to+\infty} k(\hat{y}_k - \hat{z}_k)^2 = 0.$$

□

6.4 Examples and numerical results

6.4.1 Linear price impact

Suppose that $\psi(y) = y$ and $h(y) = \rho y$. Observe that $\Phi(y) = \frac{y^2}{2}$. In this case, we are under the framework of Alfonsi and Blanc [4].
Suppose further that the process

$$N_t = \int_0^t \sigma^2(Y_s)\, dW_s$$

is a martingale, where Y is the solution of (6.4) with $X \equiv 0$ and $\overline{M} \equiv 0$. The argument in the proof of Theorem 2.1 of Alfonsi and Blanc [4] shows that in this case there are no Price Manipulation Strategies and the control problem is equivalent to the liquidation problem of Obizhaeva and Wang [53]. Indeed, we can verify that in this case the function

$$v(t,x,y) = \frac{(y + (\overline{X} - x))^2}{2 + \rho(T-t)} - \frac{y^2}{2} = \frac{2}{2 + \rho(T-t)}\Phi(y + (\overline{X} - x)) - \Phi(y) \qquad (6.48)$$

solves the variational inequality (6.25) and satisfies the terminal condition (6.12). Notice that the solution does not depend on $\sigma(y)$. Observe that there is no No Trading region because

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \psi = 0.$$

In addition

$$\frac{\partial v}{\partial t} + \mathcal{L}v = 0$$

if and only if $\sigma(y) \equiv 0$ and

$$(\overline{X} - x) = (1 + \rho(T-t))y. \qquad (6.49)$$

Hence only if $\sigma(y) = 0$ there is absolutely continuous trading on the linear frontier (6.49). Otherwise, the optimal trading strategy should be represented by a "local time" on the frontier.

6.4.2 Deterministic resilience

Let's assume that $\sigma = q = 0$. In particular the volume effect process Y is of finite variation. Let's further assume that $0 \le t < T$, $X \in \mathcal{A}(t,x)$ and $Y_{t-} = y$. Then one can show that

$$\Phi(Y_T) - \Phi(y) = \int_t^T \psi(Y_s)(dX_s^c - h(Y_s)ds) + \sum_{t \le s \le T} \Phi(Y_s) - \Phi(Y_{s-}).$$

Hence,

$$v(t,x,y) = \inf_{X \in \mathcal{A}(t,x)} \mathbb{E}\left[\Phi(E_T^{t,y,X}) + \int_t^T g(h(E_s^{t,y,X}))ds\right] - \Phi(y),$$

where

$$g(y) = y\psi(h^{-1}(y)).$$

If one further assumes that g is a convex function, by Jensen's inequality

$$\int_t^T g(h(E_s^{t,y,X}))ds \ge (T-t)g\left(\frac{1}{T-t}\int_t^T h(Y_s^{t,y,X})\, ds\right).$$

But we also know that

$$Y_T^{t,y,X} = y + (\overline{X} - x) - \int_t^T h(Y_s^{t,y,X})\, ds.$$

Hence we have that

$$v(t,x,y) \geq \inf_{X \in \mathcal{A}(t,x)} \mathbb{E}\left[G_{t,x,y}(Y_T^{t,y,X}))\right] - \Phi(y),$$

where

$$G_{t,x,y}(e) = \Phi(e) + (T-t)g\left(\frac{\overline{X} - x + y - e}{T-t}\right).$$

Assume that $e^*_{t,x,y}$ minimizes $G_{t,x,y}$ and

$$e^*_{t,x,y} \in \{Y_T^{t,y,X} : X \in \mathcal{A}(t,x) \text{ is of type } A\}. \tag{6.50}$$

Type A strategies are those that jump only at times t and T and such that $dX_s = dX_s^c = h(Y_t)ds$ on (t,T). Then we would have that

$$v(t,x,y) = G(e^*_{t,x,y}) - \Phi(y).$$

If furthermore $e^*_{t,x,y}$ is smooth in the variables t,x,y and $G_{t,x,y}$ is smooth, so that $G'(e^*_{t,x,y}) = 0$, then

$$\frac{\partial v}{\partial t} + \mathcal{L}v = \frac{\partial v}{\partial t} - h\frac{\partial v}{\partial y} = h\psi;$$

and,

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \psi = -\psi + \psi = 0.$$

Assumption (6.50) is not always satisfied. For instance, suppose that we have a block shape for the order book with $F(x) = \psi(x) = h(x) = x$. In this case $\Phi(x) = \frac{x^2}{2}$ and $g(x) = x^2$. One can show that

$$\{Y_T^{t,y,X} : X \in \mathcal{A}(t,x) \text{ is of type } A\} = \{\overline{X} - x + y - (y+\xi)(T-t) : \xi \geq 0\}.$$

Also in this case

$$e^*_{t,x,y} = \frac{2(\overline{X} - x + y)}{T-t+2}.$$

We conclude that (6.50) holds if and only if

$$\frac{2(\overline{X} - x + y)}{T-t+2} \leq \overline{X} - x + y - y(T-t);$$

if and only if

$$\overline{X} - x + y - ((T-t)+2)y \geq 0;$$

if and only if

$$\overline{X} - x - (1 + (T - t))y \geq 0.$$

If this condition is not satisfied then one can show that among the type A strategies the best one, does not jump at time t. For this strategy the corresponding cost is

$$\frac{(\overline{X} - x + y(1 - (T - t)))^2}{2} + ((T - t) - \frac{1}{2})y^2.$$

Now consider a strategy of the following form : Do not buy any shares on $[t, t+\delta) \subset [t, T]$. After time $t + \delta$ follow a type A strategy with jump at time $t + \delta$ equal to 0. The cost of this strategy is

$$\frac{(\overline{X} - x + ye^{-\delta}(1 - (T - t - \delta)))^2}{2} + (T - t - \delta)(y^2 e^{-2\delta}) + y^2(1 - e^{-2\delta}) - \frac{1}{2}y^2.$$

It can be shown that for certain choices of the parameters $x, \overline{X}, y, t, T$, e.g. $T - t = 2, \overline{X} - x = 1, y = 1$, the last expression as a function of δ decreases around 0. Hence $\delta = 0$ is not optimal and a strategy as the one proposed above for some $\delta > 0$ outperforms any type A strategy. Therefore in this case type A strategies do not longer contain optimal strategies even under the condition of convexity for g.

6.4.3 Numerical Results

In this section, we present some numerical results obtained by a numerical approximation of (6.26). We have implemented a finite difference method for singular control problem as studied in [42]. Since the scheme has monotonicity, consistency and stability properties, it converges to the viscosity solution of (6.26) (see [11] and [10]).

6.4.3.1 Parameters

Time parameters :

— Time step : Δt=0.002 hour=7.2 seconds

— Horizon time T : $T = 1$ hour

Space parameters :

— Space step : $\Delta x = \Delta y = 0.01$

— Asset position $\bar{X}$: $\bar{X} = 1$

— Bound of Y $\bar{Y}$:$\bar{Y} = 3$

Diffusion and jumps parameters

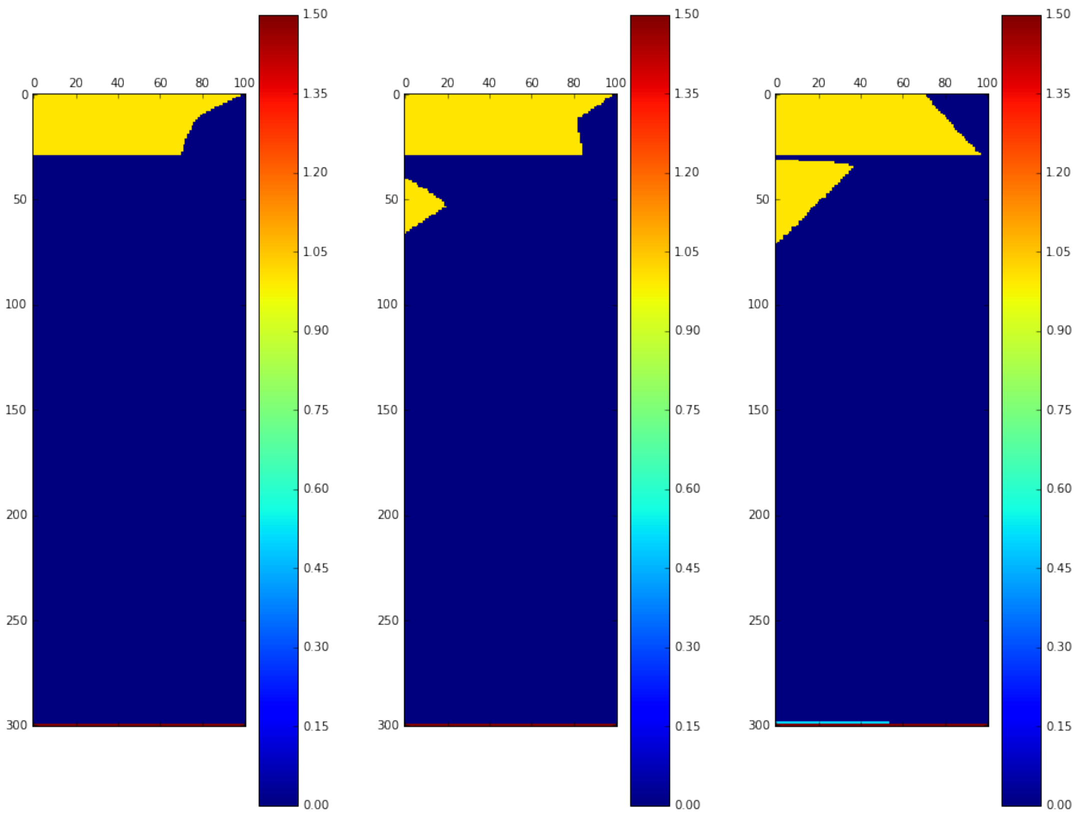
— Resilience function $h(y) = y$

— Volatility function $\sigma^2(y) = 0.25y(\bar{Y} - y)$

— Jump sizes $q(y,z) = ye^z$, mean measure : $\gamma_t = 1$

6.4.3.2 Modified Block Order Book

$$\Phi(y) = \begin{cases} \frac{1}{2}y^2 & \text{if } y \leq a \\ \frac{1}{2}\left((y+b-a)^2 + a^2 - b^2\right) & \text{else} \end{cases} \quad \text{with } a = 30,\ b = 70.$$



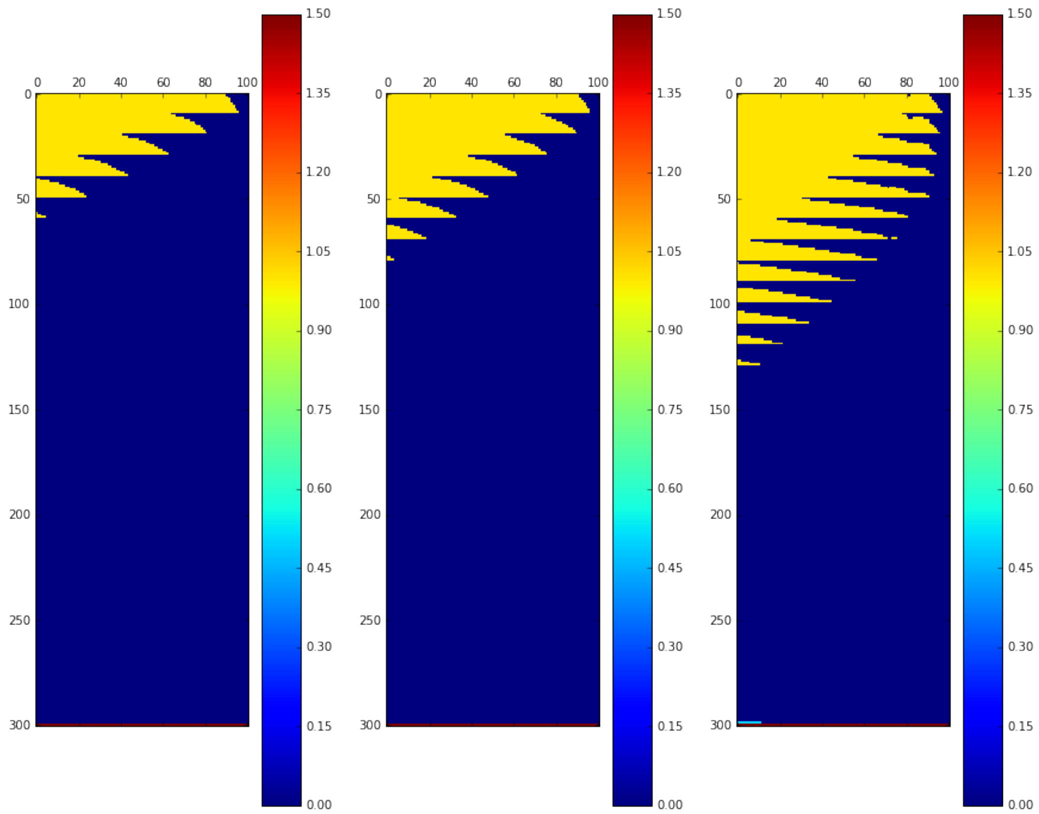
(a) t=0 (b) t=250 (c) t=490

FIGURE 6.3 – Modified Block Order Book

In the previous and the following illustration, we represent the buying and the waiting regions respectively in yellow and in blue. On the x-axis we have the quantity of already bought and the present value of Y is the second coordinate.
When the order book follows the so-called modified block shape, there is a gap between two prices a and b. Therefore, as soon as the present price is greater than a, there is a jump in the cost of purchasing new shares and the agent has better to wait that new limit sell order arrive and fill the gap. Obviously, close to maturity, the agent cannot wait any longer and this is illustrated in figures (6.3b) and (6.3c).

6.4.3.3 Discrete Order Book

$$\Phi(y) = \sum_{k=0}^{\infty} k(y - \frac{1}{2}k - \frac{1}{2})\mathbb{1}_{(k,k+1]} = \lfloor y \rfloor (y - \frac{1}{2}\lfloor y \rfloor - \frac{1}{2})$$



(a) t=0 (b) t=250 (c) t=450

FIGURE 6.4 – Discrete Order Book

For a discrete order book, there are gaps with sizes of one or several ticks between two consecutive prices. Therefore, as soon as the agent has consumed the liquidity at a given price, there is a jump in the cost of purchasing new shares and the agent may prefer to wait that new limit sell orders arrive and fill the gap. The closer to maturity, she will be, the less she should wait. This extends the ideas of the example of modified block order books.

Conclusion

In this chapter, we solved the optimal execution problem for the limit order book shape model, with general order book shape and resilience function form. We used the Dynamic Programming Principle and viscosity solution to find a variational inequality that the value function satisfies. We approach numerically the value function and find for a fixed time, that the strategy is completely linked to the gap of the order book shape.

Conclusion

In the first part of the thesis, in chapter 2, we designed a limit order book model with one side of the limit order book, the bid side. We relaxed the constant size event and allowed size randomness. It gave a flexible model for the state dependence for size random variables. We represented the state of the limit order book by the depth n. With this representation, the gain function is easier to use for optimal liquidation problem. We constructed a theory around the space of depth $\mathcal{N}$ by defining operators of $\mathcal{N}$. Through the Markovian framework, we described the transition probabilities and the Markov chain generator. We specified the space of the random variables δ, α^S, α^B, α^C, β^B, β^C by using the empirical studies in the literature on the limit order book. We explained the concept of homogeneity and gave an important lemma 2.3.1 based on the strong law of large number theorem for martingale.

In chapter 3, we found that our model is enough flexible for letting the Markov Chain to be recurrent or transient. For the transient case, the introduction of the scope process is an original way to express the fact that all limit orders are not not too far to the bid price. With the fact that the volume process is bounded, the total volume H doesn't explode to infinity which is the financial reality. We proved in all cases, the existence of parameters and we derived in particular cases the set of the parameters which lead to the properties. Since the scope is not available in financial data, we need to find another characteristics for bounding the volume distribution. Instead of using the scope $\mathfrak{b}(\mathsf{n}) - \mathfrak{d}(\mathsf{n})$, we can define another scope as $\mathfrak{b}(\mathsf{n}) - \mathfrak{b}^S(\mathsf{n}, a)$ with an appropriate $a \in \mathbb{N}$.

In chapter 4, we explain the interest of the current state n dependence of the random variable α^C for the recurrence problem. The random variable α^C should not follow a uniform law and binomial law lead to a recurrent Markov chain. We conclude an interesting result for the liquidity criteria by showing the interest of the the current state n dependence of the random variable α^S. A more general problem of calibration would be to calibrate according to the stylized facts and the liquidity criteria as metrics. Then the calibration problem would be to find the parameters in the set of probabilities $\mathbb{Q} \in \Theta^{calib}$. The explicit calculation of the metrics, depending on the model parameters, is necessary in this calibration problem.

In chapter 5, the main result is the strategy is reduced to choose the price impact instead of the quantity, in the case of the one step model. In order to find this result, we just use the dynamical programming principle and the different relations between the operators of $\mathcal{N}$. Further research needs to be done for solving completely the optimal liquidation problem.

In the second part of the thesis, in chapter 6 we solved the optimal liquidation problem for the limit order book shape model, with general order book shape and resilience function form. We used the Dynamic Programming Principle and viscosity solution to find a variational inequality that the value function satisfies. We approach numerically the value function and find for a fixed time, that the strategy is completely linked to the gaps of the order book shape. Further research needs to be done for the calibration of the model. If we assume that the order book shape μ is the mean average profile, we need to determine the reference price. We can use Delattre et al [27] for computing the reference price (A_t) in the financial data. Simultaneously, we need to compute the volume effect (Y_t^{data}) in the financial data. Since the diffusion Y^{model} is a jump diffusion, we need to distinguish the continuous and the jump part. In the case of Level 1 financial data, we only have the volume difference $\Delta^{data}Y$ between two consecutive events at the ask ($\Delta^{data}Y > 0$ for limit order arrival and $\Delta^{data}Y < 0$ for canceling or market order arrival). There is no ambiguity for the volume effect change, we can assume for the calibration $\Delta^{data}Y = \Delta^{model}Y$. In the case of Level 2 financial data, we only have the volume change for 10 price levels. In a naive consideration, the volume effect evolve for any event in the 10 price levels. It means that a volume change at the bid has the same impact as a volume change at six ticks lower to the bid. In this case, we can add another structures such as $\Delta^{data}\psi(Y)$ and $\Delta^{data}\phi(Y)$ for taking account the event level price.

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Titre : Modélisation de carnet d'ordres et gestion de risque de liquidité

Mots clés : Carnet d'ordres, Microstructure des marchés financiers, Chaîne de Markov dénombrable, Données à haute fréquence, Contrôle stochastique, Stratégie optimale de liquidation

Résumé : Cette thèse porte sur l'étude de modélisation stochastique de carnet d'ordres, et de deux problèmes de contrôle stochastique dans un contexte de risque de liquidité et d'impact sur le prix des actifs. La thèse est constituée de deux parties distinctes.
Dans la première partie, nous traitons, sous différents aspects, un modèle markovien de carnet d'ordres. En particulier, dans le chapitre 2, nous introduisons un modèle de représentation par profondeur cumulée. Nous considérons différents types d'arrivées d'évènements avec une dépendance de l'état courant. Le chapitre 3 traite le problème de stabilité du modèle à travers une approche semi-martingale pour la classification d'une chaîne de Markov dénombrable. Nous donnons, pour chaque problème de classification, une calibration du modèle à partir des faits empiriques comme le profil moyen de la densité du carnet d'ordres. Le chapitre 4 est consacré à l'estimation et à la calibration de notre modèle à partir des flux de données du marché. Ainsi, nous comparons notre modèle et les données au moyen des faits stylisés et des critères de liquidité. Nous donnons une calibration concrète aux différents problèmes de classification. Puis, dans le chapitre 5, nous traitons le problème de liquidation optimale dans le cadre du modèle de représentation par profondeur cumulée.
Dans la deuxième partie, dans le chapitre 6, nous proposons une modélisation d'un problème de liquidation optimale d'un investisseur avec une résilience stochastique. Nous nous ramenons à un problème de contrôle stochastique singulier. Nous montrons que la fonction valeur associée est l'unique solution de viscosité d'une équation d'Hamilton-Jacobi-Bellman. De plus, nous utilisons une méthode numérique itérative pour calculer la stratégie optimale. La convergence de ce schéma numérique est obtenue via des critères de monotonicité, de stabilité et de consistance.

Title : Limit order book modeling and liquidity risk management

Keywords : Limit order book, Financial market microstructure, High frequency data, Denumerable Markov Chain, Stochastic control, Liquidation optimal strategy

Abstract : This thesis deals with the study of stochastic modeling of limit order book and two stochastic control problems under liquidity risk and price impact. The thesis is made of two distinct parts.
In the first part, we investigate Markovian limit order book model under different aspects. In particular, in chapter 2, we introduce a model of cumulative depth representation. We consider different arrival events with dependencies on current state. Chapter 3 handles the model stability problem through a semi-martingale approach for the denumerable Markov Chain classification. We give for each problem a model calibration from empirical facts such as mean average profile of limit order book density. Chapter 4 is dedicated to model estimation and calibration by means of market data flow. Thus, we compare our model to market data through stylized facts and market liquidity criteria. We give a concrete calibration to different stability problems. Finally, in chapter 5, we handle an optimal liquidation problem in the cumulative depth representation model framework.
We study, in the second part, in chapter 6, an optimal liquidation problem of an investor under stochastic resilience. This problem may be formulated as a stochastic singular control problem. We show that the associated value function is the unique viscosity solution of an Hamilton-Jacobi-Bellman equation. We suggest an iterative numerical method to compute the optimal strategy. The numerical scheme convergence is obtained through the monotonicity, stability and consistency criteria.

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