**UNIVERSITE D'EVRY-VAL-D'ESSONNE**

UFR de Sciences Economiques et Juridiques

Année 2011 Numéro attribué par la bibliothèque

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# T H E S E

pour le Doctorat en Sciences Economiques
(arrêté du 30 mars 1992, arrêté du 18 janvier 1994)
présentée et soutenue publiquement par

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Novembre 2011

## Market Imperfections and Macroeconomic Instability

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*To my family and Palestine*

# Acknowledgments

I would like to express my gratitude to all those who contributed to the completion of this thesis. My utmost gratitude goes to my principal supervisor, Stefano Bosi, for his encouragement and continued valuable suggestions during this research. He first brought me into the frontiers of 'dynamic macroeconomic' research and shared with me a lot of his expertise as well as research insight. Working with him has been a very enriching experience.

I thank the EPEE and in particular, Michel Guillard, for welcoming me. I also thank all EPEE seminar organizers and participants for their beneficial comments and discussions.

I am very grateful to Thomas Seegmuller and Francesco Magris for their helpful comments and the participants of the T2M 2011 Conference held in Montreal.

I thank my family and all the friends that have always been there for me during good times and hard times of the Ph.D., even when they are far away. Their warmth and affection have been essential ingredients of this thesis. Especially, I thank my parents, Enareta Kurtbegu, Saloua El-Bouzaidi and special thanks for Riham Barbar.

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# Contents

# Chapter 1

# Introduction

Business cycle fluctuations are unstable movements in macroeconomic variables of a country over several periods. These fluctuations present a successive change between low economic growth phase and high growth phase. The interpretation of economic fluctuations associated with the business cycles has become one of the main topics treated in macroeconomic theory.

The analysis of business cycles has been initially considered by Smith (1776), Ricardo (1810) and Juglar (1862) who argue that if prices are fully flexible in the market then real output never deviate from potential one and so business cycle fluctuations never emerge. After that, Keynes (1936) in his book "*The General Theory of Employment Interest and Money*" states that the entrepreneurs' expectations about the future, which might be either optimistic or pessimistic, are the main causes of any fluctuations in investment and output (animal spirits). In other words, the distribution of expectations among economic agents influences both consumption and capital accumulation and determines the equilibrium allocation. Keynes identifies this point by saying:[1] "*Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits – a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities*".

Keynesian approach was criticized by the New Classical economists, led by Lucas (1972)

[1] In the modern macroeconomics, Shiller (2009) comments the recent crisis by going back to Keynesian thought. He says: "*Animal spirits offer an explanation for why we get into recessions in the first place- for why the economy fluctuates as it does. [...] The business cycle is in good part driven by animal spirits*".

and Sargent (1973). In particular, their main criticism is that Keynesian models are not microfounded and do not take in consideration the general equilibrium theory. Therefore, the New Classical thought proposes an alternative approach which is based upon the Rational Expectations Hypothesis[2] proposed by Muth (1961) and explains cycles by monetary shocks. Further, the New Classical approach supposes that market clearing conditions are applied for all periods and agents, who maximize their utility, adopt the rational expectation hypothesis. In addition, these agents have incomplete information about the distribution of prices across markets. Following a money supply shock, workers and firms would react by declining or augmenting labor supply and production. These variations make the appearance of fluctuations around the natural level of employment and production.

In the eighties, a new version of business cycles analysis appeared and interested in studying the economic fluctuations as a result of real shocks (technology shocks). Contrary to New Classical economists, who suppose that monetary shocks are the source of business cycles, the real business cycles approach (thereafter RBC), pioneered by Kydland and Prescott (1982), Long and Plosser (1983) and Stadler (1994), argues that cyclical fluctuations emerge due to productivity shocks. However, this approach maintains the same assumptions concerning market clearing prices and the rational expectations hypothesis. Based on above explanation, RBC theory shows that cycles appear as a result of exogenous productivity shock while money supply shocks do have small effects on cyclical fluctuations.

Based on rational expectation hypothesis, RBC theory gets more realistic cycles by introducing market imperfections. At the same time, the presence of market imperfections in macroeconomic models can generate cycles endogenously from intrinsic processes that destabilize the economy.

There is no doubt that exogenous shocks are important elements in explaining business cycle fluctuations, especially, after the high economic growth of 1990s which was motivated by the presence of novel technology shocks. However, nobody can disagree with the importance of cyclical fluctuations emerged endogenously through unstable responses inside the economy itself. Day (1982), Grandmont (1985), Gale (1973) and Benhabib and Nishimura (1979), among

[2] According to rational expectation hypothesis, agents make expectations about the future using all available current and past information about the economy.

others, show that endogenous cycles appear through both wealth effect and variations in savings and interest rate.

There are two types of endogenous cyclical fluctuations that arise in standard models: on the one side, there are some models which display deterministic cycles for different initial conditions. The resulted equilibrium is locally unique and exhibits perpetual fluctuations, see among others, Benhabib and Day (1982), Benhabib and Nishimura (1985), Grandmont (1985) and Woodford (1984). On the other side, models jointed with market imperfections can generate an indeterminate stationary solution. The presence of indeterminacy allows for the appearance of expectation-driven cycles. These expectations are not related to economic fundamentals (technology, preference, and endowments); instead, they depend on current and previous data. In such a case, the achievement of Pareto-optimality configuration is impossible. Grandmont (1998) confirms this point while stating "*... expectations of the agents of a socioeconomic system can influence the observed trajectories of that system ... through a mechanism of self-fulfilling prophecies ...*" and "*... one significant cause of economic fluctuations is to be found in the more or less erratic variations of the "state of expectations" of economic agents ...*" Further, Grandmont (1985) considers a general equilibrium model with perfect competition hypothesis and optimization behaviour of agents. He shows that persistent deterministic fluctuations emerge only endogenously without any exogenous shocks to fundamentals as in RBC. In particular, cycles result from a conflict between substitution effect and the wealth effect associated with interest rate variations.

The above literature does mainly imply that cycles are determined either by the presence of exogenous shocks on the fundamentals or as money supply shock, or by endogenous fluctuations associated with self-fulfilling expectations (Benhabib and Farmer 1994, 1996; Benhabib and Nishimura 1998; Farmer and Guo 1994; Harrison and Weder 2000 and Weder 2001, 2005). However, variations in agents' expectations together with productivity/monetary exogenous shocks might coexist. This is why Dos Santos Ferreira and Dufourt (2006) and Dos Santos Ferreira and Lloyd-Braga (2008) consider these two approaches as complementary rather than substitutable in the sense that the presence of a positive productivity shock creates an optimistic expectation about the future while a negative shock is followed by a pessimist behaviour and so these agents' responses amplify the influence of shocks.

An important progress in endogenous business cycles theory is based on the methodological work of Shell (1977) and Cass and Shell (1983) where they originate the concept of "sunspot equilibria". They show that a small depart from Arrow-Debreu assumptions gives a steady state where agents can choose different allocations for different states of nature even if the fundamentals are fix. The basic idea of sunspot equilibrium is that if agents presume that quantities and prices are influenced by an arbitrary factor (sunspot) while fundamentals are constant, then, the self-fulfilling anticipation is realized and agents get sunspot equilibria if the resulting quantities and prices satisfy their presumption.[3]Farmer (1999) states that: "*I like to think of the sunspot as the prediction for the economy of the Wall Street Journal*".

The famous work of Shell (1977), Azariadis and Guesnerie (1982), Farmer and Woodford (1984) and Woodford (1986) demonstrate that the existence of sunspot equilibria necessarily requires the presence of indeterminate steady state. Accordingly, Benhabib and Perli (1994) state: "*it is possible to view cultural and non-economic factors in such model not as affecting the fundamentals like technology or preferences, but simply as selection devices for equilibria which differ on the transition paths. It is of course well known that "sunspots" which act as selection devices among equilibria, may arise in models that exhibit indeterminacies*".

Indeterminacy means that there exists a continuum of distinct paths starting from the same initial level for the predetermined variables, respecting optimality and equilibrium conditions, and converging to the same steady state. In the past, the emergence of indeterminate stationary equilibrium in a dynamic model represents a weak point of the theory where economists have to provide careful assumptions to avoid it. Conversely, there are many recent works[4]in macroeconomic modelling which describe the occurrence of indeterminacy as a way to understand the origin and the persistency of economic fluctuations.

This thesis is mainly interested in the appearance of endogenous cycles generated by agents' self-fulfilling expectations. Two models are usually employed to study the emergence of endogenous fluctuations: the optimal growth model and the overlapping generations (OLG) model. The former, initiated by Ramsey (1928) and later by Cass (1965) and Koopmans (1965), de-

[3]The work of Azariadis (1981), Azariadis and Guesnerie (1986) and Grandmont and Malgrange (1986) supported by the works of Cass and shell have outlined the existing view about stochastic endogenous cycles.

[4]For more literature, see Benhabib and Farmer (1996, 1999).

scribes a real economy with perfect foresight and infinite-lived identical agents. This model provides an equilibrium which is unique, Pareto optimal and locally determinate. However, the OLG model, where agents live two periods, may exhibit a sub-optimal equilibrium and deterministic cycles (Allais 1947; Samuelson 1958; Diamond 1965).

Hence, in order to get endogenous cycles, while keeping the fundamentals constant, we have to depart from standard models. Departures consist of market incompleteness (such as the presence of transaction costs), the presence of distortionary taxes (consumption or income taxes), non-competitive market (fixed or entry costs), the existence of externality (in production or in consumption) or by introducing money as an exchange medium. Furthermore, market imperfections make the social optimal equilibrium and the market solution different. Consequently, the First Welfare Theorem, which shows that the centralized and the decentralized equilibrium are the same, is violated. It is worth to notice that standard OLG framework can promote cycles and violates the First Welfare Theorem because of the presence of an infinite number of agents and goods (De la Croix and Michel 2002).

As the emergence of endogenous fluctuations depends primarily on the social interactions between households, the interrelations among agents in the market can be better captured when they are heterogeneous. Further, from a dynamic view point, the heterogeneity provides a richer environment than the representative-agent's framework. In fact, there are two types of heterogeneity that can be introduced in the economic models. On the one hand, heterogeneity across households that stems from different initial endowments and preferences. On the other hand, the coexistence of two generations, young and old, gives rise to an economy with two types of agents.

Taking both types of heterogeneity into considerations, the main objective of this thesis is to provide an explanation for endogenous cycles and the existence of indeterminate stationary solution due to self-fulfilling expectations in four different economies: (i) with consumption externalities; (ii) with transaction costs and collaterals; (iii) with flexible transaction costs and (iv) with progressive labor-income tax. In order to achieve this objective, we use the local dynamic analysis method based on geometrical arguments and pioneered by Grandmont *et al.* (1998).

At the beginning, consumption externalities are defined as a non-market phenomenon and

considered as external costs or benefits arising from a consumption activity. The existence of these externalities is confirmed empirically. These externalities, which are either positive or negative, affect agent's behaviour in the sense that one wants either to imitate or to be different from others in their consumption behaviour. The introduction of consumption externalities makes the equilibrium of the model inefficient because the individual does not take them in account when he chooses his consumption path (Alonso-Carrera *et al.* 2005). Further, consumption externalities can also be considered as a source of endogenous fluctuations and economic instability (Chen and Hsu 2007; Alonso-Carrera *et al.* 2008). However, even if the empirical evidence confirms the existence of consumption externalities and their effect on long-run equilibrium can not be ignored, they have not been largely treated in literature relative to production externalities. Therefore, chapter two stresses on the dynamic effects of consumption externalities on equilibrium in a Ramsey model where there exist poor and rich agents.

The model in chapter two describes a non-monetary economy. Namely, when transactions take place, there is no money that serves to facilitate these transactions. In order to employ Ramsey model for the analysis of monetary issues, a role for money should be formulated. One method that is followed to integrate money into general equilibrium models is the Cash-in-Advance (CIA) approach (Clower 1967). Accordingly, money is required to purchase certain types of goods and for a given positive interest rate, agents face an opportunity cost of holding money. Thus, in order to reduce the burden of the CIA constraint, the requirement of CIA is introduced only to a subset of consumption goods, *i.e.*, partial CIA constraint (Grandmont and Younès 1972). In particular, it is assumed that consumers pay a fraction of their consumption goods in cash while the remaining part is paid by credit. The amount of credit payments carried out by an agent depends on the capital available to him, *i.e.*, collateral. Such a framework gives the households the possibility of substituting between two ways of getting consumption goods. More specifically, when the opportunity cost of holding money goes up, consumers can pay their consumption by credit. This monetary framework with credit market imperfection can be a source of economic fluctuations (Barbar and Bosi 2010). In this line, chapter three takes one further step to study the dynamic effects of credit market imperfection in a segmented market where poor agents without collaterals have limited access to credit while rich agents can pay by credit.

While consumers face the opportunity cost of holding money (nominal interest rate) in monetary economies whenever they use money as a medium of exchange, other costs are also incurred in making exchange in non-monetary economies. These costs are called transaction costs (entry costs) and represent the costs of participation in the market. That is, when households make investment, they should not pay only the price of capital, but also other costs associated with getting this capital. These costs consist of, among others, government fees, information and search costs and brokerage commissions, etc. As these costs affect the decision of consumers about his saving (investment), they may be considered fundamental factors that generate economic fluctuations and instability. Such an issue is the main objective of chapter four which introduce transaction costs in a simple OLG model.

In order to perform any investments, agents should pay related transaction costs. At the same time, after doing investments and agents receive their income (either capital or labor income), they have to pay income taxes. As provided by the U.S. tax code, labor-income taxation is more progressive than capital-income taxation. Thus, by introducing the progressive labor income taxation policy in an OLG model, chapter five aims at examining its role on the stability properties of the economy. Furthermore, according to the existing literature (Cazzavillan and Pintus 2004), OLG framework with endogenous labor supply makes indeterminacy more likely to appear. This justifies why, in chapter five, we endogenize the labor-leisure choice.

This thesis is organized as follows. The remaining part of this introduction presents a detailed description of market imperfections considered in the thesis. The second chapter is based on a joint work with Riham Barbar focusing on the effect of consumption spillover on steady state stability in an infinite-horizon heterogeneous agents' framework. Chapter three, which is the fruit of collaboration with Stefano Bosi and Alain Venditti, considers the role of transaction costs and collaterals in a monetary economy with credit market imperfection. However, chapter four studies the role of flexible transaction costs in a simple OLG model with exogenous labor supply while chapter five focuses on the role of taxation policy in economic stability in an OLG framework with endogenous labor supply. Finally, chapter six provides concluding remarks.

## 1.1 Consumption Externalities

The fact that agents do not care only about their own consumption but also about others' consumption is contributed by the early work of Veblen (1899) and later in the relative income hypothesis of Duesenberry (1949). Their argument is that agents' activity is shared with others in the market and so one's consumption behavior affects others' consumption decision, giving rise to interdependent preferences (the jealousy and the admiring feelings). In other words, agents care more about relative consumptions rather than absolute level of consumption.

There are some empirical works which show the importance of external effects and the interdependent preferences in the real life. Easterlin (1995) suggests a model where preferences depend positively on individual's income and inversely related to others' average income. He deduces that raising income of all agents does not increase the happiness of all.[5]Kosicki (1987) finds that agent's income ranking in his region influences considerably their saving and consumption behaviour. Based on British data, Clark and Oswald (1996) test the hypothesis that preferences depend on income relative to a reference or a comparison level. They deduce that workers' happiness is inversely related to their comparison wages. Recently, Tian and Yang (2009) give clear explanations for the results obtained by Easterlin (1995) where average happiness levels do not increase as the economic growth of a country raises.

Theoretically, a large number of works introduces the consumption external effects as some reference consumption level in the agent's felicity function. Such a reference is given by the average consumption level of the entire or a subset of the population and is exogenously taken by an individual. Dupor and Liu (2003) suppose that an agent's felicity function depends on own consumption in addition to the consumption level of some reference group. They show that externalities can affect agent's utility in two different manners. On the one side, externalities have direct impact on agent's utility, *i.e.*, an individual feels "admiration" (resp., "jealousy") if consumption externalities increase (resp., decrease) his utility. On the other side, externalities have an influence on marginal levels: if consumption externalities raise the marginal utility from own consumption, then preferences display "Keeping-up with the Joneses" feature. This means

[5]For more literature concerning empirical evidence, see Brekke and Howarth (2002) and Frank (1985). Moreover, Zizzo and Oswald (2001) show, by experiment, that agents' satisfaction depends negatively on others' level of income.

that others' consumptions make the marginal utility from my own consumption more valuable. Conversely, if consumption externalities reduce the marginal utility from own consumption, then agent's preferences exhibit "Running-away from the Joneses" feature. In this case, an increase in others' consumption reduces the satisfaction obtained from own consumption, inducing the agent to reduce his consumption.

Consumption externalities have been widely studied in the existing literature. Among others, Abel (1990) shows that the presence of consumption externalities in asset-pricing model can explain the divergence between empirical and theoretical findings. Galî (1994) demonstrates that consumption externalities influence the investment choices of individuals under the assumption that investment in financial assets is considered as a social activity. The impact of consumption external effects has been also examined in the context of taxation policy (Alonso-Carrera *et al.* 2005; Ljungqvist and Uhlig 2000; Nakamoto 2009), long-run macroeconomic efficiency (Alonso-Carrera *et al.* 2004; Turnovsky and Monteiro 2007; Liu and Turnovsky 2005) and long-term economic growth (Fisher and Hof 2000; Mino 2008).

Further, the dynamic effects of consumption externalities have been extensively treated in literature. In a representative-agent Ramsey model, Liu and Turnovsky (2005) show that the dynamic equilibrium is always unique, and thus locally determinate, whenever labor supply is inelastic. However, consumption externalities can be considered as a source of local indeterminacy if the labor is endogenously supplied (Alonso-Carrera *et al.* 2008). In particular, local indeterminacy appears if the interest rate increases when agents accumulate more capital (higher savings). They demonstrate that the positive relation between saving and capital returns occurs whenever an increase in saving gives a higher level of next-period labor supply, which necessitates that consumption externalities have an influence on labor supply. Conversely, if labor supply is inelastic, an increase in savings has a negative effect on the marginal product of capital, and thus agents' initial expectations are not self-fulfilled. Chen and Hsu (2007) also consider a one-sector neoclassical model with consumption externalities and prove that it is possible to display indeterminate stationary equilibrium, even when the labor supply is inelastic. The authors demonstrate that local indeterminacy appears if the rate of time preference is decreasing in individual consumption.

In light with the above literature, consumption externalities play a fundamental role on the

appearance of local indeterminacy. Along this direction, the second chapter of this thesis studies the dynamic effect of consumption externalities. However, we depart from the mainstream of the existing literature about consumption externalities by assuming that households are heterogeneous.

By considering the heterogeneity across agents, we focus on a more realistic and more relevant framework than the representative-agent framework. Obviously, the societies are populated of households that differ in their wealth and preferences. In addition, such an assumption allows stressing on the dynamic interaction between agents in the economy which in turn makes the role of external effects more significant than in the representative-agent framework. More specifically, consider for instance an economy with two groups of agents: poor and rich. Then, one can observe that an agent in the rich group does not care about the consumption behaviour of the poor agents whereas he is jealous from or want to keep-up with the consumption level of other members of his group. Therefore, interestingly, the heterogeneous-agent framework allows covering more configurations than that examined in the representative-agent framework.

In literature, several authors have examined the impact of consumption externalities in heterogeneous-agent models on the long-run distributions of income and wealth as well as the appearance of endogenous cycles and local indeterminacy. For example, Garcia-Peñalosa and Turnovsky (2008) consider a neoclassical growth model with inelastic labor supply and assume that agents' heterogeneity stems from initial wealth and the reference group. They show that whenever the preferences display "keeping up with the Joneses" feature, then wealth inequality is lower than that in the model without consumption externalities. Mino and Nakamoto (2008a, 2009) consider also a neoclassical growth model with inelastic labor supply with two types (groups) of agents who have different felicity function and initial wealth. Further, they distinguish between intergroup and intragroup consumption externalities. On the one side, Mino and Nakamoto (2008a) introduce nonlinear income taxation and show that the symmetric steady state is either unstable or locally indeterminate if the intergroup external effects dominate the intragroup externalities. On the other side, Mino and Nakamoto (2009) demonstrate that the long-run wealth distribution is highly sensitive to the strength of intergroup and intragroup externalities. Further, Mino (2008) develops an endogenous growth version of Diamond's (1965) overlapping generations model with intragenerational and intergenerational

consumption externalities. Mino demonstrates that balanced growth equilibrium and transitional dynamics depend on the preference structure that characterizes consumption external effects. Finally, Barbar and Barinci (2010) introduce intergroup and intragroup consumption externalities in a Ramsey model with inelastic labor supply à la Becker (1980) where agents differ in their initial wealth, discounting rates and felicity functions, and can not borrow against their future income. The authors show that whenever the intergroup externalities are high enough and display keeping-up with the Joneses feature, then they represent the channel through which deterministic cycles of period two appear.

Along this line of research, chapter two considers a heterogeneous-agent Ramsey model with consumption externalities and aims at examining the conditions under which endogenous fluctuations due to self-fulfilling expectations can emerge. For this purpose, we focus on a simple and realistic configuration by assuming that there exist two types (or groups) of households: poor and rich. Further, it is assumed that the agent from the poor group wants to keep-up with the members of the rich group whereas a rich person does not care at all about the consumption behaviour of the poor. It is also assumed that agents in the same group are identical and their consumption does not create any external effects on other members of the same group. In other terms, chapter two only focus on intergroup externalities and ignore the intragroup external effects for two reasons. First, this simplifies the analysis. Second, as previous literature demonstrates, it is more likely the intergroup externalities that play a crucial role on instability and the emergence of cycles.

In the absence of consumption externalities, Ramsey model with heterogeneous agents and inelastic labor supply has been treated in two different economies, where heterogeneity across consumers stems from different initial capital, discounting rates and felicity functions. On the one hand, in an economy where households have different discount factors and impatient agents are allowed to borrow against their future income (*i.e.*, markets are complete), impatient agents' consumption asymptotically vanishes and the steady state does not exist (Le Van and Vailakis 2003). On the other hand, in another economy with incomplete markets as represented by borrowing constraints, *i.e.*, agents cannot borrow against their future income; Becker (1980) has shown the existence of a steady state, contrary to Le Van and Vailakis (2003), at which only the most patient agent holds the entire of capital in the economy while the impatient works to

finance his consumption. After that, Becker and Foias (1987) have demonstrated that this type of heterogeneity plays a crucial role on the appearance of deterministic cycles of period two. Such a result has been later discussed in more details by Becker and Foias (1994) and further elaborated by Sorger (1994).

As the borrowing constraint matters whenever agents are heterogeneous concerning the existence of a steady state and the long-run wealth inequality, chapter two adopts the Ramsey model initiated by Becker (1980) to analyze the social interaction between poor and rich. More precisely, it is assumed that there are two types (groups) of consumers who are differently endowed with capital and have different preferences and are not allowed to borrow against future income. It is also assumed that the poor (impatient) wants to keep-up with the consumption of the rich (patient) while this later does not care about the consumption of the poor. In this framework, we assume that the labor supply is endogenous. In other terms, chapter two is an extension of Barbar and Barinci (2010) by allowing labor-leisure choice and restricting attention to the most realistic configuration that may emerge in the economy between poor and rich.

The inclusion of endogenous labor supply is motivated as follows. An endogenous labor supply in a heterogeneous model allows stressing heterogeneity across agents as well as the external effects on the stability properties of the steady state. In fact, Barbar and Barinci (2010) do not capture the role of heterogeneity in the sense that only the patient agent's preferences generate cycles. In particular, contrary to Becker and Foias (1994)[6], Barbar and Barinci (2010) show that the negative response of the patient agent's income to the capital stock is no longer required for the appearance of these cycles if the patient agent's preferences exhibit KUJ feature. Therefore, the endogeneity of labor supply in Barbar-Barinci is expected to stress on the interaction across agents, making the role of external effect more transparent. More precisely, the impact of consumption externalities arises through the interplay between the consumption-labor arbitrage and the mechanism of capital accumulation. Thus, the heterogeneous preferences, and particularly, the impatient agent's preferences, will have a great influence on the local dynamic. In the absence of consumption externalities, Bosi and Seegmuller (2010a) have shown that the inclusion of elastic labor supply in Becker and Foias (1994) implies that the impatient agent's

[6]Becker and Foias (1994) show that the existence of periodic orbit of period two requires a negative response of the patient agent's income to the capital stock. In other terms, Income Monotonocity Assumption is sufficient to rule out these cycles.

preferences play an important role on the emergence of cycles of period two. Further, the endogeneity of labor choice is known to have a crucial effect on the dynamic properties of the steady state in a representative-agent Ramsey model with consumption externalities (Liu and Turnovsky 2005 and Alonso-Carrera *et al.* 2008). As a result, by including endogenous labor supply in Barbar-Barinci, we aim at examining whether it provides a richer dynamic than in the representative model.

In chapter two, it is shown that only the patient agent owns capital stock in the long-run while the other (impatient) agent only consumes his wage-income as he does not hold capital. In addition, we show the existence of two different configurations according to agents' labor supply. The first is where both agents supply labor and the second is with only impatient agents supply labor but the patient agents enjoy leisure.

Our main result states that there is a room for local indeterminacy and endogenous fluctuations due to self-fulfilling expectations. This result stands to the contrary of all the variants of Becker (1980), mentioned above, in which the steady state is always determinate but changes its stability through the emergence of deterministic cycles of period two.

Intuitively, the appearance of local indeterminacy requires that when agents coordinate to enhance their savings, the interest rate increases. In models with elastic labor supply, the positive relationship between saving and interest rate may arise when an increase in capital accumulation causes an augmentation in next-period labor supply which in turn raises future interest rate. In our model, the next-period labor supply goes up due to two mechanisms. The first one is through the elasticity of labor supply with respect to the real wage. A positive elasticity acts as a destabilizing factor. The second mechanism is through the patient agent's income. When it increases with the capital, then rich agents increase their consumptions. The presence of consumption externalities allows poor agents to increase their consumption as well which requires a higher level in labor supply. Higher labor supply implies a higher interest rate. Thus, the increase of patient agents' income with respect to capital acts as a destabilizing factor. As a result, local indeterminacy appears if and only if both effects work in the same direction or, when they are opposite, the destabilizing effect should dominate.

One can note that whenever labor supply is inelastic, the above intuition does not hold. In particular, following an increase in impatient agent's consumption, they do not change their

labor supply which implies no effects on real interest rate. Self-fulfilling expectations by patient agents would be never satisfied.

Finally, we show that, under specific parameterization values, these endogenous fluctuations can be ruled out even when the elasticity of input substitution is close to zero.

## 1.2 Collaterals

Chapter three represents the unique contribution of this thesis that considers a monetary environment. The objective is to study the influence of collateral on macroeconomic fluctuations and economic growth in a monetary framework with credit market imperfection and heterogeneous agents.

The idea of this chapter is motivated by the U.S. subprime crises which have not only influenced the American economy but also many developed and under-developed countries. As a result, stock markets around the world have fallen, housing market and large financial institutions have distorted and governments in even the wealthiest countries have created rescue packages to support their financial systems. One major problem to these crises is that banks, financial institutions, and mortgage lenders have issued housing loans with very high risk, *i.e.*, mortgage loans to households with low income and collateral or with bad credit history.

The statistics show that the collateralized lending was not applied in U.S. financial system over the last decade of 20th century. Namely, the annual growth rate in mortgage debt outstanding was 10.2 percent from 2001-2005 while the annual growth rate of non-mortgage debt outstanding was only 4.6 percent (Mian and Sufi 2008). Throughout these years, the homeownership rate has increased from 64 percent in 1994 to about 69 percent in 2005, because of a raise in the mortgage credit rate and housing market development (Chambers *et al.* 2009).

Following the high interest rates, borrowers find it more difficult to repay their credits, resulting in high levels of delinquencies and foreclosures. Therefore, lenders were forced to take borrowers' houses and to sell it which resulted in sharp decline in house prices. Reductions in the value of housing imply a decrease in the homeowner, banks and financial institutions' wealth, which justifies losses for households' consumption expenditures that affect the whole economy. Papadimitriou *et al.* (2007) show that the index of median prices of existing houses

increased by 19 percent between 1995 and the first quarter of 2000 and by 20 percent till the third quarter of 2005, where the house prices reached their peak. In the first quarter of 2007, this index dropped about 9 percent of its value and since the second quarter of 2007 it continues to decline. However, average house prices in U.S. rose by almost 200 percent between 1996-2005 during the boom period of the mortgage loan market (Mian and Sufi 2008).

Subsequently, the fall in American house prices affects the consumption of wealthy households and so their investment, which results in a decline in GDP. Meanwhile, industry and maintenance sectors, which are related to houses, are affected by the crisis (such as, machines, furniture, painting, etc). According to estimates released by the National Bureau of Economic Analysis, quarterly real GDP of the U.S. declined at rate of 3.8 percent in the fourth quarter of 2008 compared to a decrease of 0.5 percent in the third quarter. In the fourth quarter of 2007, GDP declined by 0.2 percent compared to an increase of 1.5 percent in 2006. As a result, the lack of collateral with borrowers can be considered as one main reason for macroeconomic volatility and economic instability.

In light of the financial crises, this chapter is interested in examining the role of collateral on the emergence of endogenous fluctuations. For this purpose, we introduce the collateral via a partial cash-in-advance (thereafter, CIA) constraint in the spirit of Grandmont and Younès (1972). In particular, we assume that only a part of consumptions is paid in cash while the remaining part is paid by credit. It is also assumed that the share paid with credit depends positively on the amount of collateral available to the consumer, where the collateral is represented by agent's current capital. More capital held by consumer, more goods can be paid by credit. In view of cash-credit models such as, Lucas and Stokey (1987), Ireland (1994) and Hromcova (2003), the share paid by credit can be referred to "credit good" while the part paid in cash is called "cash good".

By considering a partial CIA constraint, this chapter deviates from models that introduce collateral through the credit constraint, such as Kiyotaki and Moore (1997), Kiyotaki (1998), Cordoba and Ripoll (2003), Suárez-Lledó (2007, 2009), Longstaff and Wang (2008) and Chakraborty and Hu (2006). In this literature, agent's credit is endogenously determined by the amount of collateral assets owned by them. Particularly, Kiyotaki and Moore (1997) and Kiyotaki (1998) suppose that lenders can not force borrowers to repay their debts unless the debts

are secured by collateral. In such economies, durable goods –like land, buildings and machines- do not represent only factors of productions but also serve as collateral for loans. Kiyotaki aims to show that when credits are limited and endogenously determined, the impact of small temporary shocks on either technology or wealth distribution has large persistent fluctuations in aggregate outputs and prices. In a monetary framework, Cordoba and Ripoll (2003) focus on the role of collateral in the propagation of aggregate output fluctuations following small monetary shocks. In a small open economy, Kocherlakota (2000) demonstrates how introducing credit constraints can have amplifications and propagation mechanism which transforms the shocks into observed fluctuations in aggregate output. Chakraborty and Hu (2006) consider two alternative approaches concerning credit constraints: exogenous and endogenous constraints. His objective is to show their effects on the amplification on business cycles. Calibrating his model to the Japanese economy, they show that the productivity fluctuations in Japanese data in 1980–2000 can be explained better once credit constraint is endogenous.

In addition, our work differs from the above literature in the sense that aggregate economic fluctuations and cycles do not emerge because of productivity/monetary exogenous shocks. Instead, our model exhibits endogenous fluctuations that are generated by agent's self-fulfilling expectations. In fact, the partial CIA constraint is known to be a source of endogenous fluctuation and indeterminacy as shown by Bosi and Magris (2003), Bosi and Dufourt (2008) and Barbar and Bosi (2010). In particular, Bosi and Magris (2003) suppose that an exogenous fraction of consumption goods is paid in cash and the remainders by credit. It is demonstrated that under sufficiently slight liquidity constraints (small departure from standard Ramsey model) the steady state exhibits indeterminacy for all values of elasticity of intertemporal substitution in consumption. This work is extended by Bosi and Dufourt (2008) who show that the result of Bosi and Magris (2003) is robust even when the partial cash-in-advance is applied to both consumption and investment. Moreover, Barbar and Bosi (2010) also extend Bosi and Magris (2003) by considering an endogenous growth framework à la Romer (1986) and assuming that the share of consumption goods financed by credit depends positively on households' capital-consumption ratio. That is, if household's level of capital is high relative to consumption level, he can easily pay his consumption by credits. The authors show that a high elasticity of credit market with respect to collateral has a stabilizing effect in the sense that fluctuations are less

likely to appear. Regarding the monetary rule, a higher money growth rate is associated with a higher economic growth.

In line with Barbar and Bosi (2010), this chapter adopts an endogenous growth à la Romer (1986) in order to study the effect of collateral on economic growth. However, contrary to the existing partial CIA models, we assume that agents are heterogeneous à la Becker and Foias (1994). The introduction of heterogeneity across agents allows covering a richer class of situations than that treated in the representative-agent economy. In particular, the heterogeneity in discounting gives rise to two classes of households: patient capitalists and impatient workers.

We prove that two different configurations can occur concerning the patient agent's portfolio. The first is where patient agent holds only capital but no balances whereas the second consists of a portfolio of both capital and money.

Our results are the following: we show that heterogeneous discounting implies that the most patient agent holds the entire capital in the market (capitalist) while impatient agents hold zero capital (workers).

Provided that patient agents do not pay in cash in the first regime, and thus liquidity constraint does not exist, the economy jumps to the balanced growth path. However, in the second regime, where money influences the accumulation of capital, endogenous fluctuations and deterministic cycles might emerge around the balanced growth path.

In the first regime, the effect of monetary policy on welfare is different among agents. In particular, increasing money supply enhances the well-being of the capitalist because he will spend this money in the current period not in the future period. So he avoids any opportunity costs of holding money. However, since higher money supply is accompanied by increasing nominal interest rate, workers who use only money to consume bear a higher opportunity of costs of holding money. Workers' welfare loss appears since the interest rate effect is much higher than the benefits generated from additional monetary transfer received from the monetary authority.

In the second regime, money supplied by monetary authority has a similar influence on both agents' welfare even if capitalist use both money and credit to consume. Given that the capitalist holds all capital in the market, which can be used as collateral, this reduces the burden of liquidity constraint and hence he bears a low opportunity cost of holding money. Moreover,

it is shown that increasing the sensitivity of credit to collateral raises both agents' welfare due to its' positive effect on economic growth.

From the stability point of view, contrary to Barbar and Bosi (2010), higher money growth rate promotes the appearance of endogenous fluctuations. This is because rising money supply involves a higher opportunity cost of holding money which in turns makes agents more restricted to the liquidity constraint. The presence of a larger credit market imperfection is associated with a higher probability of endogenous cycles to emergence due to self-fulfilling expectations. Contrarily, increasing the credit market regulation policy reduces the burden of the liquidity constraint faced by agents and makes indeterminacy less likely to appear. Therefore, the sensitivity of credit to collateral performs as a stabilizing factor.

## 1.3 Variable transaction costs

In the financial market, transaction costs are considered as main elements that determine the fraction of households who can participate in financial market and their investment portfolio choice. Intuitively, if these costs are sufficiently high comparing to interest rates, households do not participate, and vice versa. These costs consist of, among others, government fees, communication and time costs, information and search costs and brokerage commissions.

Several empirical works have provided strong evidence about the importance of transaction costs and particularly the effect of these costs on asset prices or on household investment portfolio. For instance, using Italian Survey of Household Income and Wealth, Paiella and Tiseno (2004) find that the risky asset portfolios of Italian households are coherent with a fixed cost of participating in the stock market around 150 Euros per year. Based on Panel Study of income Dynamics data in 1984, 1989 and 1994, it is observed that 49.66 percent of households with positive wealth do not hold positions in either bonds or stocks. In addition, 23.04 percent of households hold stocks but no bonds; whereas 14.79 percent hold bonds but not stocks and only 12.5 percent hold both. According with these observations, Vissing-Jorgensen (2002) introduces several types of costs related to stock market participations - fixed transaction costs, proportional transaction costs, entry cost and per period participation costs – into an econometric model to explain the above data. One main result is that he estimates that 50

dollar of transaction costs are sufficient to explain the choices of half of the non-participants in the stock market, further, 260 dollar of transaction costs are enough to explain the choices of 75 percent of the nonparticipants.

Moreover, based on 1998 OECD data, countries with the highest level of stockholdings are U.S. and Sweden where about 50 percent of households do not invest in stock market. This percentage is about twice as high as in France, Germany and Italy. Guiso *et al.* (2003) seek to know empirically the economic reasons behind the variations of stockholding across countries observed in the data and show that the main indicator of stockholding variations is the presence of costs which are not related to stock prices. Further, Alan (2006) tries to explain the observed limited stock market participation in the data. To do so, he executes a basic life-cycle model with portfolio choice and participation costs. These costs are supposed to be paid only once over the life cycle. Once they are paid, investors can freely participate. These costs are formulated as a proportional to the wage income. It is obtained, based on a simulation estimation technique, that in the absence of costs, household stockholding is about 80 percent to 90 percent, but with only one percent cost, this percentage reduces to around 30 percent.

Transaction costs have been also treated in theoretical literature. For example, in an open market framework, Carmichael and Coën (2003) study the implications for international portfolio diversification with transaction costs and show that the introduction of very small transaction costs is sufficient to reproduce the large home bias observed in the composition of portfolio in the data. In a two-asset portfolio selection, Constantinides (1986) proves that the demand for assets is very sensitive to transaction costs and proportional costs have only a small impact on asset prices. In an OLG model, Chau (2002) considers liquid and illiquid financial assets. The liquid assets, which have zero return, are traded without transaction costs while trading in illiquid assets, which have positive return, involves fixed costs which are independent of the volume of transactions. It is shown that agents use illiquid assets for long-term savings in order to avoid paying costs each time they trade. Lo *et al.* (2004) consider a dynamic equilibrium model of asset prices and trading where agents face fixed costs. It is found that the presence of these costs allows agents to trade infrequently. Similar results are obtained by Vayanos (1998) who demonstrates that increasing in costs has an ambiguous effect on stock prices; on one hand, agents buy fewer stocks and on the other hand, they hold them for longer periods.

In view of the existing literature, the study of transaction costs seems to be plausible and significant. Further, it worth noticing that this literature investigates the role of transaction costs in partial equilibrium frameworks and mainly, focuses on their effects on investment portfolio choice and on stock demand. In addition, the influence of transaction costs on stability properties of the steady state has not been studied yet. This paper thus fills this gap by considering an extension of the Diamond's (1965) two-period overlapping generation economy through introducing transaction costs for young agents. The benchmark Diamond's model is based on the coexistence of young and old agents at each date. Young agents work, consume, and save during the first period and old agents are retired and consume their saving returns during the second period. It is assumed that old agents have no altruistic concern their children and thus they consume the entire of their capital income. The transaction costs are assumed to depend positively on agents' saving amount of young agents.

In fact, contrary to infinite-horizon Ramsey model, the OLG framework allows modelling the different periods of life explicitly and is usually exploited by literature to study the allocation of resources across different generations. Thus, using a two-period OLG model enables us to formulate and to focus on the effect of costs on agents capital accumulation (saving) explicitly. No restriction is imposed on saving function or on substitutability of consumption goods.

The main objective of chapter four is to study the role of transaction costs on capital accumulation and on the appearance of endogenous cycles in a two-period OLG model where agents pay transaction costs on their savings when young.

Along this line of research, the emergence of endogenous cycles and local indeterminacy has been examined in the OLG models. In Diamond model, the steady state exhibits unique equilibrium if the saving rate is increasing in real interest rate. In this type of models, it has been shown the possibility of achievement of oscillating trajectories and chaotic dynamic under mild assumptions concerning the curvature of the utility function. In addition, it may exhibit a continuum of equilibria and displays deterministic cycles of every periodicity (see de La Croix and Michel 2002). There exist equilibria that are dynamically inefficient, *i.e.*, it is possible to increase the total consumption at least in one period without reducing it in any other period. A variant of Diamond model which has been considered by Geanakoplos and Polemarchakis (1986), Reichlin (1986) and Azariadis (1993) assumes that agents are not allowed to consume

in the first-period of their life. They show that the model exhibits Hopf cycles. Grandmont (1992) considers a monetary version of this OLG model and shows that it is possible to obtain deterministic cycles together with local bifurcations.

More recent literature is interested in studying the stability properties of the steady state in an OLG with endogenous labor supply such as, Nourry (2001), Cazzavillan and Pintus (2004, 2006), Nourry and Venditti (2006) and Lloyd-Braga *et al.* (2007). They show that the elastic labor supply may be a channel through which expectations matter and therefore generate self-fulfilling fluctuations. Nourry (2001) extends Diamond's model by endogenizing the labor supply. Using a non-separable utility function, the author shows that the steady state exhibits deterministic saddle-path stability if agents' savings are increasing with respect to real interest rate and if the stationary equilibrium is unique. In addition, there is a room for local indeterminate steady state to emerge. However, Cazzavillan and Pintus (2004) show that, whenever agents' preferences are additively separable and both-period consumption goods and leisure are substitutable, then there is a room for local indeterminacy if young agents' consumption share is sufficiently high. Therefore, endogenous cycles appear for unrealistic shares of young agents' savings out of wage income in addition to a low elasticity of capital-labor substitution. Subsequently, Nourry and Venditti (2006) assume a general non-separable utility function with no restriction concerning the substitutability property. The authors demonstrate that gross substitutability in consumption and leisure implies a unique stationary equilibrium if only one steady state exists. However, if the gross substitutability is violated, local indeterminacy appears if and only if labor supply function is elastic enough with respect to interest factor. After that, Cazzavillan and Pintus (2006) show that for small enough level of capital externalities, local indeterminacy requires unrealistically high saving rates. Then, Lloyd-Braga *et al.* (2007) introduce capital and labor externalities into a standard OLG with two-period consumptions. They demonstrate that, if the share of the first-period consumption out of wage income is sufficiently high then local indeterminacy can emerge with small labor externalities if the elasticity of input substitution and the elasticity of labor supply are adequately high.

Chapter four provides the following results. At the beginning, we show that the steady state may be dynamically inefficient (resp. efficient) if there is an over-accumulation (resp. under-accumulation) of capital with respect to the modified Golden Rule. In other words, the net

return of capital is lower (resp. higher) than the gross rate of population growth. Moreover, it is shown that the steady state exhibits a determinate equilibrium which is characterized by an under-accumulation of capital comparing to standard Diamond model without costs. Finally, we demonstrate that the steady state changes its stability through the emergence of cycles of period two.

The intuition of flip bifurcation is the following. Assume that the level of capital increases from its steady state value. This yields a higher wage income and therefore induces more capital accumulation. However, there are some factors that affect capital accumulation negatively. Namely, the presence of high transaction costs, low elasticity of marginal utility of future consumption, high sensitivity of marginal utility of current consumption and high first-period consumption share induce agents to accumulate lower capital. Cycles of period two appears whenever the latter effect dominates the former one. This requires a sufficiently high sensitivity of transaction cost with respect to capital accumulation, low elasticity of marginal utility with respect to future consumption, high elasticity of marginal utility with respect to current consumption and high first period consumption share out of wage income.

## 1.4 Progressive income tax

The statistics provide strong evidence about the existence and the importance of individual income taxes. In particular, these taxes are considered as one important resource for U.S. government. These taxes stand for 8 percent of American GDP and about 45 percent of Federal Tax Revenue in 2008.[7]However, comparing to 2007, the amount of income taxes paid declined by about $84 billion and the average tax rate reduced from 12.68 percent in 2007 to 12.24 percent in 2008 (see Prante and Robyn 2010, Fiscal Fact, Tax Foundation). In addition, in 2008, the top 1 percent of tax returns earned 20 percent of adjusted gross income and 38 percent of all federal individual income taxes compared to 22.8 percent and 40.4 percent respectively in 2007.

From 2005 to 2007, the share of income earned and taxes paid by the top 1 percent is always rising but this trend reversed in 2008. Particularly, in 2008, the share of the top 1 percent of

[7]Source: Congressional Budget Office, a preliminary analysis of the president's budget and an update of CBO's budget and economic outlook, March 2009.

tax returns was the lowest during the last decade which can be principally justified by the low of capital gain. At the same time, according to federal income tax in 2007, the income and tax shares of the top 1 percent are higher than that of the bottom 95 percent. But, this comparison is no longer the same in 2008 in the sense that the bottom 95 percent paid 41.3 percent of total collected by federal income tax which is more than the top 1 percent shared around 38 percent (Prante and Robyn 2010).

Most OECD countries apply income taxes with progressive features where tax rate increases as the taxable base amount gets higher. For instance, the average income tax rate in 2008 ranged from roughly 2.6 percent of income for the bottom half of tax returns to about 23.27 percent for the top 1 percent (see "Summary of Latest Federal Individual Income Tax Data" 2010, Fiscal Facts No. 249, Tax Foundation).

Given the importance of income taxes in U.S. economy, chapter five introduces progressive income taxation policy in a two-period Diamond's (1965) OLG model with endogenous labor supply. In this framework, when agents are young, they supply labor elastically and allocate their labor-income between consumption and saving while, when old, they are retired and consume the entire saving returns. Such a framework is extended to include a progressive tax which is imposed on labor income. In fact, we only focus on labor-tax since it is more progressive than capital-income tax according to U.S. tax code.

The main objective of this chapter is to examine the impact of progressive tax on young agents' labor-income on the stability properties of the steady state in a two-period OLG model with consumption in both periods.

In the existing literature, the role of progressive tax policy on macroeconomic stability has been considered in OLG model. For instance, Dromel and Pintus (2006) introduce a progressive capital-income taxation policy (Musgrave and Thin 1948) into Reichlin's (1986) model where agents consume only in the second period. The authors show that this fiscal policy acts as a stabilizing factor. Further the stability of the fiscal policy against endogenous fluctuations depends mainly on the tax base. Namely, if agents' consumptions are mostly financed by capital (resp. labor) income, then in order to stabilize the economy and prohibited any fluctuations, government has to apply progressive taxation policy on capital (resp. labor) income.

Moreover, Chen and Zhang (2009a, 2009b) study a two-period OLG model with separable

utility function where agents consume in both periods and propose different endogenous linear tax policies. Mainly, Chen and Zhang (2009a) focus on an endogenous labor-income tax rule and show that under numerical calibrations of the parameters, local indeterminacy easily appears with small distortionary taxes, provided an elasticity of input substitution less than capital share in total income and an elasticity of labor supply is sufficiently high. In addition, labor income tax rates are helpful for the emergence of indeterminate steady state. This result is motivated by the fact that labor-income tax rates make the ratio of consumption while young to saving smaller and therefore, endogenous cycles are more likely to appear. Chen and Zhang (2009b) introduce endogenous capital-income tax policy into the same OLG model and show contrarily that this tax policy prevents the appearance of endogenous cycles. Putting it differently, increasing the tax rates shrinks the range of values of the consumption share out of wage income associated with local indeterminacy.

In this line of research, considering a progressive labor-income taxation policy in an OLG allows studying the robustness of the results obtained by Dromel and Pintus (2006).

The impact of income tax has been also studies in a standard infinite-horizon model. Schmitt-Grohé and Uribe (1997) is the pioneering paper that initially investigated the relationship between fiscal policy and long term economic stability when constant government expenditure is financed by either labor income or total income taxation. It is shown that the steady state is indeterminate for plausible values of labor/total income tax rates. Intuitively, if agents are optimistic about the future, they choose to supply more labor in order to raise investment. Since the government spending is constant, the government has to decline the tax rate as total output rises. The presence of this counter-cyclical fiscal policy (regressive income tax policy) facilitates agents' expectations to be self-fulfilling and so endogenous cycles to appear.

The work of Schmitt-Grohé and Uribe (1997) has been later extended by Guo and Harrison (2004) who consider different tax rates applied to capital and labor income together with an endogenous government spending in a standard infinite-horizon model. They show that, contrary to Schmitt-Grohé and Uribe, this fiscal policy rules out indeterminacy emergence. Further, Guo and Lansing (1998) extend the model of Benhabib and Farmer (1994) by introducing income taxes. It is shown that high progressive income taxes prevent the occurrence of endogenous fluctuations. Their model confirms that a progressive (resp. a regressive) tax increases (resp.

reduces) the increasing rate of returns required for indeterminacy to arise. In addition, several contributions have focused on the capital distribution in the long run when consumers are heterogeneous and progressive tax applied to capital and labor income (Sarte 1997; Sorger 2002; Bosi and Seegmuller 2010b; Mino and Nakamoto 2008b).

Our main result states that progressive labor-income taxation operates as a destabilizing factor. In particular, local indeterminacy is more likely to appear under a high tax progressivity, a very low consumption share out of wage income and a non-realistic high savings rate, provided that the taxed income finances only a small fraction of total consumptions made by agents. Therefore, an increase in tax progressivity makes the normalized steady state more likely to be indeterminate because of conflicting effects on savings that operate through wage and real interest rate. Therefore, this confirms the result obtained by Dromel and Pintus (2006).

In this type of models, the emergence of indeterminacy is equivalent to the appearance of deterministic fluctuations (Lloyd-Braga *et al.* 2007). Thus the intuition goes as follows. Assume that current capital stock increases from its steady state. This generates two different effects on future capital. On one hand, an increase in current capital raises wage income and therefore agents supply more labor. According to the budget constraint, future capital increases. On the other hand, a high expected increase in future capital stock declines its real return. Based on real interest rate and real wage equations, the decrease in future real return is followed by a decrease in current-period labor supply and according to the budget constraint future capital declines. A cyclical path emerges whenever the second effect dominates the first one. This requires sufficiently low values of elasticity of input substitution and elasticity of marginal disutility of labor together with high values of tax progressivity and elasticity of intertemporal substitution in consumption.

# Chapter 2

# Consumption Externalities in a Ramsey Model with Endogenous Labor Supply

## 2.1 Introduction

In modern economies, it is observed that individuals care about the social aspect of their consumption in the sense that, it is not the absolute level of consumption that matters, but rather one's consumption relative to other individuals in the society. The idea that one's relative consumption represents his social position is not new issue. It has been initially discussed by Veblen (1899) and Duesenberry (1949) who show that individual's satisfaction depends on his position in the society, *i.e.*, his social status.

Such an idea is supported in numerous empirical studies. For instance, Easterlin (1995) deduces, based on American data, that increasing income of all people in a society does not increase their happiness because agents are caring more about their relative income. Moreover, based on British data, Clark and Oswald (1996) find that workers' happiness is inversely related to their comparison wages. Using psychological evidence, Frank (1997) shows that agent's satisfaction depends on his relative position.

The influence of social forces on individual's satisfaction is formulated in theoretical litera-

ture by introducing some reference (external) consumption level in the utility function. Such a reference is exogenously given and equal to the average consumption of some reference group which may comprise the entire or a subset of the population. According to Dupor and Liu (2003), when consumption spillovers augment (or decline) agent's marginal rate of substitution between agents' own consumption and leisure, then his preferences exhibit "Keeping-up with the Joneses", hereafter KUJ, (or "Running-away from the Joneses", hereafter RAJ). When KUJ, then agent is willing to imitate the consumption of others while when RAJ, he chooses his consumption in such a way that makes him different from others.

In the real world, such situations can emerge. Consider for example an economy in which the households differ in their wealth. Then, it can be observed that poor agents usually try to imitate the consumption behaviour. of rich while the later cares only about the consumption of other rich agents and does not pay any attention to the consumption level of the poor. We thus deduce that, on the one side, the existence of these consumption external effects is plausible and relevant. On the other side, studying these externalities in an economy with heterogeneous agents can provide a richer environment to cover more configurations than that studied in the representative-agent framework.

Along this direction, this paper considers an economy populated with rich and poor and focuses on the most realistic configuration where the consumption behaviour. of rich affects the poor's consumption while the inverse does not hold. For this purpose, a simple Ramsey model with heterogeneous agents is considered. Several works have focused on the long-run capital distribution when consumers differ in their initial capital level, discounting and felicity function. In a framework with heterogeneous agents, the borrowing constraint matters. Whenever consumers are allowed to borrow against their income, the steady state does not exist and the impatient agents' consumption asymptotically vanishes (Le Van and Vailakis 2003). Conversely, whenever a borrowing constraint is imposed, *i.e.*, market is incomplete, then a steady state exists and around it, the capital is concentrated in the hands of the most patient agents (Becker 1980; Becker and Foias 1987; 1994; 2007; and Sorger 1994; 2002). Obviously, it is more likely the later framework gives rise to two types of agents (poor and rich) in the long-run. As a result, this paper adopts the framework initiated by Becker (1980) to analyze the social interaction between poor and rich.

The objective of this paper is to study the role of consumption externalities on the appearance of local indeterminacy and endogenous cycles generated by self-fulfilling expectations in a Ramsey model with heterogeneous agents.

Consumption externalities have been widely studied in various contexts. Among others, Abel (1990) and Galî (1994) introduce externality in consumption in asset pricing model to clarify the divergence between theoretical and empirical findings in the data. Fisher and Hof (2000) and Alvarez-Cuadrado *et al.* (2004) consider consumption externalities to study the economic growth and show that considering consumption externalities increases the speed of convergence. Tian and Yang (2009) give formal explanation to Easterlin's (1995) empirical findings and analyze the impact of consumption externalities on agents' happiness.

Further, the impact of consumption externalities on the appearance of local indeterminacy and endogenous fluctuations has also been treated. In a representative-agent Ramsey model, the dynamic equilibrium is always unique and locally determinate whenever the labor supply is inelastic (Liu and Turnovsky 2005). However, Alonso-Carrera *et al.* (2008) demonstrate that the endogeneity of labor supply is a key requirement for the emergence of local indeterminacy. Chen and Hsu (2007) show that the dynamic equilibrium can be locally indeterminate, even when labor supply is inelastic, if agent's time-preference exhibits decreasing marginal impatience in private consumption.

Another strand of literature has focused on the dynamic effects of consumption externalities in Ramsey model with heterogeneous agents. For example, Garcia-Peñalosa and Turnovsky (2008) show that consumption externalities affect the speed of convergence and reduces wealth inequality. Further, intergroup and intragroup externalities have been introduced by several authors, such as Mino and Nakamoto (2008a) who show that whenever the intergroup dominate the intragroup external effects, the symmetric steady state can be locally indeterminate. In addition, Mino and Nakamoto (2009) demonstrate that the long-run wealth distribution is highly sensitive to the strength of intergroup and intragroup externalities.

Unlike these papers that only focus on heterogeneity in wealth, Barbar and Barinci (2010) introduce consumption externalities in the framework of Becker and Foias (1994) and distinguish between intergroup and intragroup external effects. The authors show that it's more likely the intergroup external effects that promote instability and the emergence of deterministic cycles of

period two. However, contrary to Becker and Foias (1994), the negative response of the patient agent's income to the capital stock is no longer required for the appearance of these cycles if the patient agent's preferences exhibit KUJ feature. The result of Barbar and Barinci (2010) is interesting but has two shortcomings. On the one side, the result is obtained in terms of the patient agent's preference only and thus the heterogeneity is not explicitly captured. On the other side, instability requires an unrealistic configuration where the rich is willing to imitate the poor in his consumption behaviour.

In light of the results of Barbar and Barinci (2010), this paper extends their work by including endogenous labor supply. It is assumed that the heterogeneity is reduced to two types of agents who differ in their initial wealth, discounting and the felicity function. However, we only consider intergroup externalities.[1]Under these specifications, we avoid the shortcomings of Barbar and Barinci (2010). The endogeneity of labor supply allows to underline the role of consumption externalities as well as the heterogeneity across agents on the stability properties of the model. In particular, the impact of consumption externalities arise through the interplay between the consumption-labor arbitrage and the mechanism of capital accumulation. Thus, under elastic labor supply, the heterogeneous preferences will have a great influence on the dynamic. In particular, contrary to Barbar and Barinci (2010), the impatient agent's preferences affect the stability properties of the model. Along this direction, in the absence of consumption externalities, Bosi and Seegmuller (2010a) show that heterogeneous preferences is the channel that generate instability and flip bifurcation and the impatient agent's preferences play a crucial role on the appearance of endogenous cycles.

Our model provides the following results. We show that there exists a steady state at which only the patient agent holds capital while the impatient agent has to work in order to finance his consumption. This result goes in line with all the variants of Becker (1980). Further, we demonstrate the existence of two economic configurations according to the labor supply: the first is where both agents supply labor whereas in the second, only impatient agent supplies labor while the patient agent enjoys leisure (see Bosi and Seegmuller 2010a). For simplicity, in local dynamic analysis, it is focused on the economy where only impatient agent supplies labor.

[1]Intragroup externalities are not introduced in the economy because, as it is shown by Barbar and Barinci (2010), they fail to promote endogenous cycles and instability in a Ramsey model à la Becker (1980).

After the characterization of the steady state, we restrict our attention to a particular configuration, which is the most relevant and the most realistic one, where the poor (impatient) wants to keep-up with the consumption of the rich (patient) while the later does not care about the consumption of the poor. Contrary to Barbar and Barinci (2010), this restriction ensures a realistic result.

Our main result states that there is a room for local indeterminacy and endogenous fluctuations due to self-fulfilling expectations. This result stands to the contrary of all the variants of Becker (1980), mentioned above, in which the steady state is always determinate but changes its stability through the emergence of deterministic cycles of period two.

The intuition goes as follows, the appearance of local indeterminacy requires that an increase in savings raises the marginal product of capital, and thus the interest rate, so that the initial expectations of a higher interest rate can be self-fulfilling. In models with elastic labor supply, the positive relationship between saving and interest rate may arise when an increase in capital accumulation causes an augmentation in next-period labor supply which in turn raises future interest rate. In our model, the next-period labor supply goes up due to two mechanisms. The first one is through the elasticity of labor supply with respect to the real wage. A positive elasticity acts as a destabilizing factor. The second mechanism is through the patient agent's capital income. When it increases with the capital, then rich agents increase their consumptions. The presence of consumption externalities allows poor agents to increase their consumption which requires a higher labor supply. Higher labor supply implies a higher interest rate. Thus, an increase in patient agents' income with respect to capital acts as a destabilizing factor. As a result, local indeterminacy appears if and only if both effects work in the same direction or, when they are opposite, the destabilizing effect should dominate.

Interestingly, the heterogeneity is captured as the impatient agent's preferences play a crucial role on the emergence of local indeterminacy. This is because the elasticity of labor supply with respect to the real wage is determined by his elasticity of intertemporal substitution in consumption. When the impatient's consumption is substitutable then the elasticity of labor supply is positive and thus endogenous cycles from self-fulfilling expectations become more likely to appear. Further, the relaxation of *Income Monotonicity Assumption* stabilizes. This stands to the contrary of the existing literature à la Becker (1980).

Finally, it is shown that the steady state changes its stability through the emergence of deterministic cycles of period two. However, unlike the existing works, this paper proves that endogenous cycles and local indeterminacy can be ruled out, in certain cases, when the elasticity of input substitution is close to zero.

This paper is organized as follows. The model together with the intertemporal equilibrium and the steady state analysis is presented in section 2 and the local dynamics in section 3. In section 4, we focus on the results and the interpretations and finally in section 5 we conclude. All technical details are gathered in the Appendix.

## 2.2 The model

We consider a discrete time growth model with heterogeneous consumers, endogenous labor supply, borrowing constraint and a representative firm.

### 2.2.1 Households

The framework we consider assumes that consumers are heterogeneous with respect to their initial endowments, labor supply, discount rates and felicity functions. In order to simplify the presentation but without loss of generality, consumers' heterogeneity is reduced to two types (or groups) of agents, labeled with $i = 1, 2$. Agents are identical in each group. Population size of group $i$ is constant and denoted by $N_i > 0$ and the total population is also constant over time with size $N > 0$. Further, it is assumed that agents of type 2 are more impatient than agents of type 1, *i.e.*, they discount their future utility more heavily, namely,

**Assumption 1** $\quad 0 < \beta_2 < \beta_1 < 1$

Moreover, it is assumed that the felicity function of the representative agent of group $i$ depends on his consumption level and his labor supply as well as on the consumption level of the representative agent of the other group $j$. Let the felicity function of agent $i$ be given by $u_i(c_{it}, c_{jt}) - v_i(l_{it})$ for $i, j = 1, 2$ with $i \neq j$, where $c_{it}$ and $c_{jt}$ are the consumption of the representative agent of group $i$ and $j$ respectively while $l_{it}$ is the labor supply of agent $i$. The felicity function satisfies the following assumption

**Assumption 2** *For $i, j = 1, 2$ with $i \neq j$, the function $u_i(c_{it}, c_{jt})$ is continuous and twice differentiable and satisfies the following conditions: (i) $u_{i,1}(c_{it}, c_{jt}) > 0 > u_{i,11}(c_{it}, c_{jt})$, (ii) $u_{i,12}(c_{it}, c_{jt}) \gtrless 0$, (iii) $\lim_{c_{it} \longrightarrow 0^+} u_{i,1}(c_{it}, c_{jt}) = +\infty$ and (iv) $\lim_{c_{it} \longrightarrow +\infty} u_{i,1}(c_{it}, c_{jt}) = 0$. Moreover, $v_i(l_{it})$ is a continuous function defined on $(0, \varsigma)$ and satisfies $v_i'(l_{it}) > 0$ and $v_i''(l_{it}) > 0$. In addition, the conditions $\lim_{l_{it} \longrightarrow 0} v_i'(l_{it}) \geq 0$ and $\lim_{l_{it} \longrightarrow \varsigma} v_i'(l_{it}) = +\infty$ are verified.*[2]

Condition $(i)$ states that the utility function of agent $i$ is an increasing function in his own consumption while condition $(ii)$ gives that consumption externality generates either positive or negative effect on the marginal utility from own consumption of agent $i$. In other terms, if agent $i$ wants to be similar to others (resp., different from others), then his preference displays KUJ feature, *i.e.*, $u_{i,12}(c_{it}, c_{jt}) > 0$ (resp., RAJ feature, *i.e.*, $u_{i,12}(c_{it}, c_{jt}) < 0$).

For further reference, we present some necessary elasticities concerning agent's preference for $i, j = 1, 2$ and $i \neq j$

$$\varepsilon_{i,11} \equiv \frac{u_{i,11} c_i}{u_{i,1}} < 0 \tag{2.1}$$

$$\varepsilon_{i,12} \equiv \frac{u_{i,12} c_j}{u_{i,1}} \lessgtr 0 \tag{2.2}$$

where $\varepsilon_{i,11}$ is the elasticity of marginal utility of private consumption for agent $i$ while $\varepsilon_{i,12}$ is the elasticity of private marginal utility with respect to external effects from the group $j$. Initially, agents are endowed with capital $k_{i0} \geq 0$ such that $K_0 = \sum_{i=1}^{2} N_i k_{i0} > 0$, *i.e.*, the condition $k_{i0} > 0$ holds at least for one agent. Given the real interest rate $r_t$ and the real wage $w_t$, agent $i$ chooses the amount of labor supply $l_{it}$, consumption $c_{it}$ and capital accumulation $k_{it+1}$ to maximize the following instantaneous separable utility function

$$\sum_{t=0}^{+\infty} \beta_i^t \left[ u_i(c_{it}, c_{jt}) - v_i(l_{it}) \right] \tag{2.3}$$

subject to

[2] Notice that $u_{i,1}(c_{it}, c_{jt})$ is the partial derivative with respect to the first variable in the utility function, or equivalently, $\partial u_i(c_{it}, c_{jt}) / \partial c_{it}$. Similarly, $u_{i,12}(c_{it}, c_{jt})$ is equivalent to $\partial u_{i,1}(c_{it}, c_{jt}) / \partial c_{jt}$.

$$c_{it} + k_{it+1} - (1-\delta)\, k_{it} \leq r_t k_{it} + w_t l_{it} \tag{2.4}$$

$$l_{it} \geq 0 \tag{2.5}$$

$$k_{it+1} \geq 0 \tag{2.6}$$

where inequality (2.4) is a sequence of budget constraint, (2.5) is the positivity of labor supply and (2.6) is the borrowing constraint which states that agents are not allowed to finance current consumption by borrowing against future income. The borrowing constraint (2.6) reveals the incompleteness of the market.

First-order conditions imply

$$u_{i,1}\left(c_{it}, c_{jt}\right) - \beta_i \left[r_{t+1} + 1 - \delta\right] u_{i,1}\left(c_{it+1}, c_{jt+1}\right) \geq 0 \tag{2.7}$$

$$v_i'\left(l_{it}\right) - w_t u_{i,1}\left(c_{it}, c_{jt}\right) \geq 0 \tag{2.8}$$

where inequality (2.7) is the Euler equation and (2.8) is the consumption-labor arbitrage condition which hold with equality if $k_{it+1} > 0$ and $l_{it} > 0$, respectively. Further, the monotonicity of the utility function gives rise to a binding budget constraint

$$c_{it} + k_{it+1} - (1-\delta)\, k_{it} = r_t k_{it} + w_t l_{it} \tag{2.9}$$

### 2.2.2 Production

In contrast to the consumers' side, the production sector is homogeneous. Assume that a representative firm produces the final good using a constant return-to-scale technology $y_t = F\left(K_t, L_t\right)$, where $K_t$ and $L_t$ are the aggregate capital and labor. Let $k_t \equiv K_t / L_t$ be the capital-labor ratio, using the homogeneity feature, the production function can be written as $F\left(K_t, L_t\right) \equiv f\left(k_t\right) L_t$. Representative firm takes prices (real interest rate $r_t$ and real wage $w_t$) and technology as given and maximizes the profit $\pi_t \equiv F\left(K_t, L_t\right) - r_t K_t - w_t L_t$.

**Assumption 3** *The production function $f\left(k\right)$ is continuous in capital-labor ratio $k > 0$, differentiable with $f'\left(k\right) > 0 > f''\left(k\right)$, $f\left(0\right) = 0$ and satisfies Inada conditions* $\lim_{k \longrightarrow 0^+} f'\left(k\right) =$

$+\infty$ *and* $\lim_{k\longrightarrow+\infty} f'(k) = 0$.

Given $r_t$ and $w_t$, then profit maximization implies

$$r_t = f'(k_t) \text{ and } w_t = f(k_t) - k_t f'(k_t) \tag{2.10}$$

For further reference, we introduce the elasticity of capital-labor substitution $\sigma \equiv [kf'(k)/f(k) - 1] f'(k)/kf''(k) > 0$. The capital share of total income is given by $s \equiv kf'(k)/f(k) \in (0,1)$. Finally, the elasticities of interest rate with respect to capital and labor are $r_k k/r = -r_l l/r = -(1-s)/\sigma$, and the elasticities of the real wage with respect to capital and labor are $w_k k/w = -w_l l/w = s/\sigma$.

### 2.2.3 Intertemporal equilibrium

We start by providing a standard definition of equilibrium for the economy described above:

**Definition 1** *An equilibrium of the economy* $E = \left(F, (k_{i0}, \beta_i, u_i, v_i, N_i)_{i=1}^2\right)$ *is an intertemporal sequence* $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2\right)_{t=0}^{+\infty}$ *which satisfies the following conditions:*

**(D1)** $(r_t, w_t)_{t=0}^{+\infty}$ *is a sequence of strictly positive prices;*

**(D2)** *given* $(r_t, w_t)_{t=0}^{+\infty}$, $(K_t, L_t)$ *solves the firm's program for* $t = 0, 1, ..., +\infty$*;*

**(D3)** *given* $(r_t, w_t)_{t=0}^{+\infty}$, $(c_{it}, k_{i,t+1}, l_{it})_{t=0}^{+\infty}$ *solves the* $i$*th consumer's program for* $i = 1, 2$*;*

**(D4)** *the capital market clears* $K_t = N_1 k_{1t} + N_2 k_{2t}$*, for* $t = 0, 1, ..., +\infty$*;*

**(D5)** *the labor market clears* $L_t = N_1 l_{1t} + N_2 l_{2t}$*, for* $t = 0, 1, ..., +\infty$*;*

**(D6)** *the product market clears* $\sum_{i=1}^2 N_i [c_{it} + k_{it+1} - (1-\delta) k_{it}] = F(K_t, L_t)$.

In the following Lemma, we present a set of equilibrium conditions for our model.

**Lemma 1** *Let the economy* $E = \left(F, (k_{i0}, \beta_i, u_i, v_i, N_i)_{i=1}^2\right)$ *satisfying Assumptions* 1−3. *Consider the following conditions for* $t = 0, 1, ..., +\infty$ *and* $i, j = 1, 2$ with $i \neq j$*:*

**(L1)** $c_{it} > 0$, $k_{it} > 0$, $0 < l_{it} < \varsigma$, $K_t > 0$, $L_t > 0$*;*

**(L2)** $r_t = f'(k_t)$ *and* $w_t = f(k_t) - k_t f'(k_t)$ *with* $k_t \equiv K_t/L_t$*;*

**(L3)** $v_i'(l_{it}) \geq u_{i,1}(c_{it}, c_{jt})\, w_t$ *with equality when* $l_{it} > 0$*;*

**(L4)** $u_{i,1}(c_{it}, c_{jt}) \geq \beta_i (r_{t+1} + 1 - \delta)\, u_{i,1}(c_{it+1}, c_{jt+1})$ *with equality if* $k_{it+1} > 0$*;*

**(L5)** $c_{it} + k_{it+1} - (1-\delta) k_{it} = r_t k_{it} + w_t l_{it}$*;*

**(L6)** $K_t = N_1 k_{1t} + N_2 k_{2t}$*;*

**(L7)** $L_t = N_1 l_{1t} + N_2 l_{2t}$*.*

*Then, if the sequence* $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2\right)_{t=0}^{+\infty}$ *is a competitive equilibrium, the conditions* $(L1) - (L7)$ *hold. Further, if the sequence* $\left(r_t, w_t, K_t, L_t, (c_{it}, k_{it}, l_{it})_{i=1}^2\right)_{t=0}^{+\infty}$ *satisfies the conditions* $(L1) - (L7)$ *and the transversality condition*

$$\lim_{t \longrightarrow +\infty} \beta_i^t u_{i,1}(c_{it}, c_{jt})\, k_{it+1} = 0 \tag{2.11}$$

*for* $i, j = 1, 2$ with $i \neq j$*, it is an equilibrium for the economy* $E$*.*

**Proof.** See the Appendix. ■

In a Ramsey model with incomplete markets and without consumption externalities, the heterogeneity of discounting promotes the concentration of capital stock in the hands of the most patient agent, as demonstrated by Becker (1980) and Becker and Foias (1987; 1994). Such a result maintains either by the inclusion of endogenous labor supply (Bosi and Seegmuller 2010a) or by the introduction of intergroup and intragroup external effects in consumption (Barbar and Barinci 2010).

As the following Proposition shows, whenever the benchmark model of Becker and Foias (1994) is augmented to include both consumption externalities and endogenous labor supply, the benchmark result still holds. Namely, at a neighborhood of the steady state, the patient agent 1 owns the entire capital in the economy while agent 2 does not hold capital, provided that he is sufficiently impatient.

**Proposition 1** *There exists a steady state defined as follows:*

**(S1)** *$r$ and $w$ are constant;*

**(S2)** $r + 1 - \delta = 1/\beta_1 < 1/\beta_2$*;*

**(S3)** $k_1 > 0$ *and* $k_2 = 0$*;*

**(S4)** $u_{1,1}(c_1, c_2)\, w \leq v_1'(l_1)$ *and* $u_{2,1}(c_2, c_1)\, w = v_2'(l_2)$*;*

**(S5)** $c_1 = wl_1 + (r - \delta)\, k_1$ *and* $c_2 = wl_2$*;*

**(S6)** $K = N_1 k_1 > 0$*;*

**(S7)** $L = N_1 l_1 + N_2 l_2$.

**Proof.** See the Appendix. ■

According to Proposition 1, the real gross interest rate should be equal to $1/\beta_1$ around the steady state because firms prefer renting capital with the lowest costs, *i.e.*, from patient agent and therefore, the impatient agent ends up holding no capital. Further, it is important to mention that the presence of endogenous labor supply with different agents' preferences implies that agents could have different levels of labor supply. In particular, Proposition 1 shows that impatient agent is obliged to supply labor to finance his consumption while the patient agent has two choices either to supply labor or to enjoy leisure. In other words, Proposition 1 allows for the existence of *two* economic configurations: in the first one, both agents supply labor and, in the second one, only the impatient agent works while the patient agent enjoys leisure.

In the sequel, we consider a realistic configuration in which the poor agent (impatient) is willing to imitate the consumption of the rich (patient), *i.e.*, the impatient agent's preferences display a Keeping-Up with the Joneses feature. However, the rich does not care about the poor's consumption. Namely, consider the following Assumption:

**Assumption 4** *Let* $u_{1,12} = 0$ *and* $u_{2,12} > 0$.

In the following, we characterize the condition under which the patient agent decides to supply labor at a neighborhood of the steady state. But before proceeding, we need to ensure

the existence of the function $l_2 \equiv l_2(l_1)$ for all $l_1 \in (0, \varsigma)$, as demonstrated by the following Lemma.

**Lemma 2** *Given Assumptions* $1-4$ *and let* $\Psi(l_1, l_2) = 1$ *where*

$$\Psi(l_1, l_2) \equiv v_2'(l_2) / u_{2,1}(wl_2, c_1(l_1, l_2))\, w$$

*then there exists an implicit function* $l_2 \equiv l_2(l_1)$ *for all* $l_1 \in (0, \varsigma)$ *if*

$$\frac{1}{w} v_2'' - u_{2,11} w - \frac{1-\beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} \neq 0 \tag{2.12}$$

*Further, for* $l_1 = 0$, *there exists a unique* $l_2^*$ *that solves* $\Psi(0, l_2^*) = 1$ *if and only if*

$$\frac{1}{w} v_2'' - u_{2,11} w - \frac{1-\beta_1}{\beta_1} \frac{N_2 k}{N_1} u_{2,12} > 0 \tag{2.13}$$

**Proof.** See the Appendix. ■

Now, taking account the function $l_2(l_1)$ as characterized by Lemma 2, the next proposition presents explicitly the condition under which the patient agent supplies labor in a neighborhood of the steady state.

**Proposition 2** *Given Assumptions* $1-4$ *and let the condition (2.13) of Lemma* 2 hold, *the patient agent supplies labor at the steady state if and only if*

$$\lim_{l_1 \longrightarrow 0^+} v_1'(l_1) < \lim_{l_1 \longrightarrow 0^+} u_1' \left( \left[ w + \frac{1-\beta_1}{\beta_1} k \right] l_1 + \frac{1-\beta_1}{\beta_1} \frac{N_2 k}{N_1} l_2(l_1) \right) w \tag{2.14}$$

*where* $k = r^{-1}(1/\beta_1 - (1-\delta))$.

**Proof.** See the Appendix.

■

Inequality (2.14) states that patient agent supplies labor if and only if the marginal disutility of labor, whenever labor supply is close to zero, is lower than the marginal utility of

consumption. However, if (2.14) is not verified, then agent prefers enjoying leisure and pays their consumptions through capital income. For the sake of simplicity, we only focus on the steady state where patient agent does not supply labor, *i.e.*, inequality (2.14) is *not* verified.

Further, it is worth to notice that consumption externalities affect the steady-state through the consumption-labor arbitrage condition of the impatient agent, namely,

$$u_{2,1}\left(wl_2, c_1\right) w - v_2'\left(l_2\right) = 0 \tag{2.15}$$

This implies that the labor supply of impatient agent is determined by the consumption level of patient agent $c_1$ and the real wage $w$. Namely, agent 2's labor supply can be defined implicitly as a function of $c_1$ and $w$, *i.e.*, we have

$$l_2 \equiv l_2\left(c_1, w\right) \tag{2.16}$$

The existence of the function $l_2 \equiv l_2\left(c_1, w\right)$ is ensured by Lemma 2 and gives rise to the following elasticities:

$$\xi_{lc} \equiv \frac{\partial l_2/\partial c_1}{c_1/l_2} = \frac{\varepsilon_{2,12}}{\varphi_2 - \varepsilon_{2,11}} \tag{2.17}$$

$$\xi_{lw} \equiv \frac{\partial l_2/\partial w}{w/l_2} = \frac{1 + \varepsilon_{2,11}}{\varphi_2 - \varepsilon_{2,11}} \tag{2.18}$$

where $\varphi_2 \equiv v_2''\left(l_2\right) l_2 / v_2'\left(l_2\right)$ is the elasticity of marginal dis-utility of impatient labor supply while $\varepsilon_{2,11}$ and $\varepsilon_{2,12}$ are summarized respectively in (2.3) and (2.4). Further, using labor market equilibrium condition $(S7)$, one can easily verify that

$$\frac{L_{c_1} c_1}{L} = \frac{\xi_{lc}}{1 + \frac{s}{\sigma}\xi_{lw}} \tag{2.19}$$

$$\frac{L_{k_1} k_1}{L} = \frac{\xi_{lw}}{\frac{\sigma}{s} + \xi_{lw}} \tag{2.20}$$

Therefore, given the steady state characterized by Proposition 1, a Ramsey equilibrium is a sequence of $\{c_{1t}, k_{1t}\}_{t=0}^{+\infty}$ that is described by patient agent's Euler equation and budget

constraint which solves the following two-dimensional dynamic system

$$\frac{u_{1,1}\left(c_{1t}, w\left(N_1 k_{1t}/L\left(c_{1t}, k_{1t}\right)\right) l_2\left(c_{1t}, k_{1t}\right)\right)}{u_{1,1}\left(c_{1t+1}, w\left(N_1 k_{1t+1}/L\left(c_{1t+1}, k_{1t+1}\right)\right) l_2\left(c_{1t+1}, k_{1t+1}\right)\right)} = \beta_1\left[1-\delta+r\left(N_1 k_{1t+1}/L\left(c_{1t+1}, k_{1t+1}\right)\right)\right] \tag{2.21}$$

$$c_{1t} + k_{1t+1} - (1-\delta) k_{1t} = g\left(c_{1t}, k_{1t}\right) \tag{2.22}$$

subject to the initial endowment $k_{10} > 0$ and the transversality condition (2.11), where $g\left(c_{1t}, k_{1t}\right) \equiv r\left(N_1 k_{1t}/L\left(c_{1t}, k_{1t}\right)\right) k_{1t}$ is the capital-income of patient agent and the effect of capital on capital-income is given by

$$g_k = \frac{1}{\beta}\left[1-\beta\left(1-\delta\right)\right]\left(1-\frac{1-s}{\sigma+s\xi_{lw}}\right) \tag{2.23}$$

## 2.3 Local dynamics

In this section, we analyze the stability properties of the economy described above and study the occurrence of indeterminacy and local bifurcations. We start by linearizing the system (2.21)-(2.22) around the steady state where only the patient agent holds capital and only the impatient agent supplies labor. We get the linear system $\left(dk_{1t+1}/k_1, dc_{1t+1}/c_1\right)^T = J\left(dk_{1t}/k_1, dc_{1t}/c_1\right)^T$, where $J$ is the associated Jacobian matrix which is given by

$$J = \left[\begin{array}{cc} -\rho & \rho\xi_{lc} + \left(\sigma+s\xi_{lw}\right)\varepsilon_{1,11}/s \\ \beta_1 & 0 \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & \left(\sigma+s\xi_{lw}\right)\varepsilon_{1,11}/s \\ 1-\frac{s\rho}{\sigma+s\xi_{lw}} & \frac{s\rho\xi_{lc}}{\sigma+s\xi_{lw}} - \left(1-\beta_1\right) \end{array}\right] \tag{2.24}$$

where

$$\rho \equiv \frac{1}{s}\left(1-s\right)\left[1-\beta_1\left(1-\delta\right)\right] > 0 \tag{2.25}$$

The trace $T$ and the determinant $D$ of the Jacobian matrix (2.24) are given by:

$$T = 1 + D + \frac{s\rho\,(1-\beta_1)}{\beta_1}\frac{\xi_{lc}-1}{(\sigma + s\xi_{lw})\,\varepsilon_{1,11} + s\rho\xi_{lc}} \tag{2.26}$$

$$\begin{aligned} T &= -1 - D \\ &\quad + \frac{1}{\beta_1}\frac{(1+\beta_1)\,[2\,(\sigma + s\xi_{lw})\,\varepsilon_{1,11} + s\rho\xi_{lc}] - s\rho\,(2\varepsilon_{1,11} + 1 - \beta_1)}{(\sigma + s\xi_{lw})\,\varepsilon_{1,11} + s\rho\xi_{lc}} \end{aligned} \tag{2.27}$$

$$D = \frac{1}{\beta_1}\frac{(\sigma + s\xi_{lw} - s\rho)\,\varepsilon_{1,11}}{(\sigma + s\xi_{lw})\,\varepsilon_{1,11} + s\rho\xi_{lc}} \tag{2.28}$$

where $\xi_{lc}$ and $\xi_{lw}$ are respectively given by (2.17) and (2.18). The model of Bosi and Seegmuller (2010a) without consumption externalities is recovered by setting $\xi_{lc} = 0$ while the model of Barbar and Barinci (2010) is obtained by considering $\xi_{lc} = 0$ and $\xi_{lw} = 0$.

Using the fact that the trace $T$ and the determinant $D$ are the sum and the product of the eigenvalues of the Jacobian matrix $J$, our analysis consists of two parts: the occurrence of local indeterminacy and the saddle-path stability.

Since $k_1$ is the only predetermined variable, the stationary equilibrium is locally indeterminate if and only if both eigenvalues of the Jacobian matrix $J$ lie inside the unit circle. Namely, the conditions $D < 1$, $T < 1 + D$ and $T > -1 - D$ should be verified simultaneously. Before providing our main results, let us define the following critical values:[3]

Let $\sigma = \sigma_{AB}$ be the critical value at which $T = -1 - D$ holds and given by

$$\sigma_{AB} \equiv -\frac{2\varepsilon_{1,11}\,[(1+\beta_1)\,\xi_{lw} - \rho] - \rho\,(1-\beta_1) + \rho\,(1+\beta_1)\,\xi_{cl}}{2\frac{1}{s}\,(1+\beta_1)\,\varepsilon_{1,11}} \tag{2.29}$$

Let $\sigma = \sigma_D$ be the solution of $D = 1$ which is given by

$$\sigma_D \equiv \frac{[\rho - (1-\beta_1)\,\xi_{lw}]\,\varepsilon_{1,11} + \rho\beta_1\xi_{cl}}{\frac{1}{s}\,(1-\beta_1)\,\varepsilon_{1,11}} \tag{2.30}$$

The conditions in which local indeterminacy appears around the steady state are summarized in the next proposition.[4]

[3] More critical values are provided in the Appendix.
[4] For further detailed conditions, see the Appendix.

**Proposition 3** *Let Assumptions* $1-4$ *hold and given the critical values (2.29)-(2.30) and the other critical values provided in the Appendix, then:*

**Case 1** *Whenever* $\xi_{lw} > 0$*, then the steady state is locally indeterminate under the following conditions:*

**(1.1)** *For* $\sigma < \sigma_D$ *with* $1 < \xi_{cl} < \xi_{cl}^{AB}$ *and* $\varepsilon_{1,11} < -1$ *and* $\rho < \xi_{lw} < \ddot{\xi}_{lw}$*.*

**(1.2)** *For* $\sigma_{AB} < \sigma < \sigma_D$ *with* $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ *and* $\varepsilon_{1,11} < -1$ *and* $\rho < \xi_{lw} < \ddot{\xi}_{lw}$*.*

**(1.3)** *For* $\sigma_D < \sigma < \sigma_{AB}$ *with* $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ *and either* $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ *and* $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ *or* $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ *and* $\xi_{lw} < \ddot{\xi}_{lw}$*.*

**(1.4)** *For* $\sigma < \sigma_{AB}$ *with* $\xi_{cl}^{AB} < \xi_{cl} < 1$ *and* $\varepsilon_{1,11} > -1$ *and* $\rho < \xi_{lw} < \xi_{lw}^{Z}$*.*

**Case 2** *Whenever* $\xi_{lw} < 0$*, then the steady state is locally indeterminate under the following conditions:*

**(2.1)** *For* $\varepsilon_{111} < -1$ *and all* $\xi_{lw} < 0$ *and* $1 < \xi_{cl} < \xi_{cl}^{ABD}$ *and* $\sigma_{AB} < \sigma < \sigma_D$*.*

**(2.2)** *For* $\xi_{cl}^{ABD} < \xi_{cl} < 1$ *and* $-1 < \varepsilon_{1,11} < -\beta_1$ *and all* $\xi_{lw} < 0$ *and* $\sigma_D < \sigma < \sigma_{AB}$*;*

**(2.3)** *For* $\xi_{cl}^{ABD} < \xi_{cl} < 1$ *and* $\varepsilon_{1,11} > -\beta_1$ *and* $\xi_{lw} < \tilde{\xi}_{lw}$ *and* $\sigma_D < \sigma < \sigma_{AB}$*;*

**(2.4)** *For* $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ *and* $\sigma_D < \sigma < \sigma_{AB}$ *and either* $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ *and* $\xi_{lw} > \tilde{\xi}_{lw}$*; or* $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ *and* $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$*.*

**Proof.** See the Appendix. ■

In Proposition 3, two cases are distinguished according to the elasticity of labor supply with respect to the real wage $\xi_{lw}$. Note that $\xi_{lw} > 0$ if and only if the impatient agent's intertemporal substitution in consumption is sufficiently high, *i.e.*, $\varepsilon_{2,11} > -1$. In either of these cases, local indeterminacy occurs if consumption externalities are small, *i.e.*, $\xi_{lc} < 1$ (resp. high, *i.e.*, $\xi_{cl} > 1$) and patient agent's substitutability is high, *i.e.*, $\varepsilon_{1,11} > -1$ (resp. small, *i.e.*, $\varepsilon_{1,11} < -1$). Further, one can notice that the elasticity of input substitution can take small or moderate values.

In view of Proposition 3, a Ramsey model with heterogeneous agents and endogenous labor supply under the presence of consumption externalities displays local indeterminacy. Such a

result is of interest for two reasons: First, it provides new dynamic configurations comparing to the existing literature à la Becker (1980) in which the model only exhibits deterministic cycles of period two. Second, it stresses the role of heterogeneity, *i.e.*, even if the dynamic system (2.21)-(2.22) is described by patient agent's Euler equation and budget constraint, Proposition 3 shows that impatient agent's preferences also play a crucial role in the appearance of endogenous fluctuations. On the one side, the elasticity of labor supply with respect to the real wage is determined by the impatient agent's elasticity of intertemporal substitution in consumption. On the other side, the elasticity of consumption externalities also affect the stability properties of the model.

Further, Proposition 3 shows that the steady state changes its stability through the occurrence of flip bifurcation at $\sigma = \sigma_{AB}$ and so there is a room for deterministic cycles of period two. On the one side, such a result goes in line with previous literature, namely, Becker and Foias (1987, 1994), Bosi and Seegmuller (2010a) and Barbar and Barinci (2010). On the other side, while these papers show that a sufficiently high capital-labor substitution can rule out the occurrence of fluctuations, the next Proposition proves that endogenous cycles and local indeterminacy can be also ruled out under sufficiently *weak* elasticity of input substitution, whenever the intertemporal substitution in the patient agent's consumption is low while both of the external effects and the impatient agent's consumption substitutability are high.

**Proposition 4** *Let Assumptions* $1-5$ *hold. Whenever* $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ *and* $\varepsilon_{1,11} < -1$ *and* $\rho < \xi_{lw} < \ddot{\xi}_{lw}$*, then there exists a critical value of the elasticity of capital-labor substitution* $\sigma = \sigma_{AB}$ *such that* $\sigma < \sigma_{AB}$ *implies saddle-path stability.*

**Proof.** See the Appendix. ■

Finally, in order to complete the local analysis, consider the derivative of (2.29) and (2.30) with respect to $\xi_{cl}$, we get $\partial\sigma_D/\partial\xi_{cl} < 0$ and $\partial\sigma_{AB}/\partial\xi_{cl} > 0$. This gives rise to two cases. From one side, whenever the patient agent's intertemporal substitutability is low then the indeterminacy region is given by $\sigma_{AB} < \sigma < \sigma_D$ or $0 < \sigma < \sigma_D$ and thus, consumption externalities make the emergence of endogenous fluctuations due to self-fulfilling expectations less likely. From the other side, whenever the patient agent's intertemporal substitutability is high then the local indeterminacy region $\sigma_D < \sigma < \sigma_{AB}$ or $0 < \sigma < \sigma_{AB}$ widens with

consumption external effects and so these externalities promote the appearance of endogenous fluctuations.

## 2.4 Discussion

The model we consider in this paper is an extension of Barbar and Barinci (2010) augmented to include the endogenous labor supply. It is shown that such a framework allows to stress on the role of consumption externalities and the heterogeneity across agents. To understand the added value of this paper, consider three variants of Ramsey model when the labor supply is inelastic: i) with identical agents; ii) with heterogeneous agents; and with heterogeneous agents and consumption externalities. Finally, we discuss the results obtained in the previous section and compare them to the existing literature.

**i) Ramsey model with identical agents**

In a standard Ramsey model with identical agents and inelastic labor supply, the steady state is always a saddle point.

**ii) Ramsey model with heterogeneous agents**

Once the heterogeneity across agents is introduced in the standard Ramsey model where both agents supply work inelastically, Becker and Foias (1994) show that the heterogeneity gives rise to a new dynamic configuration. In particular, the model exhibits deterministic cycles of period two. The authors prove that a necessary condition for the appearance of these cycles is the relaxation of *Income Monotonicity Assumption*, *i.e.*, the patient agent's income should be decreasing in capital stock. Further, cycles are ruled out whenever input substitutability is sufficiently high.

**iii) Ramsey model with heterogeneous agents and consumption externalities**

Subsequently, Barbar and Barinci (2010) extend the framework of Becker and Foias (1994) by introducing consumption externalities. In particular, they distinguish between the intergroup

and intragroup consumption externalities. The purpose of such a specification is to capture the role of heterogeneity as well as the external effects. Barbar and Barinci (2010) show that the steady state is still determinate but changes its stability through the emergence of flip bifurcation. However, contrary to Becker and Foias (1994), they demonstrate that such a result is obtained without the need to relax the *Income Monotonicity Assumption*. Instead, consumption externalities are the mechanism that generate cycles. More precisely, if the patient agent is willing to keep-up with the impatient, then cycles of period two occur even under *Income Monotonicity Assumption*, *i.e.*, input substitutability is required to be high.

The added value of Barbar and Barinci (2010) is simple, clear and interesting but has two weak points. From the one side, even when the intergroup externalities that generate cycles rather than the intragroup externalities, the role of heterogeneity across agents is not explicitly captured. In other terms, the impatient agent's preferences do not play any role in the emergence of cycles. From the other side, such a result is not realistic. That is, it is based on the configuration in which rich agent imitates the poor while the contrary happens in the real world.

**iv) Ramsey model with heterogeneous agents, consumption externalities and endogenous labor supply**

This paper avoids the weakness in Barbar and Barinci (2010) by considering two main assumptions. First, we assume an elastic labor supply. From the one side, the endogeneity of labor supply gives rise to the positive relation between saving and labor supply that makes the initial expectations about higher future interest rate self-fulfilling, and thus local indeterminacy appears. From the other side, it allows to capture the heterogeneity across agent, that is, the role of the impatient agent's preferences. In particular, when impatient agent's elasticity of intertemporal substitution in consumption is higher than one, the elasticity of labor supply to the real wage becomes positive, making the emergence of endogenous fluctuations due to self-fulfilling expectations more likely, and vice versa. Along this direction, Bosi and Seegmuller (2010a) also stress on the role of heterogeneous preferences on the stability properties of the steady state, in the absence of consumption externalities. They show that when the impatient agents' elasticity of intertemporal substitution is greater than one, the occurrence of cycles is

less likely, and vice versa.

Second, we consider a realistic configuration in which the poor (impatient) agent wants to keep-up with the rich (patient) while the rich does not pay any attention to the poor's consumption. Such an assumption ensures a result that is based upon plausible conditions.

Under these specifications, we show that there is a room for local indeterminacy. More precisely, we show that the introduction of elastic labor supply is a key requirement for the appearance of local indeterminacy in a Ramsey model with heterogeneous agents and consumption externalities. The emergence of indeterminacy requires that when agents coordinate to enhance their savings, the interest rate increases, so that the initial expectations of higher interest rate can be self-fulfilling. The positive relationship between saving and interest rate may arise when an increase in capital accumulation causes an augmentation in next-period labor supply which in turn raises future interest rate. However, if labor supply is inelastic, an increase in savings has a negative effect on the marginal product of capital, and thus there is no room for self-fulfilling expectations. Then the presence of consumption externalities only gives rise to cycles of period two (Barbar and Barinci 2010). Conversely, if labor supply is elastic, the positive relationship between saving and interest rate may arise when an increase in the amount of saving causes an increase in the next-period labor supply which in turn affects the future interest rate, resulting in self-fulfilling initial expectations. Finally, this result goes in line with Alonso-Carrera *et al.* (2008) who show that consumption externalities, in a representative-agent Ramsey model, are a source of local indeterminacy if labor supply is endogenous.

It worth noticing that such a result is obtained without considering social concavity restriction *i.e.*, $\varepsilon_{i,11} + \varepsilon_{i,12} < 0$. In fact, this condition states that either the externality augments the direct effect of own consumption, or, if it is offsetting, it is dominated by the own consumption effect. Thus it imposes an upper bound on the KUJ effect. Such a restriction is essentially a stability condition and relaxing it means that we consider a KUJ effect which is too intense that it induces instability. The existing literature, such as Alonso-Carrera *et al.* (2008) and Barbar and Barinci (2010), considers this restriction. Conversely, this paper does not impose it in order for the calculations to be less complicated. In addition, it does not change the main result.

In the next section, we provide the detailed intuition for our main result in Proposition 3.

## 2.5 Intuition for local indeterminacy

The appearance of local indeterminacy requires that an increase in savings raises the marginal product of capital, and thus the interest rate, so that the initial expectations of higher interest rate can be self-fulfilling. In models with elastic labor supply, the positive relationship between saving and interest rate may arise when an increase in the amount of saving causes an augmentation in next-period labor supply which in turn raises future interest rate.

The initial expectation of a higher future interest rate induces patient agent to increase his savings and reduces his current consumption. In this framework, this has two effects on next-period labor supply:

1. The first effect is through the future wage $w_{t+1}$: If patient agent expects a higher future interest rate $r_{t+1}$, he accumulates more capital $k_{1t+1}$ which implies an increase in wage $w_{t+1}$. The effect of the rise in wage on the next-period labor supply depends on the sign of elasticity of labor supply $\xi_{lw}$.

2. The increase in savings influences next-period patient agent's capital-income $g$. This effect also depends on the sign of the elasticity of labor supply $\xi_{lw}$ and on the elasticity of input substitution $\sigma$ as follows

$$g_k = \frac{1}{\beta}\left[1-\beta\left(1-\delta\right)\right]\left(1-\frac{1-s}{\sigma+s\xi_{lw}}\right) \tag{2.31}$$

   The consumption of patient agent at $t+1$ reacts to the change in his income according to the elasticity of intertemporal substitution $-1/\varepsilon_{1,11}$. Further, the presence of consumption externalities implies a positive influence of $c_{1t+1}$ on next-period labor supply, namely,

$$\xi_{lc} \equiv \frac{\partial l_2/\partial c_1}{c_1/l_2} = \frac{\varepsilon_{2,12}}{\varphi_2-\varepsilon_{2,11}} > 0 \tag{2.32}$$

According to these effects, three cases should be distinguished in Proposition 3:

**Whenever** $\xi_{lw} > 0$ **and** $g_k < 0$**,** then from the one side, following the augmentation in the wage, the next-period labor supply $L_{t+1}$ increases and does the future interest rate $r_{t+1}$.

Therefore, the positivity of $\xi_{lw}$ acts as a destabilizing factor. Such an effect is sufficiently large because the elasticity of input substitution is low.

From the other side, the increase in next-period capital $k_{1t+1}$, due to the expectation of a higher interest rate $r_{t+1}$, is followed by a decrease in future capital income $g$, then according to the budget constraint, patient agent's future consumption falls but slightly because the intertemporal substitutability is very low. As the impatient agent wants to keep up ($\varepsilon_{2,12} > 0$) and so to reduce his consumption, he will thus work less, *i.e.*, $l_{2t+1}$ decreases. This results in a decrease in future interest rate $r_{t+1}$. Therefore, the negative response of capital-income to $k_1$ plays a stabilizing role and makes the appearance of fluctuations due to self-fulfilling expectations less likely.

In this case, in order for the initial expectations to be self-fulfilling, the first effect should dominate the second one. Such a requirement is ensured because of the small elasticity of input substitutability that generates large wage effect and the low sensitivity of the patient agent's consumption to income that results in a small second effect.

**Whenever $\xi_{lw} > 0$ and $g_k > 0$,** then from the one side, following the augmentation in the wage, the next-period labor supply $L_{t+1}$ increases and does the future interest rate $r_{t+1}$. Therefore, the positivity of $\xi_{lw}$ acts as a destabilizing factor. As the elasticity of input substitution is high, such an effect is not large.

From the other side, as the elasticity of input substitutability is high, then $g_k > 0$ and thus future capital income $g$ goes up in response to the increase in next-period capital $k_{1t+1}$. Then patient agent's future consumption augments. Since $\varepsilon_{2,12} > 0$, then impatient agent is willing to imitate patient agent and raise his consumption. This induces him to increase labor supply $l_{2t+1}$ which yields to a rise in $r_{t+1}$. Therefore, such an effect works in the same direction as the first effect (due to the positive $\xi_{lw}$). In other terms, the positive response of capital-income to $k_1$ destabilizes and promotes the appearance of local indeterminacy.

In this case, both effects work in the same direction, *i.e.*, both act as destabilizing effects and thus give rise to local indeterminacy.

**Whenever $\xi_{lw} < 0$ and $g_k > 0$,** then from the one side, a rise in $k_{1t+1}$ raises $w_{t+1}$ which in

turn affects $l_{2t+1}$ negatively (since $\xi_{lw} < 0$). Then $L_{t+1}$ decreases and thus the future interest rate $r_{t+1}$ falls as well. In other terms, negative elasticity of labor supply $\xi_{lw}$ acts as a stabilizing effect and makes the emergence of local indeterminacy less likely.

From the other side, the increase in $k_{1t+1}$ will raise $g_{t+1}$, because $g_k > 0$. Then patient agent augments his future consumption $c_{1t+1}$ as the intertemporal substitutability of the patient agent's consumption is high. Since $\varepsilon_{2,12} > 0$, then impatient agent is willing to raise his future consumption in response to the increase in $c_{1t+1}$ and so he works more, *i.e.*, $l_{2t+1}$ goes up. This implies that $L_{t+1}$ increases, resulting in a higher future interest rate $r_{t+1}$. So the positive effect of capital on patient agent's income destabilizes and promotes the occurrence of endogenous cycles.

In this case, endogenous fluctuations driven from self-fulfilling expectation appear if and only if the second effect dominates the first one. Such a requirement is ensured because of the high input substitutability $\sigma$ which provides a small effect of $K_{t+1}$ on $w_{t+1}$, and both high intertemporal substitutability of patient agent's consumption and high consumption externalities result in large second effect.

Notice that the counter-intuitive configuration for the appearance of local indeterminacy, that is, $\xi_{lw} < 0$, $g_k < 0$ can also emerge in proposition 3.

## 2.6 Conclusion

This chapter considers a Ramsey heterogeneous agents model of Becker (1980), Becker and Foias (1987, 1994) with borrowing constraint and consumption externalities augmented to include endogenous labor supply. For simplicity, it is supposed the existence of two types of identical agents. Agents are different in their initial endowment of capital, labor supply, time preference and felicity function. Agents utility function depends on his own consumption as well as on consumptions of the other group. It is shown the existence of two steady states: the first one consists of both agents supply labor and the second one with only impatient labor supply while the patient agents enjoy leisure. As in the literature, the entire capital market is held by the most patient agents while the impatient agents have to supply labor always to finance

their consumptions. Along this chapter, it is focused on the steady state where patient agents enjoy leisure. In the stationary equilibrium, it is assumed that patient agents are not affected by impatient agents consumptions but impatient agents have KUJ feeling toward the patient agents. The main contribution of this chapter is the existence of local indeterminacy due to self-fulfilling expectations. These expectations emerge since the presence of endogenous labor supply together with consumption spillover give rise to an intratemporal effect which can generate endogenous fluctuations.

## 2.7 Appendix

### 2.7.1 Proof of Lemma 1

*Necessity.* Conditions $(L1)$, $(L6)$ and $(L7)$ directly come from the equilibrium definition. In addition, $(L2)$ corresponds to the first-order necessary condition of profit maximization of the firm. $(L3)$ and $(L4)$ are the first-order necessary conditions of the utility maximization of household $i$ and $(L5)$ is the corresponding budget constraint.

*Sufficiency.* Condition $(D1)$ is guaranteed by condition $(L2)$ and Assumption 4. In order to show that $(D2)$ is verified, notice that for every alternative pair $\left(\tilde{K}_t, \tilde{L}_t\right)$, we have:

$$
\begin{aligned}
& F\left(K_t, L_t\right) - w_t L_t - r_t K_t - \left[F\left(\tilde{K}_t, \tilde{L}_t\right) - w_t \tilde{L}_t - r_t \tilde{K}_t\right] \\
& f\left(K_t/L_t\right) L_t - w_t L_t - r_t K_t - \left[f\left(\tilde{K}_t/\tilde{L}_t\right) \tilde{L}_t - w_t \tilde{L}_t - r_t \tilde{K}_t\right] \\
= \; & f\left(K_t/L_t\right) L_t - f\left(\tilde{K}_t/\tilde{L}_t\right) \tilde{L}_t - r_t\left(K_t - \tilde{K}_t\right) - w_t\left(L_t - \tilde{L}_t\right) \\
\geq \; & f'\left(K_t/L_t\right)\left(K_t - \tilde{K}_t\right) + \left[f\left(K_t/L_t\right) - f'\left(K_t/L_t\right)\left(K_t/L_t\right)\right]\left(L_t - \tilde{L}_t\right) \\
& -r_t\left(K_t - \tilde{K}_t\right) - w_t\left(L_t - \tilde{L}_t\right) \\
= \; & 0
\end{aligned}
$$

Let us now consider a sequence $\left(\tilde{k}_{it}, \tilde{l}_{it}, \tilde{c}_{it}\right)$ satisfying the constraints of agent's $i$'s maximization problem and the initial condition. Then, we have

$$
\begin{aligned}
& \sum_{t=0}^{\infty} \beta_i^t \left(\left[u_i\left(c_{it}, c_{jt}\right) - v_i\left(l_{it}\right)\right] - \left[u_i\left(\tilde{c}_{it}, c_{jt}\right) - v_i\left(\tilde{l}_{it}\right)\right]\right) \\
= \; & \sum_{t=0}^{\infty} \beta_i^t \left[u_i\left(c_{it}, c_{jt}\right) - u_i\left(\tilde{c}_{it}, c_{jt}\right) + v_i\left(\tilde{l}_{it}\right) - v_i\left(l_{it}\right)\right] \\
\geq \; & \sum_{t=0}^{\infty} \beta_i^t \left[u_i'\left(c_{it}, c_{jt}\right)\left(c_{it} - \tilde{c}_{it}\right) - v_i'\left(l_{it}\right)\left(l_{it} - \tilde{l}_{it}\right)\right]
\end{aligned}
$$

$$= \sum_{t=0}^{\infty} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) \left[ \left(r_t + \delta - 1\right) \left( k_{it} - \tilde{k}_{it} \right) + w_t \left( l_{it} - \tilde{l}_{it} \right) - \left( k_{it+1} - \tilde{k}_{it+1} \right) \right] - v_i'\left(l_{it}\right) \left( l_{it} - \tilde{l}_{it} \right) \right]$$

$$= \sum_{t=0}^{\infty} \beta_i^t u_i'\left(c_{it}, c_{jt}\right) \left[ \left(r_t + \delta - 1\right) \left( k_{it} - \tilde{k}_{it} \right) - \left( k_{it+1} - \tilde{k}_{it+1} \right) \right] + \sum_{t=0}^{\infty} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) w_t - v_i'\left(l_{it}\right) \right] \left( l_{it} - \tilde{l}_{it} \right)$$

$$= \lim_{T \longrightarrow \infty} \left[ \begin{array}{c} \sum_{t=0}^{T} \beta_i^t u_i'\left(c_{it}, c_{jt}\right) \left(r_t + \delta - 1\right) \left( k_{it} - \tilde{k}_{it} \right) - \sum_{t=0}^{T} \beta_i^t u_i'\left(c_{it}, c_{jt}\right) \left( k_{it+1} - \tilde{k}_{it+1} \right) \\ + \sum_{t=0}^{T} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) w_t - v_i'\left(l_{it}\right) \right] \left( l_{it} - \tilde{l}_{it} \right) \end{array} \right]$$

$$= \lim_{T \longrightarrow \infty} \left[ \begin{array}{c} u_i'\left(c_{i0}, c_{j0}\right) \left(r_0 + \delta - 1\right) \left( k_{i0} - \tilde{k}_{i0} \right) + \sum_{t=1}^{T} \beta_i^t u_i'\left(c_{it}, c_{jt}\right) \left(r_t + \delta - 1\right) \left( k_{it} - \tilde{k}_{it} \right) \\ - \sum_{t=0}^{T-1} \beta_i^t u_i'\left(c_{it}, c_{jt}\right) \left( k_{it+1} - \tilde{k}_{it+1} \right) - \beta_i^T u_i'\left(c_{iT}, c_{jT}\right) \left( k_{iT+1} - \tilde{k}_{iT+1} \right) \\ + \sum_{t=0}^{T} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) w_t - v_i'\left(l_{it}\right) \right] \left( l_{it} - \tilde{l}_{it} \right) \end{array} \right]$$

$$= \lim_{T \longrightarrow \infty} \left[ \begin{array}{c} \sum_{t=0}^{T-1} \beta_i^t \left[ \beta_i u_i'\left(c_{it+1}, c_{jt+1}\right) \left(r_{t+1} + \delta - 1\right) - u_i'\left(c_{it}, c_{jt}\right) \right] \left( k_{it+1} - \tilde{k}_{it+1} \right) \\ - \beta_i^T u_i'\left(c_{iT}, c_{jT}\right) \left( k_{iT+1} - \tilde{k}_{iT+1} \right) \\ + \sum_{t=0}^{T} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) w_t - v_i'\left(l_{it}\right) \right] \left( l_{it} - \tilde{l}_{it} \right) \end{array} \right]$$

$$\begin{aligned} &\geq \lim_{T \longrightarrow \infty} \left[ \begin{array}{c} \sum_{t=0}^{T-1} \beta_i^t \left[ \beta_i u_i'\left(c_{it+1}, c_{jt+1}\right) \left(r_{t+1} + \delta - 1\right) - u_i'\left(c_{it}, c_{jt}\right) \right] k_{it+1} \\ - \beta_i^T u_i'\left(c_{iT}, c_{jT}\right) k_{iT+1} \\ + \sum_{t=0}^{T} \beta_i^t \left[ u_i'\left(c_{it}, c_{jt}\right) w_t - v_i'\left(l_{it}\right) \right] l_{it} \end{array} \right] \\ &= 0 \end{aligned}$$

This proves that condition $(D3)$ holds. Finally, conditions $(D4)$, $(D5)$ and $(D6)$ are easily obtained using $(L2)$, $(L5)$, $(L6)$ and $(L7)$.

### 2.7.2 Proof of Proposition 1

This proof consists of four steps:

**1.** For $i = 1$, conditions $(S2) - (S5)$ satisfy the optimality conditions in Lemma 1. Further, the transversality condition $\lim_{t \longrightarrow +\infty} \beta_1^t u_{11}(c_1, c_2) k_1 = 0$ holds since $\beta_1^t \in (0,1)$ and $c_1$ and $c_2$ are constant.

**2.** For $i = 2$, we show that it is optimal that the impatient agent holds no capital $k_2 = 0$ and consume the entire of his wage-income, *i.e.*, $c_2 = wl_2$. For this purpose, let us consider a feasible sequence $\left(\tilde{k}_2, \tilde{l}_2, \tilde{c}_2\right)$ starting from $\tilde{k}_{20} = 0$ and compare this path to the steady state solution $(l_2, c_2)$ with $l_2 \in (0, \varsigma)$ and $c_2 = wl_2$. Given the external effect $c_1 = wl_1 + (r - \delta) k_1$, then in the following proof, we show that the stationary solution is optimal. We have:

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta_2^t \left[ u_2(c_2, c_1) - v_2(l_2) - \left( u_2(\tilde{c}_2, c_1) - v_2\left(\tilde{l}_2\right)\right)\right] \\
= &\sum_{t=0}^{\infty} \beta_2^t \left[ u_2(c_2, c_1) - u_2(\tilde{c}_2, c_1) - \left( v_2(l_2) - v_2\left(\tilde{l}_2\right)\right)\right] \\
\geq &\sum_{t=0}^{\infty} \beta_2^t \left[ u_{2,1}(c_2, c_1)[c_2 - \tilde{c}_2] - v_2'(l_{2t})\left(l_2 - \tilde{l}_2\right)\right] \\
&= \sum_{t=0}^{\infty} \beta_2^t \left[ u_{2,1}(wl_2, c_1)[wl_2 - \tilde{c}_2] - v_2'(l_{2t})\left(l_2 - \tilde{l}_2\right)\right] \\
&= u_{2,1}(w_2 l_2, c_1) \sum_{t=0}^{\infty} \beta_2^t \left[ w\tilde{l}_2 - \tilde{c}_2\right] = u_{2,1}(w_2 l_2, c_1) \sum_{t=0}^{\infty} \beta_2^t \left[ \tilde{k}_{2t+1} - (1/\beta_1)\tilde{k}_{2t}\right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \longrightarrow \infty} \left[ \sum_{t=0}^{T} \beta_2^t \tilde{k}_{2t+1} - \sum_{t=0}^{T} \beta_2^t (1/\beta_1)\tilde{k}_{2t}\right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \longrightarrow \infty} \left[ \beta_2^T \tilde{k}_{2T+1} + \sum_{t=0}^{T-1} \beta_2^t \tilde{k}_{2t+1} - \sum_{t=1}^{T} \beta_2^t (1/\beta_1)\tilde{k}_{2t} - (1/\beta_1)\tilde{k}_{20}\right] \\
&= u_{2,1}(w_2 l_2, c_1) \lim_{T \longrightarrow \infty} \left[ \beta_2^T \tilde{k}_{2T+1} + \sum_{t=1}^{T} \frac{\beta_2^t \tilde{k}_{2t}}{\beta_2} - \sum_{t=1}^{T} \beta_2^t (1/\beta_1)\tilde{k}_{2t} - (1/\beta_1)\tilde{k}_{20}\right]
\end{aligned}$$

$$
\begin{aligned}
&= u_{2,1}\left(w_2 l_2, c_1\right) \lim_{T \longrightarrow \infty} \left[\beta_2^T \tilde{k}_{2T+1} - \left(1/\beta_1\right) \tilde{k}_{20} + \left(\frac{1}{\beta_2} - \frac{1}{\beta_1}\right) \sum_{t=1}^{T} \beta_2^t \tilde{k}_{2t}\right] \\
&\geq -u_{2,1}\left(w_2 l_2, c_1\right) \left(1/\beta_1\right) \tilde{k}_{20} \\
&= 0
\end{aligned}
$$

Because $1/\beta_2 > 1/\beta_1$ and $\tilde{k}_{2t} \geq 0$. Notice that $l_2 = 0$ is not optimal and since $c_2 = w_2 l_2$ and $\lim_{c_2 \longrightarrow 0} u'\left(c_2\right) = +\infty$, thus a small increase of labor supply $l_2$ will generate a very huge increase in the welfare.

**3.** Under Assumption 3, there is a unique finite and strictly positive value of $k$ such that $r = f'\left(k\right) = 1/\beta_1 - (1 - \delta)$.

**4.** In the last point, we show that $r + 1 - \delta = 1/\beta_1$ is the only and unique steady state solution with $K = N_1 k_1 > 0$ and $k_2 = 0$. From one hand, if $r + 1 - \delta > 1/\beta_1$, then it is optimal for agent 1 to accumulate more capital. However, this can not be a stationary solution because of the decreasing returns. From the other hand, if $r + 1 - \delta < 1/\beta_1 < 1/\beta_2$, then it is optimal for both agents to deaccumulate capital to zero in a finite time, *i.e.*, $k \longrightarrow 0$ and $\lim_{k \longrightarrow 0} f'\left(k\right) = +\infty$ which contradicts the stationarity.

### 2.7.3 Proof of Lemma 2

At the steady state, the prices $r$ and $w$ are constant and taken as given by households. Further, at the steady state, we have $r = 1/\beta_1 - (1 - \delta)$ which determines the capital-labor ratio $k = r^{-1}\left(1/\beta_1 - (1 - \delta)\right)$. Thus $k$ is obtained independently of the labor choice of agents.

Moreover, the budget constraints for patient and impatient agent are respectively given by $c_1 = w l_1 + (r - \delta) k_1$ and $c_2 = w l_2$, with $r = 1/\beta_1 - (1 - \delta)$ and using $K = kL$ with $K = N_1 k_1$ and $L = N_1 l_1 + N_2 l_2$, we get $k_1 = \left[l_1 + N_2 l_2 / N_1\right] k$. Therefore, patient agent's consumption can be written in terms of both agents' labor supply as follows

$$
c_1 = \left(w + \frac{1 - \beta_1}{\beta_1} k\right) l_1 + \frac{1 - \beta_1}{\beta_1} \frac{N_2 k}{N_1} l_2 \equiv c_1\left(l_1, l_2\right) \tag{2.33}
$$

Then using consumption-leisure arbitrage conditions

$$u_1'\left(c_1\left(l_1,l_2\right)\right)w-v_1'\left(l_1\right)=0 \tag{2.34}$$

$$u_{2,1}\left(wl_2,c_1\left(l_1,l_2\right)\right)w-v_2'\left(l_2\right)=0 \tag{2.35}$$

we obtain $l_1$ and $l_2$ as implicit functions.

$(i)$ In order to demonstrate the existence of the function $l_2\left(l_1\right)$, we use impatient agent's consumption-leisure arbitrage condition (2.35) which can be written as follows

$$\Psi\left(l_1,l_2\right)=1 \tag{2.36}$$

where $\Psi\left(l_1,l_2\right)\equiv v_2'\left(l_2\right)/u_{2,1}\left(wl_2,c_1\right)w$ is a continuous function. Moreover, consider

$$\frac{\partial\Psi\left(l_1,l_2\right)}{\partial l_2}=\frac{1}{u_{2,1}}\left[\frac{1}{w}v_2''-u_{2,11}w-\frac{1-\beta_1}{\beta_1}\frac{N_2k}{N_1}u_{2,12}\right]$$

with $\partial\Psi\left(l_1,l_2\right)/\partial l_2\neq 0$ if and only if

$$\frac{1}{w}v_2''-u_{2,11}w-\frac{1-\beta_1}{\beta_1}\frac{N_2k}{N_1}u_{2,12}\neq 0$$

Given that condition (2.12) is verified for all $l_1,l_2\in(0,\varsigma)$. In the sequel, for simplicity, we consider the following assumption

$$\frac{1}{w}v_2''-u_{2,11}w-\frac{1-\beta_1}{\beta_1}\frac{N_2k}{N_1}u_{2,12}>0 \tag{2.37}$$

Then, by implicit function theorem, $l_2$ can be expressed as a function of $l_1$, *i.e.*, there exists a function $l_2\equiv l_2\left(l_1\right)$ defined over the domain $l_1\in(0,\varsigma)$.

$(ii)$ Whenever $l_1=0$ then impatient agent's consumption-leisure arbitrage condition implicitly becomes $\Psi\left(l_2\right)=1$ with $\Psi\left(l_2\right)\equiv v_2'\left(l_2\right)/wu_{2,1}\left(wl_2,c_1\right)$. Using Assumption 2, we get that $\lim_{l_2\longrightarrow\varsigma}\Psi\left(l_2\right)=+\infty$ and $\lim_{l_2\longrightarrow 0^+}\Psi\left(l_2\right)=0$. Further,

$$\frac{\partial\Psi\left(l_2\right)}{\partial l_2}=\frac{1}{u_{2,1}}\left[\frac{1}{w}v_2''-wu_{2,11}-\frac{1-\beta_1}{\beta_1}\frac{N_2k}{N_1}u_{2,12}\right]$$

is positive under the condition (2.13). As a result, given any $w$, there exists a unique value of $l_2$ which solves $\Psi(l_2) = 1$.

### 2.7.4 Proof of Proposition 2

Given the function $l_2 \equiv l_2(l_1)$, for all $l_1 \in (0, \varsigma)$, the consumption-leisure arbitrage equation (2.34) can be written as $\Phi(l_1) = 1$, with $\Phi(l_1) \equiv v_1'(l_1)/u_1'(c_1(l_1, l_2(l_1)))w$. Under Assumption 2, we have $\lim_{l_1 \longrightarrow \varsigma} \Phi(l_1) = +\infty$. Further, the condition (2.13) implies that $\partial l_2/\partial l_1 > 0$ and thus $\Phi'(l_1) > 0$. Then there exists a unique $l_1$ that solves $\Phi(l_1) = 1$ (and so equation (2.34)) if and only if $\lim_{l_1 \longrightarrow 0^+} \Phi(l_1) < 1$, namely, the following inequality should hold

$$\lim_{l_1 \longrightarrow 0^+} v_1'(l_1) < \lim_{l_1 \longrightarrow 0^+} u_1'\left(\left[w + \frac{1-\beta_1}{\beta_1}k\right] l_1 + \frac{1-\beta_1}{\beta_1}\frac{N_2 k}{N_1} l_2(l_1)\right) w$$

provided that $l_2(0)$ is well-defined in Lemma 2.

### 2.7.5 Proof of Proposition 3

Conditions of local indeterminacy are $D < 1$, $T < 1 + D$ and $T > -1 - D$. We first study the sign of the denominator:

$$DEN \equiv (\sigma + s\xi_{lw})\varepsilon_{1,11} + s\rho\xi_{lc} \tag{2.38}$$

then $DEN > 0$ iff $\sigma < \sigma^*$. We observe that if $\xi_{lw} < 0$ then $\sigma^* > 0$. However, if $\xi_{lw} > 0$ then $\sigma^* > 0$ if and only if $\xi_{cl} > \xi_{cl}^*$, where

$$\sigma^* \equiv -s\left(\xi_{lw} + \frac{1}{\varepsilon_{1,11}}\rho\xi_{cl}\right) \tag{2.39}$$

$$\xi_{cl}^* \equiv -\frac{1}{\rho}\varepsilon_{1,11}\xi_{lw} \tag{2.40}$$

This implies that $DEN > 0$ if either $\xi_{lw} < 0$ and $\sigma < \sigma^*$ or $\xi_{lw} > 0$ and $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \sigma^*$. However, $DEN < 0$ if either $\xi_{lw} < 0$ and $\sigma > \sigma^*$ or $\xi_{lw} > 0$ and either $\xi_{cl} < \xi_{cl}^*$ and all $\sigma > 0$ or $\xi_{cl} > \xi_{cl}^*$ and $\sigma > \sigma^*$.

Then the proof is divided into two parts: The first one assumes that $DEN > 0$ and the other assumes that $DEN < 0$.

**Case 1:** $DEN > 0$

First, the condition $D < 1$ holds iff $\sigma > \sigma_D$ with $\sigma_D > 0$ iff $\xi_{cl} < \xi_{cl}^D$ and $\xi_{cl}^D > 0$ iff $\xi_{lw} < \rho/(1-\beta_1)$, where

$$\sigma_D \equiv \frac{[\rho - (1-\beta_1)\xi_{lw}]\varepsilon_{1,11} + \rho\beta_1\xi_{cl}}{\frac{1}{s}(1-\beta_1)\varepsilon_{1,11}} \tag{2.41}$$

$$\xi_{cl}^D \equiv -\frac{1}{\rho\beta_1}[\rho - (1-\beta_1)\xi_{lw}]\varepsilon_{1,11} \tag{2.42}$$

This implies that $D < 1$ is verified if either $(i)$ $\xi_{lw} < \rho/(1-\beta_1)$ and either $\xi_{cl} < \xi_{cl}^D$ and $\sigma > \sigma_D$ or $\xi_{cl} > \xi_{cl}^D$ and all $\sigma > 0$; or $(ii)$ $\xi_{lw} > \rho/(1-\beta_1)$ and $\xi_{cl} > 0$ and all $\sigma > 0$. Therefore, the condition $D < 1$ holds under $DEN > 0$ if

1. whenever $\xi_{lw} < 0$ and either $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma^*$; or $\xi_{cl} > \xi_{cl}^D$ and $\sigma < \sigma^*$;
2. whenever $0 < \xi_{lw} < \rho$ and either $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma^*$; or $\xi_{cl} > \xi_{cl}^D$ and $\sigma < \sigma^*$.
3. whenever $\xi_{lw} > \rho$ and $\xi_{cl} > \xi_{cl}^*$ and all $\sigma < \sigma^*$.

Second, the condition $T < 1 + D$ holds iff $\xi_{cl} < 1$. Given that $DEN > 0$, the condition $T < 1 + D$ holds is verified if $\sigma < \sigma^*$ and either $\xi_{lw} < 0$ and $0 < \xi_{cl} < 1$ or $\xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^* < \xi_{cl} < 1$.

Third, the condition $T > -1 - D$ holds iff $\sigma < \sigma_{AB}$ with $\sigma_{AB} > 0$ iff $\xi_{cl} > \xi_{cl}^{AB}$; and $\xi_{cl}^{AB} > 0$ iff $\xi_{lw} > \xi_{lw}^{AB}$ and $\xi_{lw}^{AB} > 0$ iff $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$, where

$$\sigma_{AB} \equiv -\frac{2\varepsilon_{1,11}[(1+\beta_1)\xi_{lw} - \rho] - \rho(1-\beta_1) + \rho(1+\beta_1)\xi_{cl}}{2\frac{1}{s}(1+\beta_1)\varepsilon_{1,11}} \tag{2.43}$$

$$\xi_{cl}^{AB} \equiv -\frac{2\varepsilon_{1,11}[(1+\beta_1)\xi_{lw} - \rho] - \rho(1-\beta_1)}{\rho(1+\beta_1)} \tag{2.44}$$

$$\xi_{lw}^{AB} \equiv \frac{\rho}{(1+\beta_1)\,\varepsilon_{1,11}}\left[\varepsilon_{1,11}+\frac{1}{2}(1-\beta_1)\right] \tag{2.45}$$

Then taking the conditions under which $DEN > 0$, we consider

$$\sigma_{AB} - \sigma^* = \frac{1}{2}\frac{s\rho}{\varepsilon_{1,11}}\left(\xi_{cl} - \hat{\xi}_{cl}\right)$$

$$\hat{\xi}_{cl} \equiv -\frac{2}{1+\beta_1}\left[\varepsilon_{1,11}+\frac{1}{2}(1-\beta_1)\right] \tag{2.46}$$

and

$$\xi_{cl}^* - \hat{\xi}_{cl} = \frac{\varepsilon_{1,11}}{\rho}\left(\hat{\xi}_{lw} - \xi_{lw}\right)$$

$$\hat{\xi}_{lw} \equiv \frac{2\rho}{1+\beta_1}\left[\varepsilon_{1,11}+\frac{1}{2}(1-\beta_1)\right]\frac{1}{\varepsilon_{1,11}} \tag{2.47}$$

Whenever $\xi_{lw} < 0$, the condition $T > -1 - D$ is verified if

**(B.1)** $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and all $\xi_{lw} < 0$ and either $0 < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma < \sigma^*$ or $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma < \sigma_{AB}$;

**(B.2)** $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB} < 0$ and all $\xi_{cl} > 0$ and $\sigma < \sigma_{AB}$;

**(B.3)** $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < 0$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$.

Whenever $\xi_{lw} > 0$, the condition $T > -1 - D$ is verified if

**(D1)** $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $0 < \xi_{lw} < \xi_{lw}^{AB}$ and all $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \min\{\sigma_{AB}, \sigma^*\}$;

**(D2)** $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl} > \xi_{cl}^*$ and $\sigma < \min\{\sigma_{AB}, \sigma^*\}$;

**(D3)** $\varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$;

**(D4)** $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and all $\xi_{lw} > 0$ and $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma < \sigma_{AB}$.

**Subcase (1.1):** $\xi_{lw} < 0$

In order to characterize the conditions of local indeterminacy, we start by determining the intersection of the conditions under which both $D < 1$ and $T < 1 + D$ are simultaneously met. Since

$$\xi_{cl}^{D} - 1 = -\varepsilon_{1,11} \frac{1 - \beta_1}{\rho \beta_1} \left( \tilde{\xi}_{lw} - \xi_{lw} \right)$$

$$\tilde{\xi}_{lw} \equiv \frac{\rho}{1 - \beta_1} \frac{\varepsilon_{1,11} + \beta_1}{\varepsilon_{1,11}} \tag{2.48}$$

then $D < 1$ and $T < 1 + D$ hold if either of the following conditions holds:

**(A.1)** $-1 < \varepsilon_{1,11} < -\beta_1$ and all $\xi_{lw} < 0$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma^*$;

**(A.2)** $\varepsilon_{1,11} > -\beta_1$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma^*$;

**(A.3)** $\varepsilon_{1,11} > -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < 0$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma_D < \sigma < \sigma^*$;

**(A.4)** $\varepsilon_{1,11} > -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < 0$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $\sigma < \sigma^*$.

Now we consider the intersection of the conditions $(A.1) - (A.4)$ with the conditions $(B.1) - (B.3)$.

**The intersection of (A1) and (B1):** Since $\varepsilon_{1,11} > -1$ then $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. Further, we know that $\sigma_{AB} < \sigma^*$ iff $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$, where

$$\xi_{cl}^{ABD} \equiv \left[ (1 - \beta_1)^2 - 4\beta_1 \varepsilon_{1,11} \right] / (1 + \beta_1)^2 \tag{2.49}$$

with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $-1 < \varepsilon_{1,11} < -\beta_1$ and all $\xi_{lw} < 0$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of (A2) and (B1):** Since $\varepsilon_{1,11} > -1$ then we have $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. In addition, we note that $\sigma_{AB} < \sigma^*$ for $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of (A2) and (B2):** Note that for all $\varepsilon_{1,11} > -\beta_1$ we have $[4\beta_1(\varepsilon_{1,11}+1)-(1+\beta_1)(1-$ 0 and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. Further, since $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ for all $\varepsilon_{1,11} > -\beta_1$, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < 1$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$. As a result, local indeterminacy occurs whenever $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$ and $\sigma$ $_D < \sigma < \sigma_{AB}$.

**The intersection of (A2) and (B3):** As shown in the preceding case, for all $\varepsilon_{1,11} > -\beta_1$, we have $[4\beta_1(\varepsilon_{1,11}+1)-(1+\beta_1)(1-\beta_1)] > 0$ and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. This implies that the intersection is empty.

**The intersection of (A3) and (B1):** Since $\varepsilon_{1,11} > -1$ then we have $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. In addition, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^D$. As a result, local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of (A3) and (B2):** As shown above, for all $\varepsilon_{1,11} > -\beta_1$, we have $[4\beta_1(\varepsilon_{1,11}+1)-(1+\beta_1)(1-\beta_1)] > 0$ and thus $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. In addition, we have $\sigma_{AB} < \sigma^*$ for all $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$, where

$$\ddot{\xi}_{lw} \equiv \frac{\rho(1+3\beta_1)}{(1+\beta_1)^2}\left(\varepsilon_{1,11} + \frac{\beta_1(1-\beta_1)}{3\beta_1+1}\right)\frac{1}{\varepsilon_{1,11}} \tag{2.50}$$

with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$. Since $-\frac{1}{2}(1-\beta_1) < -\beta_1(1-\beta_1)/(3\beta_1+1)$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$, then local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ (since $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$).

After making some simplification, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^D$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of (A3) and (B3):** For $\varepsilon_{1,11} > -\beta_1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^D$ iff $\xi_{lw} < \ddot{\xi}_{lw}$, with $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$ and $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ with $-\frac{1}{2}(1-\beta_1) < -\beta_1(1-\beta_1)/(3\beta_1+1)$. Then $\xi_{cl}^{AB} < \xi_{cl}^D$ is satisfied if either $-\frac{1}{2}(1-\beta_1) <$

$\varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw} > \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw}^{AB} < \xi_{lw} < \ddot{\xi}_{lw}$ since $\xi_{lw}^{AB} < \ddot{\xi}_{lw}$. Further, $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where

$$\dot{\xi}_{lw} \equiv \frac{\rho}{2\varepsilon_{1,11}} \frac{3+\beta_1}{1+\beta_1} \left( \varepsilon_{1,11} + \frac{1-\beta_1}{3+\beta_1} \right) \tag{2.51}$$

with $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\frac{1-\beta_1}{3+\beta_1}$. Since $-\frac{1}{2}(1-\beta_1) < -(1-\beta_1)/(3+\beta_1)$ and $\xi_{lw}^{AB} < \dot{\xi}_{lw} < \ddot{\xi}_{lw}$. First, for all $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$, we have $\sigma_{AB} < \sigma^*$. In addition, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD} < \xi_{cl}^{D}$ for $\xi_{lw} < \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$. Thus we have $\xi_{lw}^{AB} < \dot{\xi}_{lw} < \ddot{\xi}_{lw}$. This implies that in all the above cases, we have $\xi_{lw} < \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} \lessgtr 0$, then $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < \xi_{cl}^{D}$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$. Therefore, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ if either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ and $\xi_{lw} > \xi_{lw}^{AB}$; or $\varepsilon_{1,11} > -\beta_1(1-\beta_1)/(3\beta_1+1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \ddot{\xi}_{lw}$.

As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \min\left\{0, \ddot{\xi}_{lw}\right\}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of (A4) and (B1):** Note that $\hat{\xi}_{cl} < 1$ and $\hat{\xi}_{cl} < \xi_{cl}^{D}$ and $\sigma_{AB} < \sigma^*$ for $\xi_{cl} > \hat{\xi}_{cl}$, then local indeterminacy occurs whenever $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} > \tilde{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of (A4) and (B2):** As shown above, for all $\varepsilon_{1,11} > -\beta_1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$. As a result, local indeterminacy occurs whenever $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $0 < \sigma < \min\{\sigma^*, \sigma_{AB}\}$.

**The intersection of (A4) and (B3):** First, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB}$ for all $\varepsilon_{1,11} > -\beta_1$. Further, $\xi_{cl}^{AB} > \xi_{cl}^{D}$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > \xi_{lw}^{AB}$ and $\xi_{cl}^{AB} < 1$ since $\xi_{lw} < 0$. Then local indeterminacy occurs for $\sigma < \min\{\sigma^*, \sigma_{AB}\}$ if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and either $\xi_{lw}^{AB} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$; or $\xi_{lw} > \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

**Subcase (1.2):** $\xi_{lw} > 0$

The conditions $D < 1$ and $T < 1 + D$ are simultaneously met in either of the following cases:

**(C1)** whenever $\sigma_D < \sigma < \sigma^*$ if

**(C1.1)** either $-1 < \varepsilon_{1,11} < -\beta_1$ and either $\xi_{lw} < \tilde{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$ or $\tilde{\xi}_{lw} < \xi_{lw} < \rho$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$;

**(C1.2)** or $\varepsilon_{1,11} > -\beta_1$ and $0 < \xi_{lw} < \rho$ and $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$.

**(C2)** whenever $\sigma < \sigma^*$ if

**(C2.1)** either $-1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^D < \xi_{cl} < 1$;

**(C2.2)** or $\varepsilon_{1,11} > -\beta_1$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^D < \xi_{cl} < 1$.

**(C3)** whenever $\sigma < \sigma^*$ if $\xi_{cl}^* < \xi_{cl} < 1$ and

**(C3.1)** either $-1 < \varepsilon_{1,11} < -(1-\beta_1)$ and $\rho < \xi_{lw} < -\rho/\varepsilon_{1,11}$;

**(C3.2)** or $\varepsilon_{1,11} > -(1-\beta_1)$ and $\rho < \xi_{lw} < \rho/(1-\beta_1)$;

**(C3.3)** or $\varepsilon_{1,11} > -(1-\beta_1)$ and $\rho/(1-\beta_1) < \xi_{lw} < -\rho/\varepsilon_{1,11}$.

Local indeterminacy occurs if the conditions $D < 1$, $T < 1 + D$ and $T > -1 - D$ hold. Thus we should now consider the intersection of the conditions $(C1) - (C3)$ and the conditions $(D1) - (D4)$.

**The intersection of** $(C1.1)$ **and** $(D1)$: First, $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$, with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. As $\varepsilon_{1,11} > -1$, we have $\tilde{\xi}_{lw} < \xi_{lw}^{AB} < \rho < -\rho/\varepsilon_{1,11}$ and thus $\xi_{cl}^* < -\varepsilon_{1,11}$ for $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$. Then we have $\xi_{cl}^* < -\varepsilon_{1,11} < 1$.

Now, consider at first $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11} < 1$. Therefore, we have $\sigma_{AB} < \sigma^*$ for $-\varepsilon_{1,11} < \xi_{cl} < \xi_{cl}^D$ and $-\varepsilon_{1,11} < \xi_{cl} < 1$. Moreover, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. However, since $-\varepsilon_{1,11} < \xi_{cl}^{ABD} < 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^D$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ with $-\beta_1 < -\beta_1(1-\beta_1)/(3\beta_1+1)$. This implies that $\ddot{\xi}_{lw} > 0$ for all $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw}^{AB} < \ddot{\xi}_{lw}$. Therefore, $\xi_{cl}^{ABD} < \xi_{cl}^D$ for $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$.

As a result, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if

1. either $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{ABD} < \xi_{cl} < 1$;

2. or $-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1<\varepsilon_{1,11}<-\beta_1$ and $\xi_{lw}<\tilde{\xi}_{lw}$ and $\xi_{cl}^{ABD}<\xi_{cl}<1$;

3. or $-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1<\varepsilon_{1,11}<-\beta_1$ and $\tilde{\xi}_{lw}<\xi_{lw}<\xi_{lw}^{AB}$ and $\xi_{cl}^{ABD}<\xi_{cl}<\xi_{cl}^{D}$.

**The intersection of** $(C1.2)$ **and** $(D1)$**:** We have $\xi_{lw}^{AB}<\rho$ and for all $\xi_{lw}<\rho$, we have $\xi_{cl}^{*}<-\varepsilon_{1,11}<\xi_{cl}^{D}$. Further, $\sigma_{AB}>\sigma^{*}$ iff $\xi_{cl}<\hat{\xi}_{cl}$ with $\hat{\xi}_{cl}<-\varepsilon_{1,11}$ which is sufficient to conclude that we have $\sigma_{AB}<\sigma^{*}$. In addition, $\sigma_{AB}>\sigma_D$ iff $\xi_{cl}>\xi_{cl}^{ABD}$. Then $-\varepsilon_{1,11}<\xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD}>\xi_{cl}^{D}$ iff $\xi_{lw}>\ddot{\xi}_{lw}$ with $\xi_{lw}^{AB}<\ddot{\xi}_{lw}$. This implies that $-\varepsilon_{1,11}<\xi_{cl}^{ABD}<\xi_{cl}^{D}$ for $\xi_{lw}<\xi_{lw}^{AB}$.

As a result, local indeterminacy occurs whenever $-\beta_1<\varepsilon_{1,11}<-\frac{1}{2}\left(1-\beta_1\right)$ and $\xi_{lw}<\xi_{lw}^{AB}$ and $\xi_{cl}^{ABD}<\xi_{cl}<\xi_{cl}^{D}$ and $\sigma_D<\sigma<\sigma_{AB}$.

**The intersection of** $(C1.1)$ **and** $(D2)$**:** First, we have $\tilde{\xi}_{lw}<\hat{\xi}_{lw}<\rho$ and $\xi_{lw}^{AB}<\rho$. Further, $\tilde{\xi}_{lw}>\xi_{lw}^{AB}$ iff $\varepsilon_{1,11}<-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1$ with $-1<-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1<-\beta_1$. In addition, $\xi_{cl}^{*}<-\varepsilon_{1,11}$ for $\xi_{lw}<\rho$. Then, $\sigma_{AB}>\sigma^{*}$ iff $\xi_{cl}<\hat{\xi}_{cl}$ with $\hat{\xi}_{cl}<-\varepsilon_{1,11}<1$. Therefore, we have $\sigma_{AB}<\sigma^{*}$. Moreover, $\sigma_{AB}>\sigma_D$ iff $\xi_{cl}>\xi_{cl}^{ABD}$. However, we have $-\varepsilon_{1,11}<\xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD}>\xi_{cl}^{D}$ iff $\xi_{lw}>\ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw}>0$ iff $\varepsilon_{1,11}<-\beta_1\left(1-\beta_1\right)/\left(3\beta_1+1\right)$. We have $\ddot{\xi}_{lw}>0$ for $-1<\varepsilon_{1,11}<-\beta_1$ and $\hat{\xi}_{lw}<\ddot{\xi}_{lw}$. Therefore, $\xi_{cl}^{ABD}<\xi_{cl}^{D}$ and thus $-\varepsilon_{1,11}<\xi_{cl}^{ABD}<\xi_{cl}^{D}$.

As a result, local indeterminacy appears for $\sigma_D<\sigma<\sigma_{AB}$ if

1. either $-1<\varepsilon_{1,11}<-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1$ and $\tilde{\xi}_{lw}<\xi_{lw}<\hat{\xi}_{lw}$ and $\xi_{cl}^{ABD}<\xi_{cl}<\xi_{cl}^{D}$;

2. or $-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1<\varepsilon_{1,11}<-\beta_1$ and $\xi_{lw}^{AB}<\xi_{lw}<\hat{\xi}_{lw}$ and $\xi_{cl}^{ABD}<\xi_{cl}<\xi_{cl}^{D}$;

3. or $-1<\varepsilon_{1,11}<-\left[\beta_1\beta_1+(4\beta_1-1)\right]/4\beta_1$ and $\xi_{lw}^{AB}<\xi_{lw}<\tilde{\xi}_{lw}$ and $-\varepsilon_{1,11}<\xi_{cl}<1$.

**The intersection of** $(C1.2)$ **and** $(D2)$**:** First, we have $\xi_{lw}^{AB}<\hat{\xi}_{lw}<\rho$ and $\xi_{cl}^{*}<-\varepsilon_{1,11}<\xi_{cl}^{D}$ for $\xi_{lw}<\rho$. In addition, $\sigma_{AB}>\sigma^{*}$ iff $\xi_{cl}<\hat{\xi}_{cl}$ with $\hat{\xi}_{cl}<-\varepsilon_{1,11}$ which is sufficient to conclude that $\sigma_{AB}<\sigma^{*}$. In addition, $\sigma_{AB}>\sigma_D$ iff $\xi_{cl}>\xi_{cl}^{ABD}$. Then $-\varepsilon_{1,11}<\xi_{cl}^{ABD}$ and $\xi_{cl}^{ABD}<\xi_{cl}^{D}$ iff $\xi_{lw}<\ddot{\xi}_{lw}$. Since $\xi_{lw}^{AB}<\hat{\xi}_{lw}<\ddot{\xi}_{lw}$, then $-\varepsilon_{1,11}<\xi_{cl}^{ABD}<\xi_{cl}^{D}$ for $\xi_{lw}^{AB}<\xi_{lw}<\hat{\xi}_{lw}$. As a result, local indeterminacy occurs whenever $-\beta_1<\varepsilon_{1,11}<-\frac{1}{2}\left(1-\beta_1\right)$ and $\xi_{lw}^{AB}<\xi_{lw}<\hat{\xi}_{lw}$ and $\xi_{cl}^{ABD}<\xi_{cl}<\xi_{cl}^{D}$ and $\sigma_D<\sigma<\sigma_{AB}$.

**The intersection of** $(C1.1)$ **and** $(D3)$**:** In this case, we have $(3\beta_1 - 1) > 0$ and $\tilde{\xi}_{lw} < \hat{\xi}_{lw} < \rho$. Note that $\xi_{cl}^{AB} < \xi_{cl}^{D}$ iff $\xi_{lw} < \ddot{\xi}_{lw}$, with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1 (1-\beta_1) / (3\beta_1 + 1)$. Since $-1 < -\beta_1 < -\frac{1}{2}(1-\beta_1) < -\beta_1 (1-\beta_1) / (3\beta_1 + 1)$, then $\ddot{\xi}_{lw} > 0$, for all $-1 < \varepsilon_{1,11} < -\beta_1$, with $\hat{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$ and thus $\xi_{cl}^{AB} < \xi_{cl}^{D}$ is verified for all $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$. In addition, we have $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -(1-\beta_1) / (3+\beta_1)$. However, since $-\beta_1 (1-\beta_1) / (3\beta_1 + 1) < -(1-\beta_1) / (3+\beta_1)$, then $\dot{\xi}_{lw} > 0$, for all $-1 < \varepsilon_{1,11} < -\beta_1$, with $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \ddot{\xi}_{lw}$.

We also observe that $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. First, we have $\xi_{cl}^{ABD} < \xi_{cl}^{D}$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ for all $\xi_{lw} < \ddot{\xi}_{lw}$. Further, $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$. Moreover, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that in both cases, we have $\sigma_{AB} < \sigma^*$.

As a result, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ and either $-1 < \varepsilon_{1,11} < -\beta_1$ and $\hat{\xi}_{lw} < \xi_{lw} < \dot{\xi}_{lw}$; or $-1 < \varepsilon_{1,11} < -\beta_1$ and $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$.

**The intersection of** $(C1.2)$ **and** $(D3)$**:** Since $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ where $\dot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -(1-\beta_1) / (3+\beta_1)$, and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\beta_1 (1-\beta_1) / (3\beta_1 + 1)$. In this case, we have $\dot{\xi}_{lw} > 0$ and $\ddot{\xi}_{lw} > 0$ for $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$. Further, $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, then this gives rise to the following cases:

1. For $\hat{\xi}_{lw} < \xi_{lw} < \dot{\xi}_{lw}$ then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} < \xi_{cl}^{D}$;
2. For $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^{D}$;
3. For $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$.

Moreover, $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. As $\xi_{cl}^{ABD} < \xi_{cl}^{D}$ and $\xi_{cl}^{AB} < \xi_{cl}^{ABD}$ for all $\xi_{lw} < \ddot{\xi}_{lw}$ and $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$, local indeterminacy occurs for $\sigma_D < \sigma < \sigma_{AB}$ if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$.

**The intersection of** $(C1.2)$ **and** $(D4)$**:** Since $\dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, the following cases arise:

1. For $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -(1-\beta_1) / (3+\beta_1)$, then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ for $\xi_{lw} < \dot{\xi}_{lw}$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ for $\xi_{lw} > \dot{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^{D}$ for $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\xi_{lw} > \ddot{\xi}_{lw}$. Since $\dot{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$, then

(a) whenever $\xi_{lw} < \dot{\xi}_{lw}$, then $\xi_{cl}^{AB} < -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} < \xi_{cl}^{D}$;

(b) whenever $\dot{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^{D}$;

(c) whenever $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$.

2. For $-(1-\beta_1)/(3+\beta_1) < \varepsilon_{1,11} < -\beta_1(1-\beta_1)/(3\beta_1+1)$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^{D}$ for $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\xi_{lw} > \ddot{\xi}_{lw}$.

(a) whenever $\xi_{lw} < \ddot{\xi}_{lw}$ then $-\varepsilon_{1,11} < \xi_{cl}^{AB} < \xi_{cl}^{D}$;

(b) whenever $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$.

3. For $\varepsilon_{1,11} > -\beta_1(1-\beta_1)/(3\beta_1+1)$ then $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\xi_{lw} < \rho$.

In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ and $\sigma_{AB} > \sigma_D$ iff $\xi_{cl} > \xi_{cl}^{ABD}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{ABD} < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma_D < \sigma < \sigma_{AB}$.

**The intersection of** $(C2.1)$ **with** $(D1)$**:** First, we have $\xi_{lw}^{AB} < \rho$ and $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. Since $\xi_{lw}^{AB} < \rho$ then $\xi_{cl}^{*} < \xi_{cl}^{D} < 1$. In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ which is sufficient to conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C2.1)$ **with** $(D2)$**:** Since $\tilde{\xi}_{lw} < \hat{\xi}_{lw}$ and $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \rho$ and $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ with $-1 < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < -\beta_1$. Note also that for $\xi_{lw} < \rho$, we have $\xi_{cl}^{*} < \xi_{cl}^{D} < 1$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < -\varepsilon_{1,11}$ and thus $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $\xi_{cl}^{D} < \xi_{cl} < 1$ and either $-1 < \varepsilon_{1,11} < -[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1$ and $\tilde{\xi}_{lw} < \xi_{lw} < \hat{\xi}_{lw}$; or $-[\beta_1\beta_1 + (4\beta_1 - 1)]/4\beta_1 < \varepsilon_{1,11} < -\beta_1$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$.

**The intersection of** $(C2.1)$ **with** $(D3)$**:** Since $\tilde{\xi}_{lw} < \hat{\xi}_{lw} < \rho$. In addition, we have $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\xi_{lw} > \ddot{\xi}_{lw}$. Since $-\frac{1}{2}(1-\beta_1) < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$, then $\ddot{\xi}_{lw} > 0$ with $\hat{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$. This implies that $\xi_{cl}^{AB} < \xi_{cl}^{D}$ for $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\ddot{\xi}_{lw} < \xi_{lw} < \rho$. Further, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$ where

$$\xi_{lw}^{Z} \equiv -\frac{\rho}{\varepsilon_{1,11}} \frac{\beta_1 - \varepsilon_{1,11}}{1+\beta_1} \tag{2.52}$$

with $\rho < \xi_{lw}^{Z}$. This implies that $\xi_{cl}^{AB} < 1$ for $\hat{\xi}_{lw} < \xi_{lw} < \rho$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^{D}$ for all $\xi_{lw} < \xi_{lw}^{Z}$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $-1 < \varepsilon_{1,11} < -\beta_1$ and either $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$; or $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

**The intersection of** $(C2.2)$ **with** $(D1)$**:** We have $\xi_{lw}^{AB} < \rho$. Further, for $\xi_{lw} < \rho$, we have $\xi_{cl}^{*} < \xi_{cl}^{D} < 1$. In addition, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^{D}$ for all $\xi_{lw} < \xi_{lw}^{Z}$ with $\rho < \xi_{lw}^{Z}$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C2.2)$ **with** $(D2)$**:** Since $\xi_{lw}^{AB} < \hat{\xi}_{lw} < \rho$. In addition, for $\xi_{lw} < \rho$, we have $\xi_{cl}^{*} < \xi_{cl}^{D}$. Further, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^{D}$ for all $\xi_{lw} < \xi_{lw}^{Z}$ with $\rho < \xi_{lw}^{Z}$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C2.2)$ **with** $(D3)$**:** Since $\hat{\xi}_{lw} < \rho$. In addition, $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\xi_{lw} > \ddot{\xi}_{lw}$. Since $-\frac{1}{2}(1-\beta_1) < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$, then $\ddot{\xi}_{lw} > 0$ with $\hat{\xi}_{lw} < \ddot{\xi}_{lw} < \rho$ and thus $\xi_{cl}^{AB} < \xi_{cl}^{D}$ for $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\ddot{\xi}_{lw} < \xi_{lw} < \rho$. Further, $\xi_{cl}^{AB} > 1$ for $\xi_{lw} > \xi_{lw}^{Z}$ with $\rho < \xi_{lw}^{Z}$. Therefore, $\xi_{cl}^{AB} < 1$ whenever $\hat{\xi}_{lw} < \xi_{lw} < \rho$. To determine $\min\{\sigma_{AB}, \sigma^*\}$, as previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^{D}$ for all $\xi_{lw} < \xi_{lw}^{Z}$ with $\rho < \xi_{lw}^{Z}$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$ in both cases. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if $-\beta_1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and either $\hat{\xi}_{lw} < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl} < 1$; or $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl}^{AB} < 1$.

**The intersection of** $(C2.2)$ **with** $(D4)$**:** We have $\xi_{cl}^{AB} > \xi_{cl}^{D}$ iff $\xi_{lw} > \ddot{\xi}_{lw}$ with $\ddot{\xi}_{lw} < \rho$. Then $\xi_{cl}^{AB} < \xi_{cl}^{D}$ for $\xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^{AB} > \xi_{cl}^{D}$ for $\ddot{\xi}_{lw} < \xi_{lw} < \rho$. In addition, we have $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$ with $\rho < \xi_{lw}^{Z}$. Then $\xi_{cl}^{AB} < 1$ for all $\xi_{lw} < \rho$. This implies three cases:

1. For $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\xi_{lw} < \ddot{\xi}_{lw}$ then $\xi_{cl}^{AB} < \xi_{cl}^{D}$ and $\xi_{cl}^{AB} < 1$;
2. For $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ then $\xi_{cl}^{D} < \xi_{cl}^{AB} < 1$;

3. For $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ then $\ddot{\xi}_{lw} < 0$ and $\xi_{cl}^D < \xi_{cl}^{AB} < 1$.

As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^D$ for all $\xi_{lw} < \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z$. Therefore, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs for $\sigma < \sigma_{AB}$ if

1. either $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $0 < \xi_{lw} < \ddot{\xi}_{lw}$ and $\xi_{cl}^D < \xi_{cl} < 1$;
2. or $-\frac{1}{2}(1-\beta_1) < \varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $\ddot{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$;
3. or $\varepsilon_{1,11} > -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$.

**The intersection of** $(C3.1)$ **and** $(D3)$**:** Since $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > \xi_{cl}^*$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho$. Therefore, $\xi_{cl}^{AB} > \xi_{cl}^*$ for $\rho < \xi_{lw} < -\rho/\varepsilon_{1,11}$. Further, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^Z$ with $\rho < \xi_{lw}^Z < -\rho/\varepsilon_{1,11}$. Then for $\rho < \xi_{lw} < \xi_{lw}^Z$ we have $\xi_{cl}^* < \xi_{cl}^{AB} < 1$ and for $\xi_{lw}^Z < \xi_{lw} < -\rho/\varepsilon_{1,11}$ we have $\xi_{cl}^{AB} > 1$ and $\xi_{cl}^{AB} > \xi_{cl}^*$. As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho$, we conclude that $\sigma_{AB} < \sigma^*$ in both cases. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -(1-\beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C3.2)$ **and** $(D3)$: As $-(1-\beta_1) < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \rho$. Then for this range of $\xi_{lw}$, we have $\xi_{cl}^{AB} > \xi_{cl}^*$. Further, $\rho < \xi_{lw}^Z < \rho/(1-\beta_1)$ whenever $-(1-\beta_1) < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$. Finally, it remains to determine $\min\{\sigma_{AB}, \sigma^*\}$. As previously shown, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Then we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $-(1-\beta_1) < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C3.2)$ **and** $(D4)$**:** First, $\xi_{cl}^{AB} < 1$ iff $\xi_{lw} < \xi_{lw}^Z$, with $\rho < \xi_{lw}^Z < \rho/(1-\beta_1)$. In addition, we have $\xi_{cl}^{AB} > \xi_{cl}^*$ iff $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho$. Finally, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho$, we conclude that $\sigma_{AB} < \sigma^*$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\rho < \xi_{lw} < \xi_{lw}^Z$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**The intersection of** $(C3.3)$ **and** $(D3)$**:** Since $-(1-\beta_1) < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \rho/(1-\beta_1)$ and $\xi_{cl}^{AB} > \xi_{cl}^*$ whenever $\xi_{lw} > \hat{\xi}_{lw}$ and thus the intersection $(\xi_{cl}^* < \xi_{cl} < 1) \cap$

$\left(\xi_{cl} > \xi_{cl}^{AB}\right)$ is non-empty if and only if $\xi_{cl}^{AB} < 1$ which requires that $\xi_{lw} < \xi_{lw}^{Z}$. Since $\xi_{lw}^{Z} < \rho / (1-\beta_1)$, this implies that $\xi_{cl}^{AB} < 1$ can NOT be satisfied. As a result, there is no room for local indeterminacy.

**The intersection of** $(C3.3)$ **and** $(D4)$**:** First, we have $\xi_{cl}^{AB} > \xi_{cl}^{*}$ whenever $\xi_{lw} > \hat{\xi}_{lw}$ with $\hat{\xi}_{lw} < \rho/(1-\beta_1)$. Then $\xi_{cl}^{AB} < 1$ iff $\xi_{lw} < \xi_{lw}^{Z}$. Since $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ then $\rho/(1-\beta_1) < \xi_{lw}^{Z} < -\rho/\varepsilon_{1,11}$. Further, $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} < \xi_{cl}^{*}$ whenever $\xi_{lw} > \hat{\xi}_{lw}$. Since $\hat{\xi}_{lw} < \rho/(1-\beta_1)$, we conclude that $\sigma_{AB} < \sigma^{*}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\rho/(1-\beta_1) < \xi_{lw} < \xi_{lw}^{Z}$ and $\xi_{cl}^{AB} < \xi_{cl} < 1$ and $\sigma < \sigma_{AB}$.

**Case 2:** $DEN < 0$

Taking the conditions under which $DEN < 0$ is verified, the condition $D < 1$ is satisfied if

1. For $\xi_{lw} < 0$ and $\xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_D$.
2. For $0 < \xi_{lw} < \rho$ and $\xi_{cl} < \xi_{cl}^{*}$ and $\sigma < \sigma_D$.
3. For $\rho < \xi_{lw} < \rho/(1-\beta_1)$ and $\xi_{cl} < \xi_{cl}^{D}$ and $\sigma < \sigma_D$.
4. For $0 < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_D$.

The condition $T < 1 + D$ holds if

1. For $\xi_{lw} < 0$ and $\xi_{cl} > 1$ and $\sigma > \sigma^{*}$
2. For $0 < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^{*} < \xi_{cl} < 1$ and $\sigma > \sigma^{*}$
3. For $\xi_{lw} > -\rho/\varepsilon_{1,11}$ and either $1 < \xi_{cl} < \xi_{cl}^{*}$ and all $\sigma > 0$ or $\xi_{cl} > \xi_{cl}^{*}$ and $\sigma > \sigma^{*}$

Then both conditions $D < 1$ and $T < 1 + D$ are simultaneously satisfied if

**(S1)** Whenever $\varepsilon_{1,11} < -1$ and $\xi_{lw} < 0$ and $1 < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_D$.

**(S2)** Whenever $\varepsilon_{1,11} < -1$ and $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $1 < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma < \sigma_D$.

**(S3)** Whenever $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \tilde{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma < \sigma_D$.

**(S4)** Whenever $\varepsilon_{1,11} < -1$ and EITHER $0 < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^{*} < \xi_{cl} < 1$ and $\sigma^{*} < \sigma < \sigma_D$; OR $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_D$.

**(S5)** Whenever $\varepsilon_{1,11} > -1$ and $0 < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_D$.

Now consider the conditions under which $T > -1 - D$ holds: First, whenever $\xi_{lw} < 0$, then $T > -1 - D$ is verified if $\sigma > \max\{\sigma^{*}, \sigma_{AB}\}$. Second, whenever $\xi_{lw} > 0$, then $T > -1 - D$ holds if

**(R1)** For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl} < \xi_{cl}^{*}$ and $\sigma > \sigma_{AB}$.

**(R2)** For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} < \xi_{lw}^{AB}$ and either $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \hat{\xi}_{cl}$ and $\sigma > \sigma^{*}$.

**(R3)** For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$, and either $\xi_{cl} < \xi_{cl}^{AB}$ and all $\sigma > 0$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \xi_{cl}^{*}$ and $\sigma > \max\{\sigma_{AB}, \sigma^{*}\}$.

**(R4)** For $\varepsilon_{1,11} < -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$, and either $\xi_{cl} < \xi_{cl}^{*}$ and all $\sigma > 0$; or $\xi_{cl}^{*} < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma > \sigma^{*}$; or $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma > \sigma^{*}$.

**(R5)** For $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $0 < \xi_{lw} < \hat{\xi}_{lw}$ and either $\xi_{cl} < \xi_{cl}^{AB}$ and all $\sigma > 0$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma > \sigma_{AB}$; or $\xi_{cl} > \xi_{cl}^{*}$ and $\sigma > \max\{\sigma_{AB}, \sigma^{*}\}$.

**(R6)** For $\varepsilon_{1,11} > -\frac{1}{2}(1 - \beta_1)$ and $\xi_{lw} > \hat{\xi}_{lw}$ and either $0 < \xi_{cl} < \xi_{cl}^{*}$ and all $\sigma > 0$; or $\xi_{cl}^{*} < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma > \sigma^{*}$; or $\xi_{cl} > \xi_{cl}^{AB}$ and $\sigma > \sigma^{*}$.

**Case (2.1):** $\xi_{lw} < 0$

We have $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} > -\varepsilon_{1,11}$ since $\varepsilon_{1,11} < -1$. This implies that $\sigma_{AB} > \sigma^{*}$ for all $-1 < \xi_{cl} < -\varepsilon_{1,11}$. Further, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$. Since $1 < \xi_{cl}^{ABD} < -\varepsilon_{1,11}$, then local indeterminacy occurs if $\varepsilon_{1,11} < -1$ and $\xi_{lw} < 0$ and $1 < \xi_{cl} < \xi_{cl}^{ABD}$ and $\sigma_{AB} < \sigma < \sigma_D$.

**Case (2.2):** $\xi_{lw} > 0$

We consider in this case the intersection of $(S2)-(S5)$ and $(R1)-(R6)$.

**The intersection of (S2) and (R1):** Note that $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$. Further, we have $-\frac{1}{2}(3+\beta_1) < -1$ and $\xi_{lw}^{AB} < \rho$. Moreover, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$, with $\xi_{cl}^{ABD} > 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^{*}$ iff $\xi_{lw} < \xi_{lw}^{F}$. Since $\xi_{lw}^{F} > -\rho/\varepsilon_{1,11}$ and $\xi_{lw}^{F} > \xi_{lw}^{AB}$, where

$$\xi_{lw}^{F} \equiv \frac{\rho}{(1+\beta_1)^2}\left[4\beta_1 - \frac{1}{\varepsilon_{1,11}}(1-\beta_1)^2\right] \tag{2.53}$$

Then we have $\xi_{cl}^{ABD} > \xi_{cl}^{*}$ for all $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$. This implies that $\sigma_{AB} < \sigma_D$ for all $1 < \xi_{cl} < \xi_{cl}^{*}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ and $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$ and $1 < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma_{AB} < \sigma < \sigma_D$.

**The intersection of (S2) and (R3):** We have $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ with $\xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. This gives rise to the following subcases:

**Subcase 1:** For $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$, then $-\rho/\varepsilon_{1,11} < \xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^{*}$ for all $\xi_{lw} < \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$ where $\xi_{lw}^{AB} < \xi_{lw}^{Z} < \hat{\xi}_{lw}$ and $-\rho/\varepsilon_{1,11} < \xi_{lw}^{Z} < \rho$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ and $\xi_{lw}^{Z} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma < \sigma_D$.

**Subcase 2:** Note that $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$, with $\xi_{cl}^{ABD} > 1$ and $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ iff $\xi_{lw} < \ddot{\xi}_{lw}$. We are in the case in which $\varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{3\beta_1+1}$ and thus $\ddot{\xi}_{lw} > 0$ with $\ddot{\xi}_{lw} > \rho$ and $\ddot{\xi}_{lw} > \xi_{lw}^{AB}$. Therefore, $\xi_{cl}^{ABD} > \xi_{cl}^{AB}$ for all $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\xi_{cl}^{ABD} > \xi_{cl}^{*}$ iff $\xi_{lw} < \xi_{lw}^{F}$ with $\xi_{lw}^{AB} < \xi_{lw}^{F} < \rho$; and $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$ with $\xi_{lw}^{AB} < \xi_{lw}^{Z} < \rho$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\max\left\{1, \xi_{cl}^{AB}\right\} < \xi_{cl} < \min\left\{\xi_{cl}^{*}, \xi_{cl}^{ABD}\right\}$ and $\sigma_{AB} < \sigma < \sigma_D$.

**Subcase 3:** For $-\frac{1}{2}(3+\beta_1) < \varepsilon_{1,11} < -1$, we have $\xi_{lw}^{AB} < -\rho/\varepsilon_{1,11} < \rho < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} < \xi_{cl}^{*}$ for all $\xi_{lw} < \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$ with $-\rho/\varepsilon_{1,11} < \xi_{lw}^{Z} < \rho$. As a result, local indeterminacy occurs if $-\frac{1}{2}(3+\beta_1) < \varepsilon_{1,11} < -1$ and $\xi_{lw}^{Z} < \xi_{lw} < \rho$ and $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{*}$ and $\sigma < \sigma_D$.

**The intersection of (S3) and (R3):** First, we have $\xi_{lw}^{AB} < \rho < \hat{\xi}_{lw}$ and $[4\beta_1\varepsilon_{1,11} + \beta_1\beta_1 + 4\beta_1 - 1] < 0$ (since $\varepsilon_{1,11} < -1$) then $\tilde{\xi}_{lw} > \xi_{lw}^{AB}$ and $\tilde{\xi}_{lw} > \hat{\xi}_{lw}$. Moreover, $\xi_{cl}^{D} < \xi_{cl}^{*}$ since $\xi_{lw} > \rho$. In addition, we have $\xi_{cl}^{AB} > 1$ iff $\xi_{lw} > \xi_{lw}^{Z}$. However, since $\xi_{lw}^{Z} < \rho$, this implies that $\xi_{cl}^{AB} > 1$

for all $\rho < \xi_{lw} < \hat{\xi}_{lw}$. Further, $\xi_{cl}^{AB} > \xi_{cl}^{D}$ iff $\xi_{lw} > \ddot{\xi}_{lw}$, with $\ddot{\xi}_{lw} > 0$ iff $\varepsilon_{1,11} < -\frac{\beta_1(1-\beta_1)}{1+3\beta_1}$, with $-1 < -\frac{\beta_1(1-\beta_1)}{1+3\beta_1}$. Then $\ddot{\xi}_{lw} > 0$ for $\varepsilon_{1,11} < -1$, with $\rho < \ddot{\xi}_{lw} < \hat{\xi}_{lw}$. As a result, local indeterminacy occurs if

1. either $\varepsilon_{1,11} < -1$ and $\rho < \xi_{lw} < \ddot{\xi}_{lw}$ and either $1 < \xi_{cl} < \xi_{cl}^{AB}$ and $\sigma < \sigma_D$; or $\xi_{cl}^{AB} < \xi_{cl} < \xi_{cl}^{ABD}$ and $\sigma_{AB} < \sigma < \sigma_D$;
2. or $\varepsilon_{1,11} < -1$ and $\ddot{\xi}_{lw} < \xi_{lw} < \hat{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma < \sigma_D$.

**The intersection of (S3) and (R4):** Since $\rho < \hat{\xi}_{lw} < \tilde{\xi}_{lw}$ and $\xi_{cl}^{D} < \xi_{cl}^{*}$ whenever $\xi_{lw} > \rho$ . As a result, local indeterminacy occurs if $\varepsilon_{1,11} < -1$ and $\hat{\xi}_{lw} < \xi_{lw} < \tilde{\xi}_{lw}$ and $1 < \xi_{cl} < \xi_{cl}^{D}$ and $\sigma < \sigma_D$.

**The intersection of (S4) and (R2):** We have $\xi_{lw}^{AB} < \rho$ and $-\rho/\varepsilon_{1,11} < \xi_{lw}^{AB}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$. Since $\hat{\xi}_{cl} > -\varepsilon_{1,11} > 1$ and $\xi_{cl} < \hat{\xi}_{cl}$ which implies that $\sigma_{AB} > \sigma^*$. In addition, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} > 1$. Further, since $\xi_{lw}^{F} > \xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$, then $\xi_{cl}^{ABD} > \xi_{cl}^{*}$ for $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$. Thus $\xi_{cl}^{*} < \xi_{cl}^{ABD} < -\varepsilon_{1,11}$. As a result, local indeterminacy occurs for $\sigma_{AB} < \sigma < \sigma_D$ if

1. For $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ and either $\xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^{*} < \xi_{cl} < 1$; or $-\rho/\varepsilon_{1,11} < \xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$;
2. For $-\frac{1}{2}(3+\beta_1) < \varepsilon_{1,11} < -1$ and $\xi_{lw} < \xi_{lw}^{AB}$ and $\xi_{cl}^{*} < \xi_{cl} < 1$.

**The intersection of (S4) and (R3):** First, $\sigma_{AB} > \sigma^*$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\hat{\xi}_{cl} > \xi_{cl}^{*}$ for $\xi_{lw} < \hat{\xi}_{lw}$. Further, since $\xi_{lw}^{AB} < \rho$ and $\hat{\xi}_{lw} > \rho > -\rho/\varepsilon_{1,11}$. In addition, $\xi_{lw}^{AB} > -\rho/\varepsilon_{1,11}$ iff $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$. Since $\hat{\xi}_{cl} > -\varepsilon_{1,11} > 1$ and $\xi_{cl} < \hat{\xi}_{cl}$, then $\sigma_{AB} > \sigma^*$. In addition, $\sigma_{AB} < \sigma_D$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} < -\varepsilon_{1,11}$ and $\xi_{cl}^{ABD} > 1 > \xi_{cl}^{*}$. As a result, local indeterminacy occurs for $\sigma_{AB} < \sigma < \sigma_D$ if

1. For $\varepsilon_{1,11} < -\frac{1}{2}(3+\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < \xi_{cl}^{ABD}$;
2. For $-\frac{1}{2}(3+\beta_1) < \varepsilon_{1,11} < -1$ and either $\xi_{lw}^{AB} < \xi_{lw} < -\rho/\varepsilon_{1,11}$ and $\xi_{cl}^{*} < \xi_{cl} < 1$; or $-\rho/\varepsilon_{1,11} < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < \xi_{cl}^{ABD}$.

**The intersection of (S5) and (R2):** We have $\hat{\xi}_{lw} < \rho$ and $\xi_{lw}^{AB} < \rho$ and $\xi_{cl}^{*} < \hat{\xi}_{cl} < -\varepsilon_{1,11}$. Further, $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma_{D}$ iff $\xi_{cl} < \xi_{cl}^{ABD}$. However, since $\xi_{cl}^{ABD} > -\varepsilon_{1,11}$, then $\sigma_{AB} < \sigma_{D}$ is satisfied for all $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $0 < \xi_{lw} < \xi_{lw}^{AB}$ and either $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_{D}$; or $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_{D}$.

**The intersection of (S5) and (R3):** First, we have $-1 < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \rho$ and $\hat{\xi}_{lw} < \rho$. Further, note that $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ with $\xi_{cl}^{*} < \hat{\xi}_{cl} < -\varepsilon_{1,11}$ for $\xi_{lw} < \hat{\xi}_{lw}$. This gives rise to two cases:

1. $\max\{\sigma_{AB}, \sigma^{*}\} = \sigma_{AB}$ for all $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$. Further, $\sigma_{AB} < \sigma_{D}$ iff $\xi_{cl} < \xi_{cl}^{ABD}$ with $\xi_{cl}^{ABD} > -\varepsilon_{1,11}$. Therefore, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_{D}$.

2. $\max\{\sigma_{AB}, \sigma^{*}\} = \sigma^{*}$ for all $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$. Then local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\xi_{lw}^{AB} < \xi_{lw} < \hat{\xi}_{lw}$ and $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_{D}$.

**The intersection of (S5) and (R4):** First, we have $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > \xi_{cl}^{*}$ for all $\xi_{lw} > \hat{\xi}_{lw}$. In addition, we have $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ with $\dot{\xi}_{lw} > 0$ for all $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \dot{\xi}_{lw} < \rho$. Further, we have $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ . Since $\hat{\xi}_{cl} < \xi_{cl}^{*}$ for $\xi_{lw} > \hat{\xi}_{lw}$, then with $\sigma_{AB} < \sigma^{*}$ for all $\xi_{cl} > \xi_{cl}^{*}$ and thus $\max\{\sigma_{AB}, \sigma^{*}\} = \sigma^{*}$. As a result, local indeterminacy occurs if $-1 < \varepsilon_{1,11} < -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_{D}$.

**The intersection of (S5) and (R5):** Further, we have $\hat{\xi}_{lw} < \rho$ and $\sigma_{AB} > \sigma^{*}$ iff $\xi_{cl} < \hat{\xi}_{cl}$ . Since $\hat{\xi}_{cl} > \xi_{cl}^{*}$ for $\xi_{lw} < \hat{\xi}_{lw}$ and $\hat{\xi}_{cl} < -\varepsilon_{1,11}$. In addition, we have $\sigma_{AB} > \sigma_{D}$ iff $\xi_{cl} > \xi_{cl}^{ABD}$ with $-\varepsilon_{1,11} < \xi_{cl}^{ABD}$. Thus $\sigma_{AB} < \sigma_{D}$ for all $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$. As a result, local indeterminacy occurs if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $0 < \xi_{lw} < \hat{\xi}_{lw}$ and either $\xi_{cl}^{*} < \xi_{cl} < \hat{\xi}_{cl}$ and $\sigma_{AB} < \sigma < \sigma_{D}$; or $\hat{\xi}_{cl} < \xi_{cl} < -\varepsilon_{1,11}$ and $\sigma^{*} < \sigma < \sigma_{D}$.

**The intersection of (S5) and (R6):** We first have $\hat{\xi}_{lw} < \rho$ and $\xi_{cl}^{AB} > -\varepsilon_{1,11}$ iff $\xi_{lw} > \dot{\xi}_{lw}$ with $\dot{\xi}_{lw} > 0$ if $\varepsilon_{1,11} < -\frac{1-\beta_1}{3+\beta_1}$ and $\dot{\xi}_{lw} < 0$ if $\varepsilon_{1,11} > -\frac{1-\beta_1}{3+\beta_1}$. As a result, local indeterminacy occurs for all $\sigma^{*} < \sigma < \sigma_{D}$ if $\varepsilon_{1,11} > -\frac{1}{2}(1-\beta_1)$ and $\hat{\xi}_{lw} < \xi_{lw} < \rho$ and $\xi_{cl}^{*} < \xi_{cl} < -\varepsilon_{1,11}$.

### 2.7.6 Proof of Proposition 4

Consider the critical value $\sigma = \sigma_{AB}$ which solves $T(\sigma_{AB}) = -1 - D(\sigma_{AB})$. Obviously, $\sigma_{AB} > 0$ and well-defined under the conditions in Proposition 3, except (1.1). Consider the case (1.2) in Proposition 3, one can easily verify that $\xi_{lw}\varepsilon_{1,11} + \rho\xi_{lc} < 0$ and $\xi_{lc} - 1 > 0$ and $(1+\beta_1)[2\xi_{lw}\varepsilon_{1,11} + \rho\xi_{lc}] - 2\rho\left[\varepsilon_{1,11} + \frac{1}{2}(1-\beta_1)\right] > 0$. This implies that $T(0) < 1 + D(0)$ and $T(0) < -1 - D(0)$ and so the locus $(T, D)$ for $\sigma$ is very closed to zero lies in the saddle-path stability region.

# Chapter 3

# Social Segmentation, Collaterals and Macroeconomic Volatility

## 3.1 Introduction

At the origin of the subprime crisis was the credit granted to consumers with insufficient collateral at high interest rates. The rapid expansion of the subprime credit market promoted an increase in the rate of defaults. The price of houses, that is the value of collaterals, lowered in turn. As the subprime crisis deepened, the financial sector was dramatically affected and a large number of financial firms shut down. Contagion was reinforced by new banking practices such as the titrization of bad credits. In the last decade, externalizing the risk allowed the banking system to elude the accounting rules, while infecting also the stock market at the end. Credit crunch and agents' bankruptcy, jointly with falling prices in the asset market, and, possibly, bursting bubbles, spread out the financial sphere to affect the growth rate of the world economy through financial rationing and more general wealth effects. This cascade was amplified by pessimistic animal spirits and coordination failures.

In the view of the subprime crisis, we address the stability issue considering the role of collaterals in a monetary economy with credit market imperfections. In the spirit of Barbar and Bosi (2010), we assume that agents pay a part of their consumption purchases in cash, while the remainder is financed by credit whose extent depends on the amount of collaterals. One of the goal of the chapter is to understand the role of collaterals in the occurrence of economic

crises viewed as self-fulfilling prophecies.

More explicitly, we consider an economy where agents have different discount rates, cannot borrow against future labor income, but are allowed to pay a part of current consumption by credit if collaterals are positive.

We know that heterogeneous discounting promotes the accumulation of capital in the hands of the most patient agents (the capitalists). If individuals are allowed to borrow against future income, the impatient agents borrow from the patient ones and spend the rest of their life to work in order to refund the debt. In the spirit of Becker (1980), Becker and Foias (1987, 1994), but in contrast to Le Van and Vailakis (2003), we introduce a borrowing constraint: we rule out the possibility of negative capital (of debt). A positive capital works as collateral to reduce the amount of balances needed to finance the current consumption. Thus market are incomplete and collaterals weaken the cash-in-advance.

Under heterogeneous discounting, the population splits in two classes: patient capitalists and impatient workers.[1]We prove that two regimes can occur: the one where the capitalist holds capital and no balances, the other where money and capital enter her/his portfolio.

In the first case, the liquidity constraint does not affect patient agents' capital accumulation and the economy jumps on the balanced growth path.

In the second case, there is a room for equilibrium transitions: endogenous fluctuations (two-period cycles and indeterminacy) may arise around the balanced growth path.

An expansionary monetary policy can be growth enhancing and welfare-improving, while a contractionary one can work as stabilization device.

More precisely, in the first regime (capitalists hold no money), money growth improves the patient agents' condition, while worsens the situations of impatient agents, who bear the opportunity cost of holding balances (nominal interest rate). It affects the initial consumption, but has no effect on the growth rate.

In the second regime (capitalists hold money), money growth has a positive impact on the growth rate and welfare of capitalists. However, it worsens the life of workers as was the case in

[1]Woodford (1986) also studies an economy with patient capitalists and impatient workers financially constrained. Even if these agents are infinite-lived, his model looks like an overlapping-generation model because of the form of the borrowing constraint. Our model is closer to more traditional Ramsey models with cash-in-advance.

the first regime (the mechanism is similar). In addition, money growth promotes the occurrence of endogenous fluctuations. Indeed, it raises the inflation rate and the nominal interest rate, that is the opportunity cost of holding money. The imperfection (liquidity constraint) has a larger impact on the consumption smoothing of patient agents and makes indeterminacy more likely. Conversely, the possibility of collateralization moderates the effect of the credit market imperfection and makes the endogenous fluctuations less likely.

Summing up, we highlight a trade-off concerning the monetary policy. Money growth enhances economic growth, but promotes the occurrence of self-fulfilling prophecies and crises.

This chapter is organized as follows. In section 2, we present the model. Section 3 studies the equilibrium properties if capitalists hold no money. Section 4 considers the case where they are also liquidity-constrained. Section 5 concludes. Mathematical details are gathered in the Appendix.

## 3.2 Fundamentals

### 3.2.1 Homogeneous firms

Firms are represented by a technology with private constant returns and increasing social returns.

There is a large number ($J$) of identical competitive firms. The representative firm rents capital and labor to produce the good under constant (private) returns to scale. External effects of capital intensity spill over the other firms. Technology is rationalized by a Romer-type (1986) production function.

**Assumption 1** $F(K_j, L_j) \equiv A\bar{k}^{1-s}K_j^s L_j^{1-s}$, $s \in (0,1)$, *where* $K_j$ *and* $L_j$ *are the* $j$*th firm's inputs and* $\bar{k}$ *is the average capital intensity in the economy.*

In the following, we denote by $K \equiv \sum_{j=1}^{J} K_j$ and $L \equiv \sum_{j=1}^{J} L_j$ the aggregate capital and labor force, respectively. The TFP ($A\bar{k}^{1-s}$) is affected by productive externalities of average capital intensity $\bar{k} \equiv K/L$, such as knowledge spill-overs. Since the firms share the same technology, we obtain at equilibrium $k = \bar{k} = K_j/L_j$ for every $j$.

Each producer is price-taker and maximizes the profit: $A\bar{k}^{1-s}K_j^s L_j^{1-s} - rK_j - wL_j$. Under Assumption 1, profit maximization gives:

$$r_t = sA \equiv r \tag{3.1}$$

$$w_t = (1-s)\,Ak_t \tag{3.2}$$

where $r_t$ and $w_t$ are the real interest rate and the wage bill at period $t$.

The equilibrium interest rate is constant over time as in many endogenous growth models *à la* Romer.

### 3.2.2 Heterogenous households

Consumers are differently endowed with capital and have different time preferences. Without loss of generality, we consider two types of heterogeneous infinite-lived agents: $i = 0, 1$, with heterogeneous discount factors.

There are $n_i$ individuals of type $i$. Preferences are represented by an intertemporal additive utility function

$$\sum_{t=0}^{\infty} \beta_i^t u\left(c_{it}\right) \tag{3.3}$$

and each consumer maximizes (3.3) with respect to a sequence $(k_{it+1}, c_{it})_{t=0}^{\infty}$ under a sequence of budget constraints:

$$M_{it+1} - M_{it} + p_t\left(k_{it+1} - \Delta k_{it}\right) + p_t c_{it} \leq p_t r_t k_{it} + p_t w_t l_{it} + p_t \tau_{it} \tag{3.4}$$

and borrowing constraints

$$p_t c_{it} - \gamma p_t k_{it} \leq M_{it} \tag{3.5}$$

Capital ($k_i$) and consumption ($c_i$) are the same good: they are produced with the same technology and have the same price ($p$).

The RHS of constraint (3.4) is the monetary income at the individual disposal. In real terms, it is the sum of capital income ($r_t k_{it}$), labor income ($w_t l_{it}$) and transfers from the monetary authority ($\tau_{it}$). This disposable income is partly saved in form of money ($M_{it+1} - M_{it}$) and

capital ($k_{it+1} - \Delta k_{it}$), and partly consumed. $\delta = 1 - \Delta \in (0, 1)$ is the depreciation rate of capital.

In addition, the representative household faces the partial cash-in-advance constraint (3.5): in the spirit of Grandmont and Younès (1972), we assume that a share of purchases is paid in cash, while the rest by credit. In a way, we are also close to Lucas and Stokey (1987), an economy with a cash and a credit good.

The individual's wealth matters in order to pay current consumption on credit. The amount of credit depends on household's collaterals. We assume for simplicity that the credit grant is proportional to nominal collaterals ($\gamma p_t k_{it}$). $\gamma$ represents either the common credit sensitivity to collaterals or the market rules. More precisely, $\gamma$ captures the lenders' or sellers' prudential attitude towards borrowers in presence of asymmetric informations, but also credit market regulation policies, that is a legal constraint to credit grants in order to ensure borrowers' solvability. To buy a constant amount of consumption on credit $d = \gamma k_{it}$, the required amount of collateral required is given by $k_{it} = d/\gamma$: so, the higher the parameter $\gamma$, the lower the needs of collateral to buy the amount $d$ on credit.

If $\gamma$ is sufficiently high, the cash-in-advance is no longer binding and the capitalist no longer needs balances to finance the consumption purchases.

Introducing collaterals in a model of capital accumulation is the added value of the chapter. We notice that the velocity of money with respect to consumption becomes endogenous. Our formulation takes into account one of the criticisms addressed to the cash-in-advance models: the implausibility of a constant velocity of money.[2]

We call 0 the patient type and 1 the impatient one. The discount factors are heterogeneous.

**Assumption 2.1** $\beta_1 < \beta_0$.

Let $\pi_{t+1} \equiv p_{t+1}/p_t$ denote the inflation factor and $\pi$ its stationary value. Let us introduce also an alternative more restrictive assumption.

[2]Money-In-the-Utility (MIU) models are immunized against this criticism: the functional equivalence highlighted by Feenstra (1986) between CIA and MIU, no longer holds with a Cash-When-I'm-Done (CWID) timing or a CIA timing and strictly positive elasticity of substitution between consumption and real balances (see Carlstrom and Fuerst, 2003).

**Assumption 2.2**

$$\beta_1 < \beta_0 (1 - \gamma\pi) \frac{\Delta + sA}{\Delta + sA - \gamma} \tag{3.6}$$

**Assumption 3** $\gamma < \min\{1/\pi, \Delta + sA\}$.

Assumption 3 means that the sensitivity factor $\gamma$ to collaterals does not exceed the gross real returns on assets (money and capital). Assumption 3 ensures the positivity of the RHS in (3.6).

We observe that (3.6) becomes $\beta_1 < \beta_0$ when $\gamma = 0$.

Notice that, according to Fisher decomposition, $\pi(\Delta + sA) - 1$ represents the nominal interest rate at the steady state. Under the Assumption 3, the positivity of the nominal interest rate (zero lower bound) implies the inequality

$$\beta_0 (1 - \gamma\pi) \frac{\Delta + sA}{\Delta + sA - \gamma} = \beta_0 \left[1 - \gamma \frac{\pi(\Delta + sA) - 1}{\Delta + sA - \gamma}\right] < \beta_0$$

that is Assumption 2.2 implies Assumption 2.1.

Each individual has an initial endowment of capital $k_{i0} \geq 0$. The utility function has a constant elasticity of intertemporal substitution $\sigma$:

**Assumption 4** $u(c_i) \equiv c_i^{1-1/\sigma}/(1 - 1/\sigma)$ *for* $\sigma \neq 1$, *with* $\sigma > 0$, *and* $u(c_i) \equiv \ln c_i$ *for* $\sigma = 1$.

Finally, we assume that patient agents supply no labor, while any impatient agent supplies one unit of labor.

**Assumption 5** $l_{0t} = 0$ *and* $l_{1t} = 1$.

### 3.2.3 Monetary authority

Each period, an amount $M_{t+1} - M_t$ of money is "helicoptered" to agents. More precisely, an exogenous share $\theta_i$ is given to agents of type $i$. We assume also $n_i M_{i0} = \theta_i M_0$, $i = 0, 1$, as initial conditions.

For simplicity, money supply grows at a constant factor $\mu \equiv M_{t+1}/M_t$. In other words, the monetary policy consists in a pair of instruments $(\mu, \theta_0)$ with $\theta_0 + \theta_1 = 1$.

In the following, we consider two regimes.

(1) The patient agent's liquidity constraint is always non-binding.

(2) The patient agent's liquidity constraint is always binding.

We do not consider the case in which the equilibrium switches from a regime to another.

## 3.3 Patient agents buy on credit

### 3.3.1 Patient agent

In the first regime, the patient agent's liquidity constraint becomes superfluous. Thus, under the assumption of positive interest rate, the patient agent holds no money ($M_{it} = 0$). Under Assumption 5, the Lagrangian simplifies:

$$\begin{aligned}&\sum_{t=1}^{\infty} \beta_0^t u\left(c_{0t}\right) \\ &+\sum_{t=0}^{\infty} \lambda_{0t} p_t \left(r_t k_{0t} + \tau_{0t} - k_{0t+1} + \Delta k_{0t} - c_{0t}\right)\end{aligned}$$

The maximization gives an intertemporal arbitrage

$$u'\left(c_{0t}\right) \geq \beta_0 u'\left(c_{0t+1}\right)\left(\Delta + r_{t+1}\right) \tag{3.7}$$

which holds with equality if $k_{0t} > 0$, and the constraint:

$$k_{0t+1} - \Delta k_{0t} + c_{0t} = r_t k_{0t} + \tau_{0t} \tag{3.8}$$

The initial and the final condition are given by the initial endowments: $M_{00}, k_{00} \geq 0$, and the transversality condition:

$$\lim_{t \to +\infty} \beta^t u'\left(c_{0t}\right) k_{0t+1} = 0$$

### 3.3.2 Impatient agent

Focus now on the impatient agent's program and consider only the case where the liquidity constraint is binding (s/he needs money to finance the future consumption). Deriving the Lagrangian function

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta_1^t u\left(c_{1t}\right) \\
&+\sum_{t=0}^{\infty} \lambda_{1t}\left(p_t r_t k_{1t}+p_t w_t+p_t \tau_{1t}-M_{1t+1}+M_{1t}-p_t k_{1t+1}+p_t \Delta k_{1t}-p_t c_{1t}\right) \\
&+\sum_{t=0}^{\infty} \mu_{1t}\left(M_{1t}-p_t c_{1t}+\gamma p_t k_{1t}\right)
\end{aligned} \tag{3.9}$$

gives Euler inequality:

$$u'\left(c_{1t}\right) \geq \beta_1 u'\left(c_{1t+1}\right) \frac{\pi_t}{\pi_{t+1}} \frac{\Delta+r_t-\gamma}{1-\gamma \pi_t} \tag{3.10}$$

which holds with equality if $k_{1t}>0$, and the constraints:

$$\begin{aligned}
m_{1t+1}\pi_{t+1}-m_{1t}+k_{1t+1}-\Delta k_{1t}+c_{1t} &= r_t k_{1t}+w_t+\tau_{1t} && (3.11)\\
c_{1t}-\gamma k_{1t} &\leq m_{1t} && (3.12)
\end{aligned}$$

where $m_{1t} \equiv M_{1t}/p_t$ is the individual demand for real balances. Notice that the budget constraint (3.4) becomes binding under the Assumption 4 (positive marginal utilities).

The initial and the final condition are given by the initial endowments: $M_{10}, k_{10} \geq 0$, and the transversality condition:

$$\lim_{t \to +\infty} \beta^t u'\left(c_{1t}\right)\left(k_{1t+1}+M_{1t+1}/p_t\right)=0 \tag{3.13}$$

A balanced growth path (BGP) is a steady state where all the real variables grow at the same rate. We will prove later the existence of a BGP. It is known that, in Ramsey models with heterogenous discounting, the patient agents will hold the entire stock of capital in the long run. We can show the same around the BGP (long-run equilibrium).

The Euler equation (3.7) and Assumption 2.2 imply

$$\frac{c_{0t+1}}{c_{0t}} = [\beta_0 (\Delta + r)]^\sigma > \left(\beta_1 \frac{r + \Delta - \gamma}{1 - \gamma\pi}\right)^\sigma$$

(3.10) becomes

$$\frac{c_{1t+1}}{c_{1t}} \geq \left(\beta_1 \frac{\Delta + r - \gamma}{1 - \gamma\pi}\right)^\sigma$$

The impatient agent wants to consume more today ($c_{1t}$) instead of tomorrow ($c_{1t+1}$) in order to lowers the LHS of inequality and restore the equality. To this purpose, s/he dissaves over time. In order to finance the current consumption at any period, s/he ends up to hold no capital. The assumption of borrowing constraints requires $k_{it} \geq 0$ (Becker 1981). Thus, in the long run $k_{1t} = 0$ for every $t$ (even during a transition). In the long run, only the patient agents hold capital.

So, around a BGP where the impatient agent holds no capital, her/his program rewrites as follows. Deriving the Lagrangian

$$\begin{aligned}
&\sum_{t=0}^{\infty} \beta_1^t u (c_{1t}) \\
&+ \sum_{t=0}^{\infty} \lambda_{1t} (p_t w_t + p_t \tau_{1t} - M_{1t+1} + M_{1t} - p_t c_{1t}) \\
&+ \sum_{t=0}^{\infty} \mu_{1t} (M_{1t} - p_t c_{1t})
\end{aligned} \tag{3.14}$$

gives

$$m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t} = w_t + \tau_{1t} \tag{3.15}$$

$$c_{1t} = m_{1t} \tag{3.16}$$

### 3.3.3 Equilibrium

#### Money market

Let $M_t$ denotes the aggregate supply of money at time $t$.

Each agent of type $i$ receives in period $t$ an amount $\theta_i (M_{t+1} - M_t) / n_i$ of nominal balances:

$$M_{t+1} - M_t = \theta_0 (M_{t+1} - M_t) + \theta_1 (M_{t+1} - M_t) = n_0 p_t \tau_{0t} + n_1 p_t \tau_{1t} \tag{3.17}$$

where $\tau_{it} = \theta_i (M_{t+1} - M_t) / (p_t n_i)$ or, equivalently, in real terms:

$$\tau_{it} = (m_{t+1}\pi_{t+1} - m_t)\, \theta_i / n_i \tag{3.18}$$

with $m_t \equiv M_t / p_t$.

Money market clears: $M_t = n_1 M_{1t}$ or, in real terms:

$$m_t = n_1 m_{1t} \tag{3.19}$$

Money growth can have inflationary and real effects (growth of real balances):

$$\mu = \pi_{t+1} \frac{m_{t+1}}{m_t} \tag{3.20}$$

(3.19) and (3.20) give:

$$\pi_{t+1} = \mu \frac{m_{1t}}{m_{1t+1}} \tag{3.21}$$

**Labor market**

According to Assumption 5, patient agents supply no labor ($l_{0t} = 0$), while impatient agents' labor supply is inelastic ($l_{1t} = 1$): $n_1 = L \equiv \sum_{j=1}^{J} L_j$.

**Goods market**

We verify that the equilibrium in all the markets, except for one, implies also the equilibrium in the latter (corollary of the Walras' law). Aggregating side by side the budget constraints (3.4) now binding across individuals, we obtain

$$\begin{aligned} & n_0 (k_{0t+1} - \Delta k_{0t} + c_{0t}) + n_1 (m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t}) \\ = \;& n_0 (r_t k_{0t} + \tau_{0t}) + n_1 (w_t + \tau_{1t}) \end{aligned}$$

Replacing (3.17) in the RHS, we get

$$\begin{aligned} & n_1 m_{1t+1}\pi_{t+1} - n_1 m_{1t} + n_0 \left(k_{0t+1} - \Delta k_{0t}\right) + n_0 c_{0t} + n_1 c_{1t} \\ = \; & n_0 r_t k_{0t} + n_1 w_t + m_{t+1}\pi_{t+1} - m_t \end{aligned}$$

After substituting (3.19) in the LHS, we cancel out the real balances:

$$n_0 \left(k_{0t+1} - \Delta k_{0t}\right) + n_0 c_{0t} + n_1 c_{1t} = n_0 r_t k_{0t} + n_1 w_t$$

Replacing the firms' equilibrium prices (3.1) and (3.10), and noticing that $k_t \equiv K_t/L_t = n_0 k_{0t}/n_1$, that is $n_0 k_{0t} = n_1 k_t$, we have

$$n_1 \left(k_{t+1} - \Delta k_t\right) + n_0 c_{0t} + n_1 c_{1t} = n_1 A k_t \tag{3.22}$$

that is, the aggregate investment plus the aggregate consumption is equal to the aggregate production.

**General equilibrium**

A definition of (dynamic) general equilibrium is given.

**Definition 2** *An equilibrium of the economy*

$$E = \left(\gamma, \mu, \theta_0, M_0, F, \left(k_{i0}, \beta_i, u_i, n_i\right)_{i=1}^{2}\right)$$

*is an intertemporal sequence* $\left(r_t, w_t, \left(K_{jt}, L_{jt}\right)_{j=1}^{J}, \left(M_{it}, k_{it}, c_{it}\right)_{i=1}^{2}\right)_{t=0}^{+\infty}$ *which satisfies the following conditions:*

*1- given the strictly positive sequence* $(r_t, w_t)_{t=0}^{+\infty}$, $(K_{jt}, L_{jt})$ *solves the* $j$*th firm's program for* $j = 1, \ldots, J$ *and* $t = 0, \ldots, +\infty$*;*

*2- given* $(r_t, w_t)_{t=0}^{+\infty}$, $(M_{it+1}, k_{it+1}, c_{it})_{t=0}^{+\infty}$ *solves the consumer's* $i$ *program for* $i = 1, 2$*;*

*3- the money market clears:* $M_t = \sum_{i=1}^{2} n_i M_{it}$ *for* $t = 0, \ldots, +\infty$*;*

*4- the capital market clears:* $\sum_{j=1}^{J} K_{jt} = \sum_{i=1}^{2} n_i k_{it}$ *for* $t = 0, \ldots, +\infty$*;*

*5- the labor market clears:* $\sum_{j=1}^{J} L_{jt} = \sum_{i=1}^{2} n_i l_{it}$ *for* $t = 0, \ldots, +\infty$*;*

*6- the product market clears:*

$$\sum_{j=1}^{J} F\left(K_{jt}, L_{jt}\right) = \sum_{i=1}^{2} \left[c_{it} + k_{it+1} - (1-\delta)\, k_{it}\right] n_i$$

*for* $t = 0, \ldots, +\infty$.

### 3.3.4 Balanced growth path

Focus on the equilibrium budget constraints (3.8) and (3.15). Replacing $k_{0t} = k_t n_1/n_0$ and (3.18) in (3.8), and (3.16) and (3.18) in (3.15) we obtain, respectively:

$$n_1 \left[k_{t+1} - (\Delta + r)\, k_t\right] + n_0 c_{0t} = n_1 \theta_0 \left(m_{1t+1}\pi_{t+1} - m_{1t}\right) \tag{3.23}$$

$$\theta_1 m_{1t} + (1-\theta_1)\, m_{1t+1}\pi_{t+1} = w_t \tag{3.24}$$

Replacing (3.21) in (3.23) and (3.24) gives

$$n_1 \left[k_{t+1} - (\Delta + r)\, k_t\right] + n_0 c_{0t} = n_1 \theta_0 \left(\mu - 1\right) m_{1t} \tag{3.25}$$

$$\left[\theta_1 + (1-\theta_1)\, \mu\right] m_{1t} = (1-s)\, A k_t \tag{3.26}$$

since $w_t = (1-s)\, A k_t$.

Let us set

$$\left(g_{t+1}, x_{0t}, x_{1t}\right) \equiv \left(\frac{k_{t+1}}{k_t}, \frac{m_{0t}}{k_t}, \frac{m_{1t}}{k_t}\right) \tag{3.27}$$

Dividing (3.26) by $k_t$, we obtain constant ratios

$$\left(x_{0t}, x_{1t}\right) = \left(0, \frac{(1-s)\, A}{1 + (\mu - 1)\, \theta_0}\right) \tag{3.28}$$

Dividing (3.25) by $k_t$, we get

$$\frac{c_{0t}}{k_t} = \frac{n_1}{n_0}\left[\Delta + A - \frac{(1-s)\, A}{1 + (\mu - 1)\, \theta_0} - g_{t+1}\right] \tag{3.29}$$

The patient agents' Euler equation (3.7) becomes

$$\frac{c_{0t+1}}{c_{0t}} = [\beta_0 (\Delta + sA)]^\sigma$$

The only possible equilibrium (compatible with the non-negativity of quantities and TC) is the BGP

$$\begin{aligned} k_t &= k_0 g^t \\ c_{0t} &= c_{00} g^t \end{aligned}$$

with

$$g = [\beta_0 (\Delta + sA)]^\sigma$$

There is no transition and the initial consumption of the patient agent jumps on the BGP: (3.29) gives

$$c_{00} = k_0 \frac{n_1}{n_0} \left[ \Delta + A - \frac{(1-s) A}{1 + (\mu - 1) \theta_0} - g \right] \tag{3.30}$$

The consumption of impatient agents along the BGP is given by

$$c_{1t} = c_{10} g^t$$

From (3.22) evaluated at time $t = 0$, we have

$$n_1 \left( \frac{k_1}{k_0} - \Delta \right) + n_0 \frac{c_{00}}{k_0} + n_1 \frac{c_{10}}{k_0} = n_1 A$$

and, using (3.30):

$$c_{10} = \frac{(1-s) A k_0}{1 + (\mu - 1) \theta_0} \tag{3.31}$$

Focus on (3.29). Notice that the regime we consider is $c_{0t}/k_{0t} < \gamma$ which is equivalent to

$$g_{t+1} > \Delta + A - \gamma - \frac{(1-s) A}{1 + (\mu - 1) \theta_0} \tag{3.32}$$

At equilibrium, the condition to ensure that capitalists hold no money becomes

$$g = [\beta_0 (\Delta + sA)]^\sigma > \Delta + A - \gamma - \frac{(1-s) A}{1 + (\mu - 1) \theta_0}$$

that is

$$\gamma > \Delta + A - g - \frac{(1-s) A}{1 + (\mu - 1) \theta_0} \tag{3.33}$$

In the logarithmic case ($\sigma = 1$), this inequality writes:

$$\gamma > (1 - \beta_0) (\Delta + sA) + (1-s) A \frac{(\mu - 1) \theta_0}{1 + (\mu - 1) \theta_0} \tag{3.34}$$

that is a lower bound for the sensitivity to collaterals. Indeed, if $\gamma$ is lower, the capitalists have not enough collaterals to pay all the consumption on credit and they partially need balances, while if $\gamma$ satisfies (3.34), they can pay all the consumption on credit.

Along the BGP, Assumption 3 becomes $\gamma < \min \{g/\mu, \Delta + sA\}$. So in the logarithmic case ($g = \beta_0 (\Delta + sA)$), the regime where collaterals do not matter, requires

$$(1 - \beta_0) (\Delta + sA) + (1-s) A \frac{(\mu - 1) \theta_0}{1 + (\mu - 1) \theta_0} < \gamma < \min \left\{ \frac{\beta_0}{\mu} (\Delta + sA), \Delta + sA \right\} \tag{3.35}$$

Noticing that the zero lower bound (positive interest rate) means $\pi (\Delta + sA) > 1$, that is $\mu > g/(\Delta + sA)$. In the logarithmic case, we require $\mu > \beta_0$. Inequalities (3.35) become

$$(1 - \beta_0) (\Delta + sA) + (1-s) A \frac{(\mu - 1) \theta_0}{1 + (\mu - 1) \theta_0} < \gamma < \frac{\beta_0}{\mu} (\Delta + sA)$$

Focus now on the monetary policy and the welfare analysis.

### 3.3.5 Monetary policy

Monetary policy does not affect the growth factor $g$ but it has an impact on the initial consumption and, so, the welfare of both types of agents:

$$\begin{aligned}\frac{\partial c_{00}}{\partial \mu} &= \theta_0 \frac{(1-s)\,Ak_0}{\left[1+(\mu-1)\,\theta_0\right]^2}\frac{n_1}{n_0} > 0\\ \frac{\partial c_{10}}{\partial \mu} &= -\theta_0 \frac{(1-s)\,Ak_0}{\left[1+(\mu-1)\,\theta_0\right]^2} < 0\end{aligned}$$

The impact on agents' welfare is qualitatively the same. Indeed

$$U_i \equiv \sum_{t=0}^{\infty} \beta_i^t u\left(c_{it}\right) = \sum_{t=0}^{\infty} \beta_i^t \frac{\left(c_{i0}g^t\right)^{1-1/\sigma}}{1-1/\sigma} = \frac{\sigma}{\sigma-1}\frac{c_{i0}^{1-1/\sigma}}{1-\beta_i g^{1-1/\sigma}}$$

under the transversality condition

$$\beta_i g^{1-1/\sigma} < 1 \tag{3.36}$$

and

$$\frac{\partial U_i}{\partial \mu} = \frac{c_{i0}^{-1/\sigma}}{1-\beta_i g^{1-1/\sigma}}\frac{\partial c_{i0}}{\partial \mu}$$

We obtain

$$\frac{\partial U_0}{\partial \mu} > 0 \text{ and } \frac{\partial U_1}{\partial \mu} < 0 \tag{3.37}$$

So, the patient agent receives more money today and bears no opportunity cost of holding money, because s/he spends these balances today (s/he does need money to finance consumption tomorrow). The impatient agents bears an opportunity cost in order to finance future consumption with money. This opportunity cost is given by the nominal interest rate and increases with the monetary growth rate because of the inflation. Her/his cost is greater than the benefit of additional monetary transfers to impatient agents from the monetary authority.

We finally observe that, if $\mu > 1$,

$$\frac{\partial c_{00}}{\partial \theta_0} > 0 \text{ and } \frac{\partial c_{10}}{\partial \theta_0} < 0$$

So, since

$$\frac{\partial U_i}{\partial \theta_0} = \frac{c_{i0}^{-1/\sigma}}{1-\beta_i g^{1-1/\sigma}} \frac{\partial c_{i0}}{\partial \theta_0}$$

we have

$$\frac{\partial U_0}{\partial \theta_0} > 0 \text{ and } \frac{\partial U_1}{\partial \theta_0} < 0$$

The interpretation is straightforward. If the share of money "helicoptered" to patient agents increases, the welfare of patient agent increases while that of impatient agents decreases.

## 3.4 Patient agents pay cash

The household is aware of the credit function, which is a sort of institutional constraint (endogenous structure of the credit market or legal constraints), and maximizes the Lagrangian also with respect to the arguments $(k_t, c_t)$ appearing in the LHS of constraint (3.5).

### 3.4.1 Patient agent

Focus now on the second regime:

$$c_{0t}/k_{0t} > \gamma \tag{3.38}$$

The liquidity constraints is binding under the assumption of positive nominal interest rate $(\pi_t (\Delta + sA) > 1)$. Deriving the Lagrangian function

$$\begin{aligned}
&\sum_{t=1}^{\infty} \beta_0^t u(c_{0t}) \\
&+ \sum_{t=0}^{\infty} \lambda_{0t} \left( p_t r_t k_{0t} + p_t \tau_{0t} - M_{0t+1} + M_{0t} - p_t k_{0t+1} + p_t \Delta k_{0t} - p_t c_{0t} \right) \\
&+ \sum_{t=0}^{\infty} \mu_{0t} \left( M_{0t} - p_t c_{0t} + \gamma p_t k_{0t} \right)
\end{aligned}$$

gives the intertemporal arbitrage (consumption smoothing):

$$u'(c_{0t}) \geq \beta_0 u'(c_{0t+1}) \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + r_t - \gamma}{1 - \gamma \pi_t} \tag{3.39}$$

which holds with equality if $k_{0t} > 0$, and the constraints:

$$m_{0t+1}\pi_{t+1} - m_{0t} + k_{0t+1} - \Delta k_{0t} + c_{0t} = r_t k_{0t} + \tau_{0t} \quad (3.40)$$

$$c_{0t} - \gamma k_{0t} \leq m_{0t} \quad (3.41)$$

### 3.4.2 Impatient agent

Focus now on the impatient agents' program and consider only the case where the liquidity constraint is binding (that is s/he needs money to finance the future consumption). Deriving the impatient agent's Lagrangian function seen above (expression (3.9)) gives the same first-order conditions (equations (3.10), (3.11) and (3.12)) and the transversality condition (3.13). Notice that the budget constraint is binding under the Assumption 4.

Along the BGP, the Euler equation (3.39) and Assumption 2.1 imply

$$\frac{c_{0t+1}}{c_{0t}} = \left(\beta_0 \frac{r + \Delta - \gamma}{1 - \gamma\pi}\right)^{\sigma} > \left(\beta_1 \frac{r + \Delta - \gamma}{1 - \gamma\pi}\right)^{\sigma}$$

(3.10) becomes

$$\frac{c_{1t+1}}{c_{1t}} \geq \left(\beta_1 \frac{\Delta + r - \gamma}{1 - \gamma\pi}\right)^{\sigma}$$

As above, the impatient agent dissaves over time to consume more today ($c_{1t}$) instead of tomorrow ($c_{1t+1}$) and ends up to hold no capital. The assumption of borrowing constraints means $k_{it} \geq 0$ (Becker, 1981). Thus, in the long run $k_{1t} = 0$ for every $t$ (even during a transition). In the long run, only the patient agent holds capital as above, but now a less restrictive assumption is required (Assumption 2.1 instead of Assumption 2.2).

So, around a BGP where the impatient agent holds no capital, her/his program rewrites as follows. Deriving the Lagrangian (3.14) gives the constraints (3.15) and (3.16) now binding.

### 3.4.3 Equilibrium

#### Money market

Equations (3.17) and (3.18) still hold. However now the capitalists hold money and the market clearing becomes $M_t = n_0 M_{0t} + n_1 M_{1t}$ or, in real terms:

$$m_t = n_0 m_{0t} + n_1 m_{1t} \tag{3.42}$$

Replacing (3.42) in (3.20) gives

$$\pi_{t+1} = \mu \frac{n_0 m_{0t} + n_1 m_{1t}}{n_0 m_{0t+1} + n_1 m_{1t+1}} \tag{3.43}$$

#### Labor market

As above, for simplicity, patient agents supply no labor, while the impatient agents supply one unit. Thus, $n_1 = L \equiv \sum_{j=1}^{J} L_j$.

#### Goods market

It is easy to check that the aggregate investment plus the aggregate consumption is equal to the aggregate production (corollary of the Walras' law):

$$n_1 \left(k_{t+1} - \Delta k_t\right) + n_0 c_{0t} + n_1 c_{1t} = n_1 A k_t \tag{3.44}$$

(similar arguments to those of section 3.3.3 apply).

The equilibrium budget constraints and the binding liquidity constraints jointly with the Euler equation of the patient agent give the intertemporal equilibrium system.

#### General equilibrium

We consider Definition 2 and we compute the dynamic system.

**Proposition 3** *Under Assumptions 1, 2.1, 3, 4 and 5, when collaterals matter and the nominal interest rate is positive, the patient agent's liquidity constraint is binding and the intertemporal*

*equilibrium satisfies the following system*

$$\mu n_1 x_{1t+1} + (1-\mu)\,\theta_1\,(n_0 x_{0t+1} + n_1 x_{1t+1}) = n_1\,(1-s)\,A\frac{n_0 x_{0t+1} + n_1 x_{1t+1}}{n_0 x_{0t} + n_1 x_{1t}} \tag{3.45}$$

$$\begin{aligned} & \frac{g_t}{g_{t+1}}\frac{\mu n_1 x_{1t} + (1-\mu)\,\theta_1\,(n_0 x_{0t} + n_1 x_{1t})}{\mu n_1 x_{1t+1} + (1-\mu)\,\theta_1\,(n_0 x_{0t+1} + n_1 x_{1t+1})} \\ & \left[1 - \gamma\frac{\mu}{g_t}\frac{n_1\,(1-s)\,A}{\mu n_1 x_{1t} + (1-\mu)\,\theta_1\,(n_0 x_{0t} + n_1 x_{1t})}\right] \\ = \quad & \beta_0\,(sA + \Delta - \gamma)\left(\frac{n_0 x_{0t} + \gamma n_1}{n_0 x_{0t+1} + \gamma n_1}\frac{1}{g_{t+1}}\right)^{1/\sigma} \end{aligned} \tag{3.46}$$

*where $g_t$ is given by*

$$g_t = g\,(x_{0t}, x_{1t}) \equiv \Delta + sA - \gamma - (1-s)\,A\frac{\mu n_0 x_{0t} + (1-\mu)\,\theta_0\,(n_0 x_{0t} + n_1 x_{1t})}{\mu n_1 x_{1t} + (1-\mu)\,\theta_1\,(n_0 x_{0t} + n_1 x_{1t})} \tag{3.47}$$

**Proof.** See the Appendix.

Equations (3.45) and (3.46) form a two-dimensional system in $(x_{0t}, x_{1t})$.

### 3.4.4 Balanced growth path

Equation (3.45) becomes at the steady state:

$$\mu n_1 x_1 + (1-\mu)\,\theta_1\,(n_0 x_0 + n_1 x_1) = n_1\,(1-s)\,A \tag{3.48}$$

Replacing (3.48) in the Euler equation (3.46) evaluated at the steady state, gives

$$g = \gamma\mu + \beta_0\,(\Delta + sA - \gamma)\,g^{\frac{\sigma-1}{\sigma}} \equiv \varphi\,(g) \tag{3.49}$$

an implicit equation which determines the balanced growth rate $g$.

Replacing (3.48) in (3.47) evaluated at the steady state, gives

$$\mu n_0 x_0 + (1-\mu)\,\theta_0\,(n_0 x_0 + n_1 x_1) = n_1\,(\Delta + sA - \gamma - g) \tag{3.50}$$

System (3.48)-(3.50) becomes

$$\begin{bmatrix} \mu + (1-\mu)\,\theta_0 & (1-\mu)\,\theta_0 \\ (1-\mu)\,\theta_1 & \mu + (1-\mu)\,\theta_1 \end{bmatrix} \begin{bmatrix} n_0 x_0 \\ n_1 x_1 \end{bmatrix} = n_1 \begin{bmatrix} \Delta + sA - \gamma - g \\ (1-s)\,A \end{bmatrix}$$

and can be solved with respect to $(x_0, x_1)$. Noticing that $\theta_0 + \theta_1 = 1$, we obtain

$$x_0 = \frac{(\Delta + A - \gamma - g)\,[1 + (\mu - 1)\,\theta_0] - (1-s)\,A}{\mu} \frac{n_1}{n_0} \tag{3.51}$$

$$x_1 = \frac{(1-s)\,A + \theta_1\,(\mu - 1)\,(A + \Delta - \gamma - g)}{\mu} \tag{3.52}$$

where $g$ is given by (3.49).

The asset ratio is given by $m/k = n_0 x_0 + n_1 x_1 = n_1\,(A + \Delta - \gamma - g)$.

The binding cash-in-advance along the BGP gives: $c_{00} = (x_0 + \gamma n_1/n_0)\,k_0$. Replacing (3.51), we get

$$c_{00} = k_0 \frac{n_1}{n_0} \left( \gamma + \frac{[1 + (\mu - 1)\,\theta_0]\,(\Delta + A - g - \gamma) - (1-s)\,A}{\mu} \right) \tag{3.53}$$

The consumption of impatient agents along the BGP is given by $c_{1t} = c_{10} g^t$. From (3.22), we obtain

$$\frac{c_{10}}{k_0} = \Delta + A - \frac{k_1}{k_0} - \frac{n_0}{n_1} \frac{c_{00}}{k_0}$$

and, using (3.53):

$$c_{10} = k_0 \left[ \frac{1}{\mu}\,(1-s)\,A + \left(1 - \frac{1}{\mu}\right) \theta_1\,(\Delta + A - g - \gamma) \right] \tag{3.54}$$

**Restrictions**

In order to have a BGP, the following restrictions are required.

(1) Positivity of $x_0$ and $x_1$.

(2) Positivity of the nominal interest rate (to get a monetary equilibrium, money is required to be a dominated asset).

(3) Endogenous growth transversality conditions (EGTC) along the balanced growth path (BGP).

Focus on these three restrictions.

(1) Positivity of variables $x_0$ and $x_1$ requires, respectively:

$$\gamma < \Delta + A - g - \frac{(1-s)A}{1+(\mu-1)\theta_0} \tag{3.55}$$

$$g(\mu-1) < (\Delta + A - \gamma)(\mu-1) + \frac{(1-s)A}{\theta_1} \tag{3.56}$$

Notice that (3.55) is exactly the converse of (3.33) (we are in the other regime).

Let, for simplicity, $\mu \geq 1$. In this case, inequality (3.55) implies (3.56). (3.55) puts also a restriction on the credit market sensitivity to collaterals:

$$\gamma < \Delta + A - \frac{(1-s)A}{1+\theta_0(\mu-1)} \tag{3.57}$$

(notice that the RHS of (3.57) is always positive).

(2) Positive nominal interest rate. We define the nominal interest factor as $i_t \equiv \pi_t(\Delta + r_t)$ and we require the zero lower bound for the interest rate: $i_t - 1 > 0$ (a productive economy exists if money is a dominated asset). Since patient agents will hold the stock of capital in the long run, we need condition (3.39) to be satisfied with equality at the steady state, that is, under Assumption 4:

$$g = \left(\beta_0 \frac{\Delta + r - \gamma}{1 - \gamma\pi}\right)^{\sigma} \tag{3.58}$$

Noticing that $i > 1$ implies $\Delta + r_t > 1/\pi$. We obtain from (3.58): $g \geq (\beta_0/\pi)^{\sigma}$. Since, along the balanced growth path, (3.20) becomes $\mu = \pi g$, the inequality which ensures a positive nominal interest rate writes

$$\mu/\beta_0 > g^{(\sigma-1)/\sigma} \tag{3.59}$$

We observe that in exogenous growth monetary model $g = 1$ and (3.59) reduces to $\mu > \beta_0$. It is also the case in the logarithmic case ($\sigma = 1$) with different growth rates.

(3) We derive the EGTC along the BGP:

$$\lim_{t\to+\infty} \beta_0^t u'(c_{0t})(\pi m_{0t+1} + k_{0t+1}) = \lim_{t\to+\infty} \beta_1^t u'(c_{1t})\pi m_{1t+1} = 0$$

Under Assumption 4, they become

$$\lim_{t\to+\infty} g\left(\pi m_{00}+k_{00}\right) c_{00}^{-1/\sigma}\left[\beta_0 g^{(\sigma-1)/\sigma}\right]^t = \lim_{t\to+\infty} g\pi m_{10} c_{10}^{-1/\sigma}\left[\beta_1 g^{(\sigma-1)/\sigma}\right]^t = 0$$

Thus, we require $g^{(\sigma-1)/\sigma} < \min\left\{1/\beta_0, 1/\beta_1\right\}$, that is

$$1/\beta_0 > g^{(\sigma-1)/\sigma} \tag{3.60}$$

(see also (3.36)).

Notice that, if $\mu \geq 1$, then (3.60) implies (3.59).

From (3.58), inequality (3.60) writes also

$$g < \frac{\Delta + r - \gamma}{1-\gamma\pi} \tag{3.61}$$

Setting $\gamma = 0$ (no room for collateralization), we recover the usual EGTC: $g < \Delta + r$ (the growth factor does not exceed the gross interest factor). Thus, (3.61) is a generalized EGTC in a world with collaterals.

**Existence and uniqueness of the BGP**

We provide simple conditions for the existence and uniqueness of the BGP.

**Proposition 4** *Under Assumptions 1, 2.1, 3, 4 and 5 and $\mu \geq 1$, if (3.55) and (3.60) are satisfied, the BGP exists and is unique.*

**Proof.** See the Appendix.

**Example** Consider the logarithmic case ($\sigma = 1$). The BGP is explicitly given by

$$\begin{aligned}
g &= \mu\gamma + \beta_0\left(\Delta + sA - \gamma\right) \\
x_0 &= \frac{\left[\left(1-\beta_0\right)\left(\Delta+sA-\gamma\right)+\left(1-s\right)A-\mu\gamma\right]\left[1-\theta_0\left(1-\mu\right)\right]-\left(1-s\right)A}{\mu}\frac{n_1}{n_0} \\
x_1 &= \frac{\left[\left(1-\beta_0\right)\left(\Delta+sA-\gamma\right)+\left(1-s\right)A-\mu\gamma\right]\theta_1\left(\mu-1\right)+\left(1-s\right)A}{\mu}
\end{aligned}$$

Focus on the restrictions considered in the previous section. Assumption 3 implies $g > 0$. The EGTC is also satisfied: $g^{(\sigma-1)/\sigma} = 1 < 1/\beta_0$ (see (3.60)). Condition (3.55) becomes

$$\mu\gamma + \beta_0 (\Delta + sA - \gamma) < \Delta + A - \gamma - \frac{(1-s) A}{1 + \theta_0 (\mu - 1)}$$

which is equivalent to

$$\gamma < \frac{(1-\beta_0)(\Delta + sA) + (1-s) A \frac{\theta_0(\mu-1)}{1+\theta_0(\mu-1)}}{1 - \beta_0 + \mu}$$

□

### 3.4.5 Monetary policy

One may wonder whether either a monetary expansion or a financial markets regulation can have a positive impact on economic growth and citizens' welfare.

#### Effects on growth

First, let us focus on the effects on the growth rate. Then, we will study the welfare impact of economic policy.

For simplicity, we consider the policy issue along the BGP.

**Proposition 5** *Under Assumption 3, the impact of monetary growth is positive:*

$$\frac{\partial g}{\partial \mu} = \frac{\sigma h \gamma}{1 + \sigma (h-1)} > 0 \tag{3.62}$$

*where*

$$h \equiv \frac{1}{1 - \gamma\pi} (> 1) \tag{3.63}$$

*The impact of sensitivity to collaterals is also positive:*

$$\frac{\partial g}{\partial \gamma} = \frac{\sigma g}{1 + \sigma (h-1)} \left( h\pi - \frac{1}{\Delta + sA - \gamma} \right) > 0 \tag{3.64}$$

**Proof.** See the Appendix.

In the logarithmic case ($\sigma = 1$), $g = \gamma(\mu - \beta_0) + \beta_0(\Delta + sA)$ and the proof is straightforward because $\gamma < \Delta + sA$. However, it holds in general for $\sigma \neq 1$, as it is shown in the Appendix.

Focus first on (3.62). Issuing money increases the inflation and the nominal interest rate. The opportunity cost of holding money increases and agents hold less money, consume less and accumulate more capital. In our model, money growth is growth-enhancing.

Let now us interpret (3.64). Under a higher $\gamma$, increasing capital lowers more money demand and the opportunity cost of holding money (which is positive because we assume a positive nominal interest rate). Then households wants to hold more capital and this enhances growth.

**Effects on welfare**

What is the impact on welfare of the two classes of agents?

Under Assumption 3 and transversality condition (3.60), we obtain

$$U_i \equiv \sum_{t=0}^{\infty} \beta_i^t u(c_{it}) = \sum_{t=0}^{\infty} \beta_i^t \frac{\left(c_{i0} g^t\right)^{1-1/\sigma}}{1 - 1/\sigma} = \frac{\sigma}{\sigma - 1} \frac{c_{i0}^{1-1/\sigma}}{1 - \beta_i g^{1-1/\sigma}}$$

with $i = 0, 1$, where $c_{00}$ and $c_{10}$ are given by (3.53) and (3.54). Notice that $1 - \beta_1 g^{1-1/\sigma} > 0$ because $\beta_1 < \beta_0$.

The impact of $\mu$ on $U_0$ is given by

$$\frac{\partial U_0}{\partial \mu} = \frac{c_{00}^{1-1/\sigma}}{1 - \beta_0 g^{1-1/\sigma}} \left[ \frac{1}{g} \frac{\partial g}{\partial \mu} \frac{\beta_0 g^{1-1/\sigma}}{1 - \beta_0 g^{1-1/\sigma}} + \frac{1}{c_{00}} \frac{\partial c_{00}}{\partial \mu} \right] \tag{3.65}$$

As usual in endogenous growth models, there is a trade-off between the initial consumption and the growth rate. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g} \frac{\partial g}{\partial \mu} \frac{\beta_0 g^{1-1/\sigma}}{1 - \beta_0 g^{1-1/\sigma}} > 0$$

(see the inequality (3.62) and the transversality condition: $1 - \beta_0 g^{1-1/\sigma} > 0$) and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (3.53), we get

$$\frac{\partial c_{00}}{\partial \mu} = k_0 \frac{n_1}{n_0} \left[ \frac{(1-s)A - (1-\theta_0)(\Delta + A - g - \gamma)}{\mu^2} - \frac{1 + (\mu - 1)\theta_0}{\mu} \frac{\partial g}{\partial \mu} \right]$$

which can be positive or negative.

The impact of $\gamma$ on $U_0$ is given by

$$\frac{\partial U_0}{\partial \gamma} = \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{1-1/\sigma}} \left[ \frac{1}{g} \frac{\partial g}{\partial \gamma} \frac{\beta_0 g^{1-1/\sigma}}{1-\beta_0 g^{1-1/\sigma}} + \frac{1}{c_{00}} \frac{\partial c_{00}}{\partial \gamma} \right] \tag{3.66}$$

There is a trade-off between the initial consumption and the growth rate. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g} \frac{\partial g}{\partial \gamma} \frac{\beta_0 g^{1-1/\sigma}}{1-\beta_0 g^{1-1/\sigma}} > 0$$

(see the inequality (3.64) and the transversality condition: $1-\beta_0 g^{1-1/\sigma} > 0$) and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (3.53), we get

$$\frac{\partial c_{00}}{\partial \gamma} = k_0 \frac{n_1}{n_0} \left[ \frac{\mu-1}{\mu} \theta_1 - \left( 1 - \frac{\mu-1}{\mu} \theta_1 \right) \frac{\partial g}{\partial \gamma} \right]$$

which can be positive or negative.

Focus now on the welfare of impatient agents.

The impact of $\mu$ on $U_1$ is given by

$$\frac{\partial U_1}{\partial \mu} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}} \left[ \frac{1}{g} \frac{\partial g}{\partial \mu} \frac{\beta_1 g^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}} + \frac{1}{c_{10}} \frac{\partial c_{10}}{\partial \mu} \right] \tag{3.67}$$

As for agents of type 0, there is a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g} \frac{\partial g}{\partial \mu} \frac{\beta_1 g^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}} > 0$$

(inequality (3.62) and transversality condition: $1-\beta_0 g^{1-1/\sigma} > 0$) and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (3.54), we get

$$\begin{aligned} \frac{\partial c_{10}}{\partial \mu} &= \left[ \frac{\theta_1 (\Delta + A - g - \gamma) - (1-s) A}{\mu^2} - \theta_1 \left( 1 - \frac{1}{\mu} \right) \frac{\partial g}{\partial \mu} \right] k_0 \\ &= -k_0 \frac{\partial g}{\partial \mu} - \frac{n_0}{n_1} \frac{\partial c_{00}}{\partial \mu} \end{aligned}$$

which can be positive or negative. We notice that $\partial c_{00}/\partial\mu$ and $\partial c_{10}/\partial\mu$ work in opposite directions.

Focus now on $\partial U_1/\partial\gamma$. We have

$$\frac{\partial U_1}{\partial\gamma} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}}\left(\frac{1}{g}\frac{\partial g}{\partial\gamma}\frac{\beta_1 g^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}} + \frac{1}{c_{10}}\frac{\partial c_{10}}{\partial\gamma}\right) \tag{3.68}$$

There is a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g}\frac{\partial g}{\partial\gamma}\frac{\beta_1 g^{1-1/\sigma}}{1-\beta_1 g^{1-1/\sigma}} > 0$$

(inequality (3.64) and transversality condition: $1-\beta_0 g^{1-1/\sigma} > 0$) and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (3.54), we get

$$\frac{\partial c_{10}}{\partial\gamma} = -\theta_1\left(1-\frac{1}{\mu}\right)\left(1+\frac{\partial g}{\partial\gamma}\right)k_0 = -\frac{n_0}{n_1}\frac{\partial c_{00}}{\partial\gamma} - \frac{\partial g}{\partial\gamma}k_0$$

which can be positive or negative. We notice that $\partial c_{00}/\partial\gamma$ and $\partial c_{10}/\partial\gamma$ work in opposite directions.

In order to have an idea about these effects, let us provide a numerical example.

**Numerical example**

Focus for simplicity on the logarithmic case:

$$U_i = \sum_{t=0}^{\infty}\beta_i^t \ln\left(c_{i0}g^t\right) = \frac{\ln c_{i0}}{1-\beta_i} + \ln g\sum_{t=0}^{\infty}t\beta_i^t = \frac{\beta_i \ln g + (1-\beta_i)\ln c_{i0}}{(1-\beta_i)^2}$$

with $g = \gamma(\mu-\beta_0)+\beta_0(\Delta+sA)$ and

$$\begin{aligned}
&c_{00}\\
&= \tfrac{n_1}{n_0}\tfrac{k_0}{\mu}\left(\left[(1-s)A-\gamma\mu\right](\mu-1)\theta_0 + (1-\beta_0)(\Delta+sA-\gamma)\left[1+(\mu-1)\theta_0\right]\right)\\
&c_{10}\\
&= \tfrac{k_0}{\mu}\left[\left((1-s)A-\gamma\mu\right)(1+(\mu-1)\theta_1) + (1-\beta_0)(\Delta+sA-\gamma)(\mu-1)\theta_1 + \gamma\mu\right]
\end{aligned}$$

Let us fix the parameters as follows according to yearly data:

| $A$ | $k_0$ | $s$ | $n_0$ | $n_1$ | $\beta_0$ | $\beta_1$ | $\Delta$ | $\theta_0$ |
|---|---|---|---|---|---|---|---|---|
| 0.21 | 1 | 2/3 | 1 | 1 | 0.96 | 0.94 | 0.9 | 1/2 |

Notice that $s$ is set as usual in endogenous growth literature (see Mankiw *et al.* 1992).

Focus on the impact of $\mu$ and $\gamma$ around

| $\mu$ | $\gamma$ |
|---|---|
| 1.04 | 0.02 |

For simplicity, these values give $g = \gamma(\mu - \beta_0) + \beta_0(\Delta + sA) = 1$.

All the restrictions are satisfied around $(\gamma, \mu) = (0.02, 1.04)$:

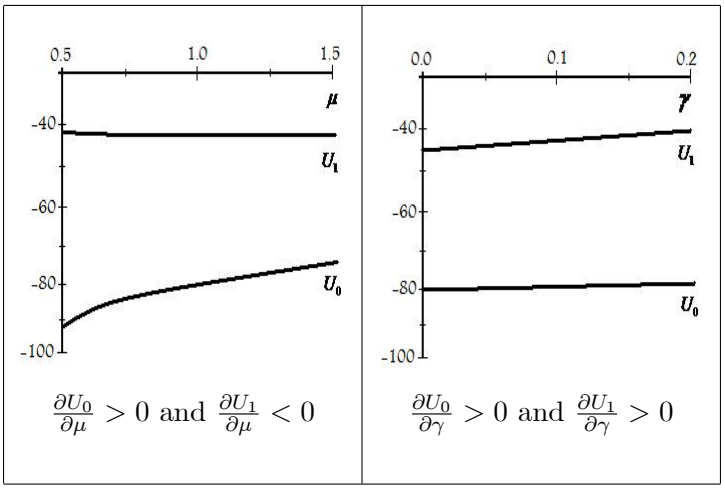
(1) Assumption 3: $\gamma = 0.02 < \min\{1/\mu, \Delta + sA\} = 0.9615$;

(2) condition (3.38) ensures that collaterals matter:

$$\begin{aligned}&\tfrac{c_{0t}}{k_{0t}}\\&= \tfrac{n_1}{n_0}\tfrac{1}{\mu}\left(\left[(1-s)A - \gamma\mu\right](\mu-1)\theta_0 + (1-\beta_0)(\Delta + sA - \gamma)\left[1 + (\mu-1)\theta_0\right]\right)\\&= 0.0410 > \gamma = 0.02\end{aligned}$$

(3) conditions (3.55) guarantees the positivity of variables $x_0$ and $x_1$: $g < \Delta + A - \gamma - (1-s)A/\left[1 + (\mu-1)\theta_0\right] = 1.0214$;

(4) conditions (3.59) and (3.60) ensure respectively the positivity of the nominal interest rate and the transversality condition: $\mu/\beta_0 > 1/\beta_0 > g^{(\sigma-1)/\sigma} = 1$.




$\frac{\partial U_0}{\partial \mu} > 0$ and $\frac{\partial U_1}{\partial \mu} < 0$ $\frac{\partial U_0}{\partial \gamma} > 0$ and $\frac{\partial U_1}{\partial \gamma} > 0$

As in the case where the capitalists' liquidity constraint is not binding, we find a positive impact of money growth of patient agents' welfare and a negative impact on impatient agents' welfare (see (3.37)).

The underlying mechanisms are substantially the same. The patient agent receives more money today and bears a low opportunity cost of holding money, because of the collaterals. The impatient agents bears a higher opportunity cost in order to finance future consumption with money because of the lack of collaterals. This opportunity cost is given by the nominal interest rate and increases with the monetary growth rate because of the inflation. Her/his cost is greater than the benefit of additional monetary transfers to impatient agents from the monetary authority.

The impact of the sensitivity degree $\gamma$ is positive on both the welfare levels essentially because of the positive effect on the growth rate.

### 3.4.6 Stabilization policy

We evaluate the characteristic polynomial $P(z) \equiv z^2 - Tz + D$ at $-1$ and 1. Along the line $AC$, one eigenvalue is equal to 1, *i.e.* $P(1) = 1 - T + D = 0$. Along the line $AB$, one eigenvalue is equal to $-1$, *i.e.* $P(-1) = 1 + T + D = 0$. On the segment $[BC]$, the two eigenvalues are complex and conjugate with unit modulus, *i.e.* $D = 1$ and $|T| < 2$. Therefore, inside the triangle $ABC$, the steady state is a sink, *i.e.* locally indeterminate ($D < 1$ and $|T| < 1 + D$). It is a saddle point if $(T, D)$ lies on the right sides of both $AB$ and $AC$ or on the left sides

of both of them ($|1+D| < |T|$). It is a source otherwise. A (local) bifurcation arises when an eigenvalue crosses the unit circle, that is, when the pair $(T, D)$ crosses one of the loci $AB$, $AC$ or $[BC]$. $(T, D)$ depends on the structural parameters. We choose and vary a parameter of interest and we observe how $(T, D)$ moves in the $(T, D)$-plane. More precisely, according to the changes in the bifurcation parameter, a saddle-node bifurcation (generically) occurs when $(T, D)$ goes through $AC$, a flip bifurcation (generically) arises when $(T, D)$ crosses $AB$, whereas a Hopf bifurcation (generically) happens when $(T, D)$ goes through the segment $[BC]$.

Let us introduce the reduced variables:

$$(y_0, z_0, a_0) \equiv \left( \frac{n_0 x_0}{n_0 x_0 + \gamma n_1}, \frac{n_0 x_0}{n_0 x_0 + n_1 x_1}, \frac{(1-\mu)\,\theta_1 z_0}{\mu z_1 + (1-\mu)\,\theta_1} \right)$$

$$\begin{aligned} y_1 &\equiv 1 - y_0 \\ z_1 &\equiv 1 - z_0 \qquad (3.69) \\ a_1 &\equiv 1 - a_0 \qquad (3.70) \end{aligned}$$

and the elasticities of the growth rate $g(x_0, x_1)$:

$$(\varepsilon_0, \varepsilon_1) \equiv \left( \frac{\partial g}{\partial x_0} \frac{x_0}{g}, \frac{\partial g}{\partial x_1} \frac{x_1}{g} \right) \qquad (3.71)$$

**Lemma 6** *The linearization of the two-dimensional system (3.45)-(3.46) around the steady state (3.52)-(3.51) yields the Jacobian matrix:*

$$\begin{aligned} J \;=\; & \begin{bmatrix} z_0 - a_0 & a_0 - z_0 \\ a_0 + \frac{\sigma-1}{\sigma}\varepsilon_0 - \frac{y_0}{\sigma} & 1 - a_0 - \frac{\sigma-1}{\sigma}\varepsilon_0 \end{bmatrix}^{-1} \\ & \begin{bmatrix} z_0 & 1 - z_0 \\ h(a_0 + \varepsilon_0) - \frac{y_0}{\sigma} & h(1 - a_0 - \varepsilon_0) \end{bmatrix} \end{aligned} \qquad (3.72)$$

**Proof.** See the Appendix.

The determinant and the trace of this Jacobian matrix are, respectively:

$$\begin{aligned} D &= \frac{y_0(1-z_0)+h\sigma(z_0-a_0-\varepsilon_0)}{(\sigma-y_0)(z_0-a_0)} \\ T &= 1+D+\frac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0 \end{aligned}$$

Consider the line $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma > 0, \text{ given } (h, \varepsilon_0, a_0, y_0, z_0)\}$ in the $(T, D)$-plane. Let

$$\begin{aligned} (T_0, D_0) &\equiv (T(0), D(0)) \\ (T_\infty, D_\infty) &\equiv \lim_{\sigma \to +\infty} (T(\sigma), D(\sigma)) \end{aligned}$$

The following lemma characterizes the behavior of alternative economies obtained by varying the elasticity of intertemporal substitution.

**Lemma 7** *In case $c_{0t}/k_{0t} > \gamma$, we have*

$$\begin{aligned} D_0 &< \min\{1, T_0 - 1\} \\ D_\infty &> \max\{1, T_\infty - 1\} \end{aligned}$$

*Moreover, $D'(\sigma) > 0$ iff $S \in (0,1)$.*

**Proof.** See the Appendix.

Let us introduce the bifurcation value:

$$\sigma^* \equiv \frac{1}{1+h}\frac{1+2y_0\left(\frac{z_1}{\varepsilon_0}+\frac{a_0-z_0}{\varepsilon_0}\right)}{1+2\frac{a_0-z_0}{\varepsilon_0}} = \frac{1}{1+h}\frac{1+2y_0\left(\frac{z_1}{\varepsilon_0}+g\frac{z_1}{x_1}\right)}{1+2g\frac{z_1}{x_1}}$$

**Proposition 8** *We always have $D'(\sigma) < 0$ with either $S < 0$ or $S > 1$. The steady state is a saddle point for $\sigma \in (0, \sigma^*)$ (equilibrium indeterminacy) and a source for $\sigma > \sigma^*$ (determinacy). Moreover, at $\sigma = \sigma^*$ the system generically undergoes a flip bifurcation and two-period cycles arise.*

**Proof.** Simply observe that $\sigma^*$ solves $D(\sigma) = -T(\sigma) - 1$ and apply Lemma 7. ■

**Numerical example**

It is possible to keep the growth rate $g$ constant by conveniently rescaling the TFP $A$. Fixing

$$A \equiv \frac{1}{s}\left[\frac{1}{\beta_0} - \Delta - \gamma\left(\frac{\mu}{\beta_0} - 1\right)\right] \tag{3.73}$$

we obtain $g = 1$, whatever $\sigma$. In this particular case, we can change $\sigma$ without affecting $(x_0, x_1, y_0, y_1, z_0, z_0, a_0, a_1)$. In order to set a positive TFP, we assume also

$$\gamma < \frac{1/\beta_0 - \Delta}{\mu/\beta_0 - 1}$$

In this case, the flip bifurcation point becomes

$$\sigma^* = \frac{1-\gamma\mu}{2-\gamma\mu}\frac{1 + 2\left(\frac{z_1}{x_1} + \frac{z_1}{\varepsilon_0}\right)y_0}{1 + 2\frac{z_1}{x_1}}$$

since $h = 1/(1-\gamma\mu)$.

Let us now address the stabilization issue. What is the impact of $\mu$ and $\gamma$ on the indeterminacy range?

Let us fix the parameters as follows according to yearly data:

| $s$ | $n_0$ | $n_1$ | $\beta_0$ | $\Delta$ | $\theta_0$ |
|---|---|---|---|---|---|
| 2/3 | 1 | 1 | 0.9 | 0.9 | 1/2 |

Notice that $s$ is set as usual in endogenous growth literature (see Mankiw *et al.* (1992)).

Focus on the impact of $\mu$ and $\gamma$ around

| $\mu$ | $\gamma$ |
|---|---|
| 1.04 | 0.01 |

All the restrictions are satisfied around $(\gamma, \mu) = (0.01, 1.04)$:

(1) Assumption 3: $\gamma = 0.01 < \min\{1/\mu, \Delta + sA\} = 0.961\,5$;

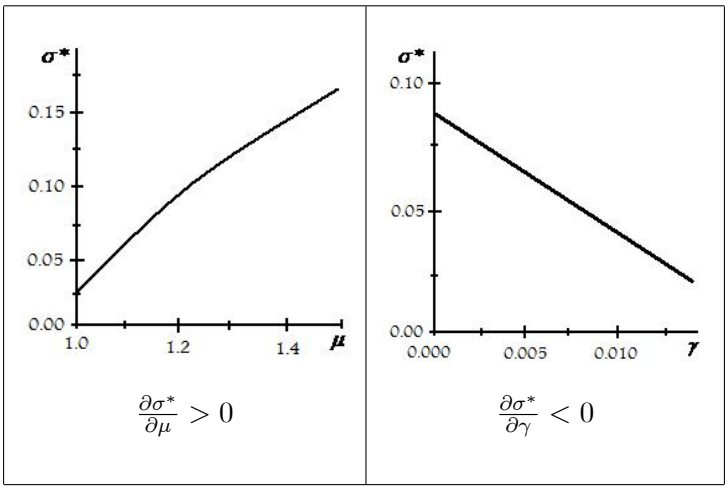
(2) condition (3.38) ensures that collaterals matter:

$$\begin{aligned}\frac{c_{0t}}{k_{0t}} &= \frac{n_1}{n_0}\left(\gamma + \frac{\left[1+(\mu-1)\,\theta_0\right](\Delta + A - 1 - \gamma) - (1-s)\,A}{\mu}\right) \\ &= 0.1097 > \gamma = 0.01\end{aligned}$$

(3) conditions (3.55) guarantees the positivity of variables $x_0$ and $x_1$: $g < \Delta + A - \gamma - (1-s)\,A/\left[1+(\mu-1)\,\theta_0\right] = 1.1016$;

(4) conditions (3.59) and (3.60) ensure respectively the positivity of the nominal interest rate and the transversality condition: $\mu/\beta_0 > 1/\beta_0 > g^{(\sigma-1)/\sigma} = 1$.

In the case of a normalized steady state (scaling factor (3.73)), we can plot $\sigma^*$ as a function of $\mu$ and $\gamma$.




$\frac{\partial \sigma^*}{\partial \mu} > 0$ $\qquad$ $\frac{\partial \sigma^*}{\partial \gamma} < 0$

This numerical exercise shows that the indeterminacy range widens with $\mu$ and shrinks with $\gamma$.

Indeed, a higher monetary growth entails a higher opportunity cost of holding money (inflation rate) and a heavier burden on liquidity-constrained agents. The larger the credit market imperfection, the more likely the occurrence of endogenous fluctuations because of the local indeterminacy.

Conversely the sensitivity of credit grants to collaterals $\gamma$ reduces the market imperfection, because, when $\gamma$ increases, collaterals matter more and the liquidity constraint becomes lighter for patient agents.

## 3.5 Conclusion

In this chapter, we have considered a Ramsey model with heterogenous agents and liquidity constraint. The heterogenous discount rates imply a social segmentation in the long-run: patient agents end up to hold the total stock of capital. Money is needed to finance future consumption when the agents has a small amount of collaterals. Thus, impatient agents hold money.

In this economy, two regimes can arise according to the possibility that patient agents hold balances or do not.

We study the role of monetary policy $(\mu, \theta)$ and market regulation $(\gamma)$ in both cases.

(1) Patient agents buy on credit and money is needed only by workers. Money growth improves the patient agents' condition (they receive balances), but worsens that of impatient agents, who bear the opportunity cost of holding money (the higher inflation, the higher the interest rate).

(2) Patient agents needs money to consume. Monetary growth is growth-enhancing because under a higher inflation, patient agents renounce to balances, consume less and accumulate more capital. The higher sensitivity of credit grant to collaterals $(\gamma)$ also promote economic growth, because patient agents accumulate more capital to collateralize the consumption purchases and to reduce money demand.

As above, an expansionary monetary policy improves the welfare of capitalists because of economic growth, and reduces that of workers because of inflation. The impact of the sensitivity degree $\gamma$ is positive on both the welfare levels essentially because of the positive effect on the growth rate.

The monetary authority can have different objectives. We have seen the growth rate and the welfare level of social classes. Stabilization is also an important concern in the authority's agenda.

We show that the indeterminacy range widens with $\mu$ and shrinks with $\gamma$. Indeed, a higher monetary growth raises the inflation rate and makes the liquidity constrained heavier, that is the credit market imperfection stronger. Conversely the sensitivity of credit grants to collaterals $\gamma$ reduces the market imperfection, because, when $\gamma$ increases, collaterals matter more and the capitalists' liquidity constraint becomes lighter.

## 3.6 Appendix

### 3.6.1 Proof of Proposition 3

Reconsider the equilibrium budget constraints. Replacing $k_{0t} = k_t n_1/n_0$, (3.18) and (3.42) in (3.40), and (3.16), (3.18) and (3.42) in (3.15) we obtain, respectively:

$$\begin{aligned} & \left[(1-\theta_0)\, n_0 m_{0t+1} - \theta_0 n_1 m_{1t+1}\right] \pi_{t+1} + \theta_0 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \;& n_1 \left[(\Delta + r)\, k_t - k_{t+1}\right] + n_0 \left(m_{0t} - c_{0t}\right) \qquad (3.74) \\ & \left[(1-\theta_1)\, n_1 m_{1t+1} - \theta_1 n_0 m_{0t+1}\right] \pi_{t+1} + \theta_1 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \;& n_1 w_t \qquad (3.75) \end{aligned}$$

In the second regime (capitalists hold money), the CIA constraint is binding:

$$c_{0t} = m_{0t} + \gamma k_t n_1/n_0 \qquad (3.76)$$

and (3.74) reduces to

$$\begin{aligned} & \left[(1-\theta_0)\, n_0 m_{0t+1} - \theta_0 n_1 m_{1t+1}\right] \pi_{t+1} + \theta_0 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \;& n_1 \left[(\Delta + r - \gamma)\, k_t - k_{t+1}\right] \qquad (3.77) \end{aligned}$$

(3.21) becomes

$$n_0 m_{0t} + n_1 m_{1t} = \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right) \pi_{t+1}/\mu \qquad (3.78)$$

Replacing (3.78) in (3.77) and (3.75), we get, respectively,

$$\begin{aligned} & \left[n_0 m_{0t+1} + \frac{1-\mu}{\mu} \theta_0 \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right)\right] \pi_{t+1} \\ = \;& n_1 \left[(\Delta + sA - \gamma)\, k_t - k_{t+1}\right] \qquad (3.79) \\ & \left[n_1 m_{1t+1} + \frac{1-\mu}{\mu} \theta_1 \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right)\right] \pi_{t+1} \\ = \;& n_1 \left(1-s\right) A k_t \qquad (3.80) \end{aligned}$$

Under Assumption 4 and constraint (3.76), patient agents' Euler equation (3.39) becomes

$$\frac{n_0 m_{0t+1} + \gamma n_1 k_{t+1}}{n_0 m_{0t} + \gamma n_1 k_t} = \left(\beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + r - \gamma}{1 - \gamma \pi_t}\right)^{\sigma} \tag{3.81}$$

Dividing (3.79), (3.80) and (3.81) by $k_{t+1}$ and replacing (3.27), we obtain at time $t$:

$$\left[n_0 x_{0t} + \frac{1-\mu}{\mu}\theta_0 \left(n_0 x_{0t} + n_1 x_{1t}\right)\right] \pi_t = n_1 \left(\frac{\Delta + sA - \gamma}{g_t} - 1\right) \tag{3.82}$$

$$\left[n_1 x_{1t} + \frac{1-\mu}{\mu}\theta_1 \left(n_0 x_{0t} + n_1 x_{1t}\right)\right] \pi_t = \frac{n_1 (1-s) A}{g_t} \tag{3.83}$$

$$\frac{n_0 x_{0t} + \gamma n_1}{n_0 x_{0t+1} + \gamma n_1} \left(\beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t}\right)^{\sigma} = g_{t+1} \tag{3.84}$$

Dividing (3.82) and (3.83) side by side, we get (3.47).

From (3.78) and (3.80), we have (3.45). Deriving $\pi_t$ from (3.83) and replacing $\pi_t$ and $\pi_{t+1}$ in (3.84), we obtain (3.46). ■

### 3.6.2 Proof of Proposition 4

Focus first on equation (3.49), that is $g = \varphi(g)$.

Let $\sigma < 1$. Under the Assumption 3, we have $\lim_{g \to 0^+} \varphi(g) = +\infty$ and $\lim_{g \to +\infty} \varphi(g) = 0^+$. Since the RHS of (3.49) is decreasing, it crosses once the LHS (the bisector of the first orthant) at $g > 0$. We immediately obtain the existence and the uniqueness of the BGP.

Let now $\sigma > 1$. In this case, $\varphi'(g) > 0$, $\varphi''(g) < 0$ for every $g > 0$; $\lim_{g \to 0^+} \varphi(g) = \gamma\mu > 0$ and $\lim_{g \to +\infty} \varphi'(g) = 0 < 1$. In other terms, $\varphi$ crosses once the LHS of (3.49).

If $\mu \geq 1$, (3.55) ensures the positivity of $x_0$ and $x_1$. Finally, (3.60) guarantees that the EGTCs are satisfied along the BGP and that, under $\mu \geq 1$, (3.59) holds, that is the nominal interest rate is positive. ■

### 3.6.3 Proof of Proposition 5

Focus first on (3.49). Totally differentiating with respect to $\gamma$, $\mu$, $g$ and noticing that

$$\beta_0 g^{1-1/\sigma} = \frac{g - \gamma\mu}{\Delta + sA - \gamma} \tag{3.85}$$

we get

$$\frac{g}{h}\left[h\pi - \frac{1}{\Delta + sA - \gamma}\right] d\gamma + \gamma d\mu + \left(\frac{\sigma - 1}{\sigma}\frac{1}{h} - 1\right) dg = 0 \tag{3.86}$$

Setting $d\gamma = 0$ in (3.86), we get the partial derivative (3.62): the impact of monetary policy on the growth rate is always positive because $h > 1$.

Setting $d\mu = 0$ in (3.86), we obtain (3.64). Under Assumption 3, the impact of deregulation is positive ($\partial g/\partial \gamma > 0$) because the nominal interest rate is positive: $\pi(\Delta + sA) > 1$. ■

### 3.6.4 Proof of Lemma 6

Equation (3.48) reduces to

$$\mu z_1 + (1 - \mu)\theta_1 = \frac{n_1(1 - s)A}{n_0 x_0 + n_1 x_1} \tag{3.87}$$

Linearizing equation (3.45) and using (3.87), we obtain

$$(z_0 - a_0)\frac{dx_{0t+1}}{x_0} - (z_0 - a_0)\frac{dx_{1t+1}}{x_1} = z_0\frac{dx_{0t}}{x_0} + z_1\frac{dx_{1t}}{x_1} \tag{3.88}$$

In order to linearize the second equation, we apply the elasticities (3.71). Under equation (3.48), equation (3.46), evaluated at the steady state, writes:

$$1 - \gamma\mu/g = \beta_0(sA + \Delta - \gamma)g^{-1/\sigma} \tag{3.89}$$

After some computations, we obtain using equations (3.69), (3.87) and (3.89):

$$\begin{aligned} & \left(a_0 + \varepsilon_0 - \frac{y_0 + \varepsilon_0}{\sigma}\right)\frac{dx_{0t+1}}{x_0} + \left(a_1 + \varepsilon_1 - \frac{\varepsilon_1}{\sigma}\right)\frac{dx_{1t+1}}{x_1} \\ = & \left[h(a_0 + \varepsilon_0) - \frac{y_0}{\sigma}\right]\frac{dx_{0t}}{x_0} + h(a_1 + \varepsilon_1)\frac{dx_{1t}}{x_1} \end{aligned} \tag{3.90}$$

where $h$ is given by (3.63).

System (3.88)-(3.90) is represented by the following Jacobian matrix:

$$J = \begin{bmatrix} z_0 - a_0 & a_0 - z_0 \\ a_0 + \frac{\sigma-1}{\sigma}\varepsilon_0 - \frac{y_0}{\sigma} & a_1 + \frac{\sigma-1}{\sigma}\varepsilon_1 \end{bmatrix}^{-1} \begin{bmatrix} z_0 & z_1 \\ h(a_0 + \varepsilon_0) - \frac{y_0}{\sigma} & h(a_1 + \varepsilon_1) \end{bmatrix} \tag{3.91}$$

From (3.47), noticing that

$$g = \Delta + sA - \gamma - (1-s)\,A\frac{\mu n_0 x_0 + (1-\mu)\,\theta_0\,(n_0 x_0 + n_1 x_1)}{\mu n_1 x_1 + (1-\mu)\,\theta_1\,(n_0 x_0 + n_1 x_1)}$$

and using $\theta_0 + \theta_1 = 1$ and (3.48), we compute the elasticities $\varepsilon_0$ and $\varepsilon_1$:

$$\varepsilon_1 = -\varepsilon_0 = \frac{x_1}{g}\frac{\mu z_0}{\mu z_1 + (1-\mu)\,\theta_1} \tag{3.92}$$

Replacing (3.69), (3.70) and (3.92) in (3.91), we get (3.72). ■

### 3.6.5 Proof of Lemma 7

A convenient parameter for the geometrical analysis is the elasticity of intertemporal substitution $\sigma$.

Fix the steady state $g$ by scaling parameters such as the TFP $A$.

The origin of the line is given by

$$\begin{aligned} D_0 &= \frac{1-z_0}{a_0 - z_0} = 1 - \frac{\theta_1 + (1-\theta_1)\,\mu}{\mu z_0} < 1 \\ T_0 &= 1 + D_0 + \frac{\varepsilon_0}{y_0\,(a_0 - z_0)} = 1 + D_0 + \frac{1}{g}\frac{x_1}{y_0 z_1} \end{aligned}$$

Then $D_0 < T_0 - 1$.

The endpoint is given by

$$\begin{aligned} T_\infty &\equiv \lim_{\sigma\to+\infty} T(\sigma) = 1 + D + \frac{1+(h-1)\,\sigma}{(\sigma - y_0)\,(z_0 - a_0)}\varepsilon_0 = 1 + h + \frac{\varepsilon_0}{a_0 - z_0} \\ &= 1 + h + \frac{1}{g}\frac{x_1}{z_1} = 1 + D_\infty - (h-1)\,\frac{1}{g}\frac{x_1}{z_1} \\ D_\infty &\equiv \lim_{\sigma\to+\infty} D(\sigma) = h\frac{z_0 - a_0 - \varepsilon_0}{z_0 - a_0} = h\left(1 + \frac{\varepsilon_0}{a_0 - z_0}\right) = h\left(1 + \frac{1}{g}\frac{x_1}{z_1}\right) > 1 \end{aligned}$$

Then

$$\begin{aligned} D_\infty &> 1 \\ D_\infty &> T_\infty - 1 \end{aligned}$$

The slope of $\Sigma$ is given by

$$\begin{aligned} S &= \frac{D'(\sigma)}{T'(\sigma)} = \frac{-\left(h\left(z_0 - a_0 - \varepsilon_0\right) + \left(1 - z_0\right)\right) y_0}{-\left(\left(h\left(z_0 - a_0 - \varepsilon_0\right) + \left(1 - z_0\right)\right) y_0 + \left(\left(h-1\right) y_0 + 1\right) \varepsilon_0\right)} \\ &= \left[1 + \frac{\frac{1}{y_0} + h - 1}{\frac{1-z_0}{\varepsilon_0} - h\left(1 + \frac{a_0 - z_0}{\varepsilon_0}\right)}\right]^{-1} = \left[1 + \frac{\frac{1}{y_0} + h - 1}{\frac{1-z_0}{\varepsilon_0} - h\left(1 + g\frac{z_1}{x_1}\right)}\right]^{-1} \end{aligned}$$

and does not depend directly on $\sigma$ (only through $g$).

There are two cases:

$$\begin{aligned} 1 &: h > \frac{z_1}{\varepsilon_0\left(1 + g\frac{z_1}{x_1}\right)} \\ 2 &: h < \frac{z_1}{\varepsilon_0\left(1 + g\frac{z_1}{x_1}\right)} \end{aligned}$$

In case (1), either $S < 0$ or $S > 1$.

In case (2), $S \in (0, 1)$.

How does $(T, D)$ move along $\Sigma$? We notice that

$$\begin{aligned} D'(\sigma) &= \frac{\varepsilon_0}{a_0 - z_0} \frac{y_0}{\left(\sigma - y_0\right)^2} \left(\frac{z_1}{\varepsilon_0} - h\left(1 + \frac{a_0 - z_0}{\varepsilon_0}\right)\right) \\ &= \frac{x_1 y_0}{g z_1 \left(\sigma - y_0\right)^2} \left(\frac{z_1}{\varepsilon_0} - h\left(1 + g\frac{z_1}{x_1}\right)\right) \\ &= \frac{S}{1 - S} \frac{x_1}{z_1} \frac{1 + \left(h - 1\right) y_0}{g\left(\sigma - y_0\right)^2} \end{aligned}$$

Then $D'(\sigma) > 0$ iff

$$\frac{S}{1 - S} > 0$$

that is iff $S \in (0, 1)$. ■

# Chapter 4

# Transaction Costs in an Overlapping Generations Model

## 4.1 Introduction

It is well known that transaction costs in asset and stock markets are considered as important factors in determining the investment portfolio. Without these costs agents can take positions in all existing assets, while the introduction of transaction costs discourages agents from purchasing these assets. For instance, if we consider two different assets without transaction costs, the portfolio choice would be a segment of these two assets. However, in the presence of transaction costs, the investment choice mainly moves to the assets with lower costs. This shows that costs have negative effect on assets' demand.

Transaction costs might consist of communication and time costs, government fees, stamp taxes, information and search costs, administration costs and brokerage commissions. Throughout the literature authors are usually interested in studying how transaction costs can influence portfolio choice and stock pricing. They also seek to explain why do not all households take positions in the stock market. They perform portfolio models based on stock market transaction costs in order to match the observed household's participation rate in the data. Among others, Alan (2006) supposes that costs are paid only one time over the entire life cycle. Once the costs paid, household is free to re-enter the stock market. He estimates that in the absence of costs, 80 to 90 percent of households participate in the stock market, while with only 1 percent costs,

the percentage of households declines to around 30 percent. Constantinides (1986) argues that proportional costs have only a small impact on asset prices. Lo *et al.* (2004) consider a dynamic equilibrium model of trade volume and asset prices when agents face fixed costs. They show that the presence of these costs allows agents to trade infrequently. Further, Vayanos (1998) demonstrates that an increase in transaction costs has two opposite effects on stock prices. On the one hand, agents buy fewer stocks and on the other hand, they hold them for longer periods. More recently, Vissing-Jorgensen (2002) considers several types of costs: fixed entry costs, proportional and per period costs. Using US data from 1989-1994, he estimates that 50 dollar of transaction costs are sufficient to explain the choices of half of the households who do not invest in the stock market. For the same period, 260 dollar of transaction costs is enough to explain the choices of 75 percent of the nonparticipants.

In the view of above papers, the study of transaction costs seems to be plausible and relevant. However, the effect of transaction costs on economic stability or capital accumulation has not been yet treated in literature. Thus, this chapter fills this gap by introducing transaction costs on saving in a standard Diamond OLG model. Contrary to infinite-horizon Ramsey model, the use of OLG framework allows formulating and studying the saving function explicitly. In the standard Diamond model, agents live two periods: youth and adulthood. In the first period, young agents supply labor inelastically and allocate their wage income between consumption and savings. When they are old, they are retired and consume their savings entirely.[1]In this chapter, it is assumed that young agents have to pay transaction costs related to their level of capital accumulation. Namely, these costs increase with the amount of savings but its marginal increase declines (concave transaction costs).

Our main objective is to analyze the impact of transaction costs on dynamic efficiency, capital accumulation and economic stability. In addition, the robustness of our results is shown by considering two forms of utility function: non-separable and separable in both periods consumption.

Our first result states that, under standard conditions, the steady state may be dynamically inefficient (efficient) if there is an over-accumulation (under-accumulation) of capital with

[1]Geanakoplos and Polemarchakis (1986), Reichlin (1986) and Cazzavillan (2001) suppose that agents consume only in the second-period.

respect to the modified Golden Rule, *i.e.* the net returns of capital are lower (higher) than the gross rate of population growth. Further, the presence of costs has a negative impact on capital accumulation.

From a stability point of view, the steady state of the standard Diamond model exhibits unique path stability if savings increase with the rate of return. However, it is possible to have a global indeterminate steady-state whenever the agent's saving is supposed to be a decreasing function of the interest rate (Galor and Ryder 1989).

This chapter is mainly interested in the local stability properties of the steady state of Diamond model augmented to include transaction costs, without any restriction on the saving function. It is demonstrated that the steady state is determinate where there is one trajectory that converges to a unique steady state. However, it changes its stability through cycles of period two. It is found that these cycles require a sufficiently high sensitivity of transaction costs with respect to savings, low elasticity of marginal utility with respect to future consumption, high elasticity of marginal utility with respect to current consumption and a high first-period consumption share.

The intuition of these cycles is given as follows: assume that the level of current capital increases from its steady state value. This leads wage income to rise which induces more capital accumulation. However, there are some factors that influence capital accumulation negatively. The presence of high transaction costs associated with savings, the existence of low elasticity of marginal utility of future consumption, high sensitivity of marginal utility of current consumption and high consumption share enforce agents to accumulate low capital. Cycles of period two are obtained whenever the latter effects dominate the former one. Therefore, higher transaction costs make the appearance of cycles of period two more likely. In particular, we show that the range of parameters giving rise to cycles of period two widens with transaction costs.

The remainder of the chapter is arranged as follows. In section 2, we present the model with a transaction costs function (the optimization problem of households and firms). The intertemporal equilibrium is presented in section 3. We present the steady state analysis in section 4. The dynamic efficiency of the intertemporal equilibrium is studied in section 5 while in section 6, we present the local dynamics. A numerical example is located in section 7 and

we conclude in section 8.

## 4.2 The model

Consider a non-monetary overlapping generations economy with identical agents who live two periods. In each period $t$, $N_t$ individuals are born and they live for two periods "young and old". In this model, at each period, there is a unique good that can be either consumed or invested. In the first period, agents are endowed with one unit of labor which is supplied inelastically to firms. They choose their amounts of consumption and saving along with income. In addition, agents have to pay variable costs "transaction costs" related to saving amount. In the second period, they do not work and their income comes from the return of first-period saving.

Given the real wage $w_t$ and the real return factor $R_{t+1}$, agents allocate savings and consumptions for both periods to maximize the intertemporal following preferences:

$$u\left(c_t, d_{t+1}\right) \tag{4.1}$$

subject to the constraints

$$\begin{aligned} c_t + s_t + \xi\left(s_t\right) &\leq w_t && (4.2)\\ d_{t+1} &\leq R_{t+1} s_t && (4.3)\\ c_t &\geq 0,\ d_{t+1} \geq 0 \text{ for all } t \geq 0 \end{aligned}$$

where $c_t, d_{t+1}$ is the consumption in first "young" and second "old" period respectively, $s_t$ is the saving, $\xi\left(s_t\right)$ is the transaction cost associated with saving.[2]

**Assumption 1** *$u\left(c_t, d_{t+1}\right)$ is strictly increasing with respect to each argument $u_1\left(c_t, d_{t+1}\right) > 0$, $u_2\left(c_t, d_{t+1}\right) > 0$, concave $u_{11}\left(c_t, d_{t+1}\right) < 0$, $u_{22}\left(c_t, d_{t+1}\right) < 0$, $C^2$ over the interior of the set $R_+^2 = [0, +\infty)\ x\ [0, +\infty)$. Additionally, $\lim_{d_{t+1} \to +\infty} u_1\left(c_t, d_{t+1}\right) / u_2\left(c_t, d_{t+1}\right) = +\infty$ and $\lim_{d_{t+1} \to 0^+} u_1\left(c_t, d_{t+1}\right) / u_2\left(c_t, d_{t+1}\right) = 0$ for all $c_t, d_{t+1} > 0$. Furthermore, $\lim_{c_t \to 0^+} u_1\left(c_t, d_{t+1}\right) / u_2\left(c_t, d_{t+1}\right) = +\infty$ and $\lim_{c_t \to +\infty} u_1\left(c_t, d_{t+1}\right) / u_2\left(c_t, d_{t+1}\right) = 0$ for all $c_t, d_{t+1} > 0$.*[3]

[2] As in standard two-period OLG models, we assume a full depreciation rate that is $s_t = K_{t+1}$.

[3] Notice that $u_1\left(c_t, d_{t+1}\right) \equiv \partial u\left(c_t, d_{t+1}\right) / \partial c_t$, $u_2\left(c_t, d_{t+1}\right) \equiv \partial u\left(c_t, d_{t+1}\right) / \partial d_{t+1}$.

For future reference, we propose some necessary elasticities: the elasticity of marginal utility with respect to first and second argument are respectively $\varepsilon_{11} \equiv u_{11}(c,d)\, c/u_1(c,d) < 0$, $\varepsilon_{22} \equiv u_{22}(c,d)\, d/u_2(c,d) < 0$. The cross elasticities in consumption are $\varepsilon_{21} \equiv u_{21}(c,d)\, c/u_2(c,d)$, $\varepsilon_{12} \equiv u_{12}(c,d)\, d/u_1(c,d)$.

**Assumption 2** *The cost function is increasing in its argument* $\xi'(s_t) > 0$ *and concave* $\xi''(s_t) < 0$.

In order to simplify the notation, we suppose[4] $\varphi(s_t) = s_t + \xi(s_t)$. The Lagrangian function for household problem is:

$$Lag = u(c_t, d_{t+1}) + \lambda_t (w_t - c_t - \varphi(s_t)) + \mu_t (R_{t+1} s_t - d_{t+1}) \tag{4.4}$$

The first-order conditions with respect to $c_t$, $d_{t+1}$ and $s_t$ are respectively:

$$u_1(c_t, d_{t+1}) = \lambda_t \tag{4.5}$$

$$u_2(c_t, d_{t+1}) = \mu_t \tag{4.6}$$

$$\mu_t R_{t+1} = \varphi'(s_t)\, \lambda_t \tag{4.7}$$

this gives:

$$\frac{u_1(c_t, d_{t+1})}{u_2(c_t, d_{t+1})} = \frac{R_{t+1}}{\varphi'(s_t)} \tag{4.8}$$

The LHS is simply the marginal rate of substitution between consumption "today" and consumption "tomorrow". Due to the existence of increasing cost, the associated marginal rate of substitution is smaller than that of the standard Diamond model. In other words, the interest rate factor is higher than that of standard Diamond without costs which implies that agents accumulate lower capital. The model of Diamond (1965) is obtained by setting $\varphi(s_t) = s_t$.

On the production side, a representative firm uses labor and capital to produce final goods using constant returns-to-scale technology $AF(K_t, L_t)$ with $A > 0$ is a productivity scaling factor. Let $a_t = K_t/L_t$ be the capital stock per labor unit, then the production function can be written as $Af(a_t) = AF(a_t, 1)$.

[4] Since $\xi(s_t)$ is increasing, the function $\varphi(s_t)$ has the same properties of $\xi(s_t)$ as mentioned in Assumption 2.

**Assumption 3** *Let $a \geq 0$, the technology $f(a)$ is continuous and differentiable. It is increasing $f'(a) > 0$ and concave $f''(a) < 0$. Furthermore, $f(0) = 0$, $\lim_{a \to 0^+} f'(a) = +\infty$ and $\lim_{a \to +\infty} f'(a) = 0$.*

Each representative firm takes real wages $w_t$ and rental prices $R_t$ as given. If we set $\rho(a_t) = f'(a_t)$ and $\omega(a_t) = f(a_t) - a_t f'(a_t)$, then the competitive equilibrium conditions for profit maximization entail that the real interest rate and the real wage satisfy:

$$R_t = A\rho(a_t) \text{ and } w_t = A\omega(a_t) \tag{4.9}$$

Thus, we can deduce that the elasticity of interest rate $a\rho'(a)/\rho(a) = -(1-\alpha)/\sigma < 0$ and the elasticity of wage $a\omega'(a)/\omega(a) = \alpha/\sigma > 0$, with $\sigma \in (0, +\infty)$ is the elasticity of capital-labor substitution while $\alpha \in (0, 1)$ is the capital share in total income.

## 4.3 Intertemporal equilibrium

The number of households at each generation grows at a constant rate $n > -1$, such that $1 + n = N_{t+1}/N_t$, where $N_t$ is the number of population born at time $t$. At equilibrium, three markets clear:

1. Capital market clears according to capital-accumulation equation: $K_{t+1} = N_t s_t$.
2. Labor market clears: $L_t = N_t$.
3. By Walras' law, output market also clears: $N_t(c_t + \varphi(s_t)) + N_{t-1}d_t = AF(K_t, L_t)$.

From market clearing conditions, one can demonstrate that:

$$s_t = (1+n)\, a_{t+1} \tag{4.10}$$

Substituting (4.10) and condition (4.9) together with the binding budget constraints (4.2) and (4.3) into (4.8) yields the following one-dimensional dynamic system of $a$.

$$\frac{u_1\left[A\omega(a_t) - \varphi\left[a_{t+1}(1+n)\right], A\rho(a_{t+1})\, a_{t+1}(1+n)\right]}{u_2\left[A\omega(a_t) - \varphi\left[a_{t+1}(1+n)\right], A\rho(a_{t+1})\, a_{t+1}(1+n)\right] A} - \frac{\rho(a_{t+1})}{\varphi'\left[a_{t+1}(1+n)\right]} = 0 \tag{4.11}$$

## 4.4 The steady state

At the steady state $a_{t+1} = a_t = a$, so the dynamic system (4.11) becomes:

$$\frac{u_1\left[A\omega\left(a\right) - \varphi\left[a\left(1+n\right)\right], A\rho\left(a\right)a\left(1+n\right)\right]}{u_2\left[A\omega\left(a\right) - \varphi\left[a\left(1+n\right)\right], A\rho\left(a\right)a\left(1+n\right)\right]A} - \frac{\rho\left(a\right)}{\varphi'\left[a\left(1+n\right)\right]} = 0 \tag{4.12}$$

To simplify the analysis, we follow the method initiated by Cazzavillan *et al.* (1998) by using a scaling parameter $A$ in order to give conditions for existence of a normalized steady state $a = 1$.

**Proposition 9** *Under Assumptions* $1-3$*,* $a = 1$ *is a steady state of the dynamic system (4.11) if and only if there exists a scaling parameter* $A$ *such that* $A > \hat{A} \equiv \varphi\left[1+n\right]/\omega\left(1\right)$ *and satisfies*

$$\frac{u_1\left[A\omega\left(1\right) - \varphi\left[1+n\right], A\rho\left(1\right)\left(1+n\right)\right]}{u_2\left[A\omega\left(1\right) - \varphi\left[1+n\right], A\rho\left(1\right)\left(1+n\right)\right]A} = \frac{\rho\left(1\right)}{\varphi'\left[1+n\right]} \tag{4.13}$$

*The scaling parameter* $A$ *is a unique solution of (4.13) if and only if:* $\left(\varepsilon_{11} - \varepsilon_{21}\right)/\gamma + \varepsilon_{12} - \varepsilon_{22} - 1 < 0$ *for all* $A$.[5]

**Proof.** The solution $a = 1$ is a steady state if and only if (4.13) is verified. Moreover, the positivity of first-period consumption requires $A > \hat{A} \equiv \varphi\left[1+n\right]/\omega\left(1\right)$, so $A \in \left(\hat{A}, +\infty\right)$. Let us call the LHS as:

$$G\left(A\right) \equiv \frac{u_1\left[A\omega\left(1\right) - \varphi\left[1+n\right], A\rho\left(1\right)\left(1+n\right)\right]}{u_2\left[A\omega\left(1\right) - \varphi\left[1+n\right], A\rho\left(1\right)\left(1+n\right)\right]A}$$

since it is a continuous function, then based on Assumption 1, it is easy to show that $\lim_{A\to\hat{A}} G\left(A\right) = +\infty$ and $\lim_{A\to+\infty} G\left(A\right) = 0$. Since the RHS is a positive constant, thus there exists a steady state at $a = 1$. Concerning the uniqueness of $A$, it is enough to show that $G\left(A\right)$ is monotonic (decreasing), *i.e.*, $\left(\varepsilon_{11} - \varepsilon_{21}\right)/\gamma + \varepsilon_{12} - \varepsilon_{22} - 1 < 0$ is satisfied for all $A$. ■

**Assumption 4** *The utility function* $u\left(c_t, d_{t+1}\right)$ *is homogenous of degree one.*

[5] *We denote* $\gamma \equiv c/A\omega\left(a\right)$ *as the share of first-period consumption over wage income.*

**Corollary 1** *Under Assumptions* $1-4$*,* $a=1$ *is a steady state for the dynamic system (4.11) if and only if there exists a scaling parameter* $A$ *such that* $A>\hat{A}\equiv\varphi\left[1+n\right]/\omega\left(1\right)$ *and satisfies*

$$\frac{u_1\left[\frac{A\omega(1)-\varphi[1+n]}{A\rho(1)(1+n)},1\right]}{u_2\left[\frac{A\omega(1)-\varphi[1+n]}{A\rho(1)(1+n)},1\right]A}=\frac{\rho\left(1\right)}{\varphi'\left[1+n\right]} \tag{4.14}$$

*In this case, the scaling parameter* $A$ *is unique.*

**Proof.** Since the utility function is homogenous of degree one, equality (4.12) can be written as (4.14). The solution $a=1$ is a steady state if and only if equality (4.14) is satisfied. Notice that the RHS does not change and $A\in\left(\hat{A},+\infty\right)$. If we denote the LHS by $Q$:

$$Q\left(A\right)\equiv\frac{u_1\left[\frac{\omega(1)}{\rho(1)(1+n)}-\frac{\varphi[1+n]}{A\rho(1)(1+n)},1\right]}{u_2\left[\frac{\omega(1)}{\rho(1)(1+n)}-\frac{\varphi[1+n]}{A\rho(1)(1+n)},1\right]A} \tag{4.15}$$

Using Assumption 1, it is easy to show that $\lim_{A\to\hat{A}}Q\left(A\right)=+\infty$ and $\lim_{A\to+\infty}Q\left(A\right)=0$. Further, from homogeneity property of the utility function, one can prove that $u_{21}>0$. Consequently, a direct inspection of (4.15) gives that $Q'\left(A\right)=\left[\left[\varphi\left[1+n\right]\left(u_2u_{11}-u_1u_{21}\right)/A\rho\left(1\right)\left(1+n\right)\right]-u_1u_2\right]/\left(u_2\right.$ 0 which implies that there is a unique scaling parameter $A$ satisfying (4.14). ■

**Assumption 5** *The utility function is separable.*

**Corollary 2** *Let Assumptions* $1-3$ *and* $5$ *be satisfied, then* $a=1$ *is a steady state for the dynamic system (4.11) if there exists a scaling parameter* $A$ *such that* $A>\hat{A}\equiv\varphi\left[1+n\right]/\omega\left(1\right)$ *and satisfies*

$$\frac{v'\left[A\omega\left(1\right)-\varphi\left[1+n\right]\right]}{v'\left[A\rho\left(1\right)\left(1+n\right)\right]}\frac{1}{A}=\frac{\beta\rho\left(1\right)}{\varphi'\left[1+n\right]} \tag{4.16}$$

where $\beta$ is the discount factor. *Further, the scaling parameter* $A$ *is unique.*

**Proof.** In a separable case, household's problem is simplified at the steady state to (4.16). As before, let us call the LHS as:

$$\Pi\left(A\right)\equiv\frac{v'\left[A\omega\left(1\right)-\varphi\left[1+n\right]\right]}{v'\left[A\rho\left(1\right)\left(1+n\right)\right]}\frac{1}{A}$$

and $A$ belongs to $\left(\hat{A}, +\infty\right)$, then based on Assumption 1 we have $\lim_{A\to\hat{A}}\Pi(A) = +\infty$ and $\lim_{A\to+\infty}\Pi(A) = 0$. In order to show the existence of a unique $A$, it is easy to demonstrate that $\Pi(A)$ is always decreasing *i.e.*, $\varepsilon(1/\gamma - 1) - 1 < 0$, where $\varepsilon$ is the elasticity of marginal utility of consumption. ■

Throughout the chapter, it is supposed that the above propositions hold for each configuration.

## 4.5 Dynamic efficiency

In this section, we analyze the dynamic efficiency of the steady state. Before passing through efficiency analysis, let us define the following useful elasticities: the elasticity of transaction costs with respect to savings $\eta_1 \equiv \varphi'(s)\, s/\varphi(s) > 0$, the elasticity of marginal transaction costs $\eta_2 \equiv \varphi''(s)\, s/\varphi'(s) < 0$.

Using above intertemporal equilibrium conditions, we obtain the following stationary resource constraint:

$$c + \frac{d}{1+n} = \Sigma(a) \tag{4.17}$$

with

$$\Sigma(a) \equiv Af(a) - \varphi[(1+n)\, a] \tag{4.18}$$

is the net production and the LHS is simply the stationary aggregate consumption.[6]

**Assumption 6** *Assume that* $-(1-\alpha)/\sigma < \eta_2$.

This assumption is necessary to confirm that the net production $\Sigma(a)$ is concave. Subsequently, this ensures the existence of a unique positive capital-labor ratio that maximizes the net production and so allocates the maximum amount of consumptions.[7]In order to characterize the modified Golden Rule capital-labor ratio, we need to make the following assumption.

[6]Similar to De la Croix and Michel (2002), we define the net production as the production minus investment and its associated transaction costs.

[7]If this assumption is violated, we can not determine the stationary capital-labor ratio that maximizes the net production.

**Assumption 7** *Assume that:*

$$\begin{aligned} \lim_{a\to 0^+} Af'(a) &> \lim_{a\to 0^+} (1+n)\,\varphi'[(1+n)\,a] \\ \lim_{a\to +\infty} Af'(a) &< \lim_{a\to +\infty} (1+n)\,\varphi'[(1+n)\,a] \end{aligned}$$

Following Phelps (1965) and Diamond (1965), we define the modified Golden Rule level of capital-labor ratio.[8]

**Definition 1 (Modified Golden Rule)** *Under Assumptions 6 and 7, there exists a unique positive capital stock per young agents such that:*

$$\frac{Af'(\bar{a})}{\varphi'[(1+n)\,\bar{a}]} = 1+n \tag{4.19}$$

*with $\bar{a}$ is the modified Golden Rule capital-labor ratio.*

In other words, the modified Golden Rule (4.19) determines the level of capital in which the net marginal productivity of capital equals the gross rate of population growth. The modified Golden Rule capital does not depend on consumption allocations in both periods. At the same time, this level of capital provides the highest level of consumptions.

**Proposition 10** *Under Assumptions $6-7$, there is a unique optimal stationary path, the modified Golden Rule, which is characterized by $a=\bar{a}$ and by $\bar{c},\bar{d}$ satisfying the following conditions:*

$$\bar{c} + \frac{\bar{d}}{1+n} = \Sigma(\bar{a}) \tag{4.20}$$

$$u_1\left(\bar{c},\bar{d}\right) = (1+n)\,u_2\left(\bar{c},\bar{d}\right) \tag{4.21}$$

**Proof.** The maximum of $\Sigma(\bar{a})$[9]is satisfied using the modified Golden Rule (4.19) and the optimal allocation of first-period and second-period consumptions $\left(\bar{c},\bar{d}\right)$ that maximize household's preferences (4.1) under the constraint (4.20) is illustrated by the first-order necessary condition (4.21). ■

[8]The term "Golden Rule" was introduced by Phelps (1961).
[9]$\Sigma(\bar{a})$ is defined in (4.18).

**Definition 2 (Feasible path of capital)** *A sequence of capital stock per young agents* $a_t \geq 0$ *is a feasible path if the corresponding production net of investment i.e.,* $\Sigma(a_t, a_{t+1}) \equiv Af(a_t) - \varphi[(1+n)a_{t+1}] \geq 0$ *is non-negative for all* $t > 0$.

**Definition 3 (Efficiency)** *A feasible sequence of capital per young agents* $\{a_t\}_{t\geq 0}$ *is efficient if it is impossible to raise agent's consumption at one date without reducing it at another date, i.e., if there does not exist another feasible path* $\{\ddot{a}_t\}_{t\geq 0}$ *with* $\ddot{a}_0 = a_0$ *such that:*

(*i*) $\Sigma(\ddot{a}_t, \ddot{a}_{t+1}) \geq \Sigma(a_t, a_{t+1})$, *for all* $t \geq 0$;

(*ii*) $\Sigma(\ddot{a}_t, \ddot{a}_{t+1}) > \Sigma(a_t, a_{t+1})$, *for some* $t \geq 0$.

Now, let us consider a feasible path of capital-labor ratio $a_t$ where this path converges to the normalized steady state value $a^* = 1$. Then, we deduce the following result:

**Proposition 11** *Under Assumptions* 6 *and* 7, *then:*

(*i*) *The steady state is characterized by over-accumulation of capital for* $Af'(a^*)/\varphi'[(1+n)a^*] < 1+n$ *and the competitive equilibrium is dynamically inefficient.*

(*ii*) *The steady state is characterized by under-accumulation of capital for* $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ *and the competitive equilibrium is dynamically efficient.*

**Proof.** See Appendix (B). ■

Dynamic efficiency in terms of aggregate consumptions states that it is not possible to raise total consumption at one date without reducing it in another date. Proposition 3 states that there is an under-accumulation (or an over-accumulation) of capital comparing to the modified Golden Rule level if the *net* capital rate of return is higher (or lower) than the gross rate of population growth.

**Corollary 3** *Let Assumptions 1-3, 6 and 7 be satisfied, then comparing to standard Diamond model the steady state is characterized by an under-accumulation of capital.*

**Proof.** Let us define the Golden-Rule level of capital in Diamond (1965) as $a^D$ where $a^D$ satisfies $Af'\left(a^D\right) = 1+n$. However, the modified Golden-Rule level of capital with the presence of costs is $\bar{a}$ such as $Af'\left(\bar{a}\right) = (1+n)\,\varphi'\left[(1+n)\,\bar{a}\right]$. Given that $\varphi'\left[(1+n)\,\bar{a}\right] > 1$, this implies that $Af'\left(\bar{a}\right) = (1+n)\,\varphi'\left[(1+n)\,\bar{a}\right] > (1+n) = Af'\left(a^D\right)$. Therefore, capital accumulation with costs $\bar{a}$ is lower than that in standard Diamond model without costs $a^D$. ■

## 4.6 Local dynamics

In this section, we study the economic stability locally around the normalized steady state $a = 1$. It is demonstrated that the introduction of transaction costs in a standard OLG à la Diamond affects the appearance of cycles of period two. Linearizing the dynamic equation (4.11) around the steady state $a = 1$ yields the following eigenvalue $J \equiv da_{t+1}/da_t$:

$$J = \frac{\alpha\left(\frac{\varepsilon_{12}}{\eta_1}\frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma}\right)}{\sigma\left(2\varepsilon_{12} - \varepsilon_{22} - \eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11} + \eta_2\right) + (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)} \tag{4.22}$$

Notice that $a_t$ is a predetermined variable, therefore the steady state of system (4.11) is determinate. Further, the steady state is stable whenever the unique eigenvalue belongs to the interior of the unit circle, *i.e.* belongs to the interval $(-1, 1)$. The second-order conditions (SOCs) associated with household problem imply that[10]

$$2\varepsilon_{12} - \varepsilon_{22} - \eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11} + \eta_2 > 0$$

A sufficient condition for the emergence of cycles of period two is generically $J(\sigma) = -1$. This holds at $\sigma = \sigma^F$, where

$$\sigma^F \equiv -\frac{\alpha\left(\frac{\varepsilon_{12}}{\eta_1}\frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma}\right) + (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)}{2\varepsilon_{12} - \varepsilon_{22} - \eta_1\frac{1-\gamma}{\gamma}\varepsilon_{11} + \eta_2} \tag{4.23}$$

[10] See Appendix (A).

Before going on, we present some critical values for $\eta_1$, $\gamma$, $\varepsilon_{22}$ and $\varepsilon_{11}$.

$$\begin{bmatrix} \eta_1^* \\ \gamma^* \\ \varepsilon_{11}^* \\ \varepsilon_{22}^* \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_{12} / \left[ (1-\gamma) \left( \varepsilon_{11}/\gamma - \frac{1-\alpha}{\alpha} (1 - \varepsilon_{12} + \varepsilon_{22}) \right) \right] \\ \alpha \varepsilon_{11} / \left[ (1-\alpha) (1 - \varepsilon_{12} + \varepsilon_{22}) \right] \\ (1-\alpha) (1 - \varepsilon_{12} + \varepsilon_{22}) / \alpha \\ \varepsilon_{12} - 1 \end{bmatrix} \tag{4.24}$$

In the next proposition, we present the sufficient conditions for the appearance of cycles of period two.

**Proposition 12** *In view of Assumptions 1 - 3 together with (4.24), flip bifurcation exists when $\sigma$ is close to $\sigma^F$ if one of the following conditions holds:*

1. *For $\varepsilon_{11} > \varepsilon_{11}^*$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\gamma > \gamma^*$ with either: $(i)$ $\varepsilon_{12} > 0$, $\eta_1 > \eta_1^*$; or $(ii)$ $\varepsilon_{12} < 0$, $\eta_1 > 0$.*

2. *For $\varepsilon_{12} < 0$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ with either: $(i)$ $\varepsilon_{11} < \varepsilon_{11}^*$, for all $\gamma > 0$; or $(ii)$ $\varepsilon_{11} > \varepsilon_{11}^*$, $\gamma < \gamma^*$.*

3. *For $\varepsilon_{12} < 0$, $\varepsilon_{22} > \varepsilon_{22}^*$ and $\eta_1 < \eta_1^*$ for all $\gamma > 0$ and $\varepsilon_{11} < 0$.*

**Proof.** See Appendix (C). ■

This proposition shows that flip cycles arise for different configurations concerning agents' preferences, the cost function and the first-period consumption share. In order to study the mechanism behind the emergence of these cycles, let us initially focus on the Benchmark model without costs.

## 4.7 Discussion of the results

In order to understand the role of transaction costs on economic stability, let us investigate the cases without costs in separable and non-separable preferences respectively.

### 4.7.1 Benchmark model

**Non-separable utility**

We recover the basic model studied by Diamond (1965) where agents are not imposed to pay transaction costs by setting $\eta_1 = 1$ and $\eta_2 = 0$. Then, $J(\sigma) = -1$ holds at $\sigma = \sigma^F_{BM}$ where

$$\sigma^F_{BM} = -\frac{\alpha\left(\varepsilon_{12}\frac{1}{1-\gamma} - \frac{\varepsilon_{11}}{\gamma}\right) + (1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})}{2\varepsilon_{12} - \varepsilon_{22} - \frac{1-\gamma}{\gamma}\varepsilon_{11}} \tag{4.25}$$

where $2\varepsilon_{12} - \varepsilon_{22} - \frac{1-\gamma}{\gamma}\varepsilon_{11} > 0$ by SOCs. Given the critical values $\varepsilon^{bb}_{22} \equiv \varepsilon_{12} - 1$, $\varepsilon^b_{12} \equiv -(1-\alpha)/(2\alpha-1)$, $\varepsilon^b_{22} \equiv -\varepsilon_{12}(2\alpha-1)/(1-\alpha) - 1$, $\varepsilon^b_{11} \equiv [\varepsilon_{12}\gamma/(1-\gamma)] + [(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})\gamma/\alpha]$ and $\gamma^b \equiv (\alpha\varepsilon_{12}/(1-\alpha)(1-\varepsilon_{12}+\varepsilon_{22})) + 1$, then, the positivity of $\sigma^F_{BM}$ in (4.25) requires one of the following conditions:[11]

1. $\varepsilon_{11} > \varepsilon^b_{11}$ and $\gamma < \gamma^b$ for *either* $(i)$ $\varepsilon_{12} > \varepsilon^b_{12}$, $\varepsilon_{22} < \varepsilon^{bb}_{22}$, *or* $(ii)$ $0 < \varepsilon_{12} < \varepsilon^b_{12}$ and $\varepsilon_{22} < \varepsilon^b_{22}$.

2. $\varepsilon_{11} > \varepsilon^b_{11}$ and $\varepsilon_{12} < 0$ for *either* $(i)$ $\varepsilon_{22} < \varepsilon^{bb}_{22}$ for all $\gamma > 0$, *or* $(ii)$ $\varepsilon_{22} > \varepsilon^b_{22}$ and $\gamma > \gamma^b$.

Cycles of period two appear if capital increases in the current period and then it decreases in the following period. One can easily observe from (4.25) that the appearance of cycles of period two depends on agents' preferences and on first-period consumption share. Notice that, a low (resp. high) $\varepsilon_{22}$ means that as second-period consumption increases, its marginal utility declines significantly (resp. slightly). In addition, a high (resp. low) $\varepsilon_{11}$ implies that as first-period consumption augments, its marginal utility declines slightly (resp. heavily).

The intuition for the existence of cycles of period two is the following: Focus on case 1 and suppose that $K_t$ increases from its steady state value, then $w_t$ augments which induces a higher capital accumulation $K_{t+1}$. The presence of a sufficiently high $\varepsilon_{11}$ and a small $\varepsilon_{22}$ encourages agents to raise current consumption and to reduce future one, and thus, to accumulate low of capital. However, the presence of a small $\gamma$ enforces agents to consume less today and to accumulate more capital. Therefore, cycles of period two require that the former effect ($\varepsilon_{11}$ and

[11] Computations are available from the author upon request.

$\varepsilon_{22}$) dominates the effects of $\gamma$ and $w_t$. As a result, an increase in $K_t$ is followed by a decline in capital accumulation $K_{t+1}$. In case 2, the economic intuition is mainly the same.

**Separable utility**

The separability of preferences can be obtained by setting $\varepsilon_{12} = 0$. For simplicity, it is supposed that agents have the same utility in both periods, then flip cycles emerge at $\sigma = \sigma^F_{BM,S}$, where

$$\sigma^F_{BM,S} \equiv \frac{\alpha\varepsilon/\gamma - (1-\alpha)(1+\varepsilon)}{-\varepsilon\left(1+\frac{1-\gamma}{\gamma}\right)} \tag{4.26}$$

Where $\varepsilon$ is the elasticity of marginal utility in consumption and so the elasticity of intertemporal substitution in consumption is given by $-1/\varepsilon$. Notice that if we assume high substitutability between consumptions in both periods, that is, $1+\varepsilon > 0$, then the result of Diamond (1965) is obtained with $\sigma^F_{BM,S} < 0$ which implies a stable steady state. However, since it is not the case here, then $\sigma^F_{BM,S} > 0$ for $\varepsilon < \min\left(-1, \varepsilon^{sb}\right)$ and $\gamma > \gamma^{sb}$ with $\varepsilon^{sb} \equiv (1-\alpha)/(2\alpha-1)$ and $\gamma^{sb} \equiv \alpha\varepsilon/\left((1-\alpha)(1+\varepsilon)\right)$. Hence, income effect dominates substitution effect and so agents are not interested in future consumptions. The presence of high consumption share together with low elasticity of substitution makes agents more incentive to accumulate low capital. This means that a rise $K_t$ in current period would be followed by a decline in $K_{t+1}$ in next period.

Furthermore, the eigenvalue (4.22) is simplified to[12]

$$J = \frac{\alpha}{\sigma}\frac{-\varepsilon}{\frac{\gamma(1-\alpha)(1+\varepsilon)}{\sigma} - \varepsilon} \tag{4.27}$$

Equation (4.27) argues that the unique equilibrium path of standard Diamond (1965) is recovered as $\sigma > \alpha$.[13]However, Nourry (2001) recovers Diamond with a range of elasticity of input substitution such that $\sigma \geq 1$. In order to clarify more, let us take logarithmic formulations for the utility functions, *i.e.*, $\varepsilon = -1$ with a Cobb-Douglas technology, $\sigma = 1$. Therefore, the eigenvalue (4.27) is simplified to $J = \alpha \in (0,1)$, so a unique-path steady state.

[12]Diamond finds that the steady state exhibits a saddle-path stability if and only if the elasticity of saving with respect to interest is not negative, which means a high elasticity of intertemporal substitution in consumption. Simply, in this model, high substitutability implies $1+\varepsilon > 0$.

[13]Cazzavillan and Pintus (2004) find that endogenous fluctuations require $\sigma < \alpha$.

### 4.7.2 Our model

#### Separable preferences

As before, it is supposed that agents have the same utility in both periods, *i.e.*, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon$. The eigenvalue (4.22) is simplified to:

$$J = \frac{-\alpha\frac{\varepsilon}{\gamma}}{\sigma\left(\eta_2 - \varepsilon - \eta_1\frac{1-\gamma}{\gamma}\varepsilon\right) + (1-\alpha)(1+\varepsilon)} \tag{4.28}$$

The necessary condition for the existence of cycles of period two is $1 + \varepsilon < 0$. Along with (4.28), the numerator is positive and the SOCs state that $\eta_2 - \varepsilon - \eta_1\frac{1-\gamma}{\gamma}\varepsilon > 0$. Flip cycles arise at $\sigma = \sigma_S^F$ where

$$\sigma_S^F \equiv \frac{\alpha\varepsilon/\gamma - (1-\alpha)(1+\varepsilon)}{\eta_2 - \varepsilon - \eta_1\frac{1-\gamma}{\gamma}\varepsilon} \tag{4.29}$$

Remark that cycles appear for the same conditions as in the Benchmark model without costs, *i.e.*, for $\varepsilon < \min\left(-1, \varepsilon^{sb}\right)$ and $\gamma > \gamma^{sb}$. This provides that transaction costs do not affect the appearance of these cycles which means that the SOCs dominate the effect of transaction costs.

#### Non-separable preferences

In the non-separable case, Proposition 4 summarizes the conditions under which cycles of period two appear. We only focus on case 1 of Proposition 4, since other cases have similar intuition.

Suppose that, at period $t$, capital stock $K_t$ increases from its value of the steady state which augments the wage $w_t$ and induces more capital accumulation. The presence of high $\varepsilon_{11}$ and $\gamma$ and small $\varepsilon_{22}$ enforces agents to consume more today and to reduce their capital accumulation and so future consumption. Then, this gives rise to two subcases:[14]

From one hand, whenever $\varepsilon_{12} > 0$, this acts as an offset effect because a reduction in future consumption decreases the marginal utility from first-period consumption and thus leads agents to reduce present consumption. Hence, in order to ensure a reduction of capital accumulation and therefore the emergence of cycles of period two, a sufficiently high sensitivity of costs is

[14] If the utility function exhibits homogenous of degree one, then $\varepsilon_{12} > 0$, as a result, the conditions under which two-period cycles appear are summarized in case (1) of Proposition (4).

required, $\eta_1 > \eta_1^*$.

From the other hand, whenever $\varepsilon_{12} < 0$, then a reduction in future consumption increases the marginal utility from the first-period consumption which provides an additional force to reduce capital accumulation, hence, cycles emerge. It is important to notice that in this subcase, cycles appear without any restriction on transaction costs.

In the numerical example, we clarify the effect of transaction costs on stability range in both separable and non-separable cases with isoelastic transaction costs formulation.

## 4.8 Numerical example

In this section, we confirm numerically our theoretical results presented in the previous section. Let us consider a CES production function $f(a) = A\left[\alpha a^{-\varkappa} + (1-\alpha)\right]^{-1/\varkappa}$ with $\alpha \in (0,1)$, $A > 0$, $\varkappa > -1$ and $\varkappa \neq 0$ and an isoelastic cost function: $\varphi(s) = s^{\eta}$ with $\eta \in (0,1)$.[15]The annual ratio of personal consumption expenditures over GDP have an average of 0.65 over the period (1959 - 2008) for US economy.[16]Our objective is to determine the critical values under which flip cycles appear, *i.e.* $\sigma^F > 0$.

### 4.8.1 Non-separable preferences

As in Venditti (2003), we consider the following utility function

$$u(c,d) = \frac{1}{\theta}\left[\varsigma c^{-\rho} + (1-\varsigma)\, d^{-\rho}\right]^{-\frac{\theta}{\rho}} \tag{4.30}$$

with $\theta \leq 1$, $\varsigma \in (0,1)$ and $\rho > -1$, the discount factor is $(1-\varsigma)/\varsigma$ and the elasticity of intertemporal substitution is $1/(1+\rho)$ and $u_{12} < 0$ if and only if $\rho + \theta < 0$. Given the above technology and the cost function, we get: $c = A(1-\alpha) - (1+n)^{\eta}$ and $d = A\alpha(1+n)$. Using the above consumption values, then the steady state value of $A$ can be implicitly obtained using

[15] Consistent with previous notations together with an isoelastic cost function $\varphi(s) = s^{\eta}$, we obtain $\eta_1 = \eta$ and $\eta_2 = \eta - 1$.

[16] For more information, see Economic Report of the President, 2008.

condition (4.13):

$$\frac{\varsigma}{1-\varsigma}\left(A\left(1-\alpha\right)-\left(1+n\right)^{\eta}\right)^{-\rho-1}\eta-\left(1+n\right)^{-\rho-\eta}A^{-\rho}\alpha^{-\rho}=0 \tag{4.31}$$

The positivity of first-period consumption requires that

$$A>\frac{\left(1+n\right)^{\eta}}{1-\alpha}\equiv\hat{A} \tag{4.32}$$

The steady state value of $\gamma$ can be endogenously obtained using:

$$\gamma=1-\frac{\left(1+n\right)^{\eta}}{A\left(1-\alpha\right)} \tag{4.33}$$

One can directly show that the elasticities of preferences as:

$$\begin{aligned}
\varepsilon_{11} &= \left(\theta+\rho\right)\frac{\varsigma c^{-\rho}}{\varsigma c^{-\rho}+\left(1-\varsigma\right)d^{-\rho}}-\left(\rho+1\right)\\
\varepsilon_{22} &= \left(\theta+\rho\right)\frac{\left(1-\varsigma\right)d^{-\rho}}{\varsigma c^{-\rho}+\left(1-\varsigma\right)d^{-\rho}}-\left(\rho+1\right)\\
\varepsilon_{12} &= \left(\theta+\rho\right)\frac{\left(1-\varsigma\right)d^{-\rho}}{\varsigma c^{-\rho}+\left(1-\varsigma\right)d^{-\rho}}
\end{aligned} \tag{4.34}$$

Let us set $\alpha=0.33$, $\varsigma=0.5$, $n=0.5175$, $\rho=7$, $\theta=0.5$, then using (4.31), we obtain $A=4.181\,6>1.691\,5=\hat{A}$. Given $A$, we obtain $\gamma=0.595\,50$. Given these values, we get $\varepsilon_{12}=1.269\,7$, $\varepsilon_{11}=-1.769\,7$ and $\varepsilon_{22}=-6.730\,3$. The SOCs is verified as well $2\varepsilon_{12}-\varepsilon_{22}-\eta\frac{1-\gamma}{\gamma}\varepsilon_{11}+\eta-1=8.930\,3>0$ and finally $\sigma^{F}=0.028\,72>0$.

| $\eta$ | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| $\sigma^{F}$ | 0.028 72 | $4.934\,4\times10^{-2}$ | $5.873\,9\times10^{-2}$ | $6.355\,2\times10^{-2}$ |

From the table, we observe that $\partial\sigma^{F}/\partial\eta>0$ and further $\varepsilon_{12}>0$ and so $\partial J/\partial\sigma<0$. Notice that the basic model without costs is recovered by setting $\eta=1$. Hence, the more $\eta$ is far from 1, the higher sensitivity of costs is. As a result, transaction costs act as a destabilizing factor in the sense that it widens the range of parameters giving rise to cycles of period two, that is, $\left(\sigma^{F},+\infty\right)$.

### 4.8.2 Separable preferences

Suppose that agents have the same utility in both periods with a CIES preferences:

$$v(x) = \frac{x^{1-\delta}}{1-\delta},\ \delta > 0 \text{ and } x = c, d \tag{4.35}$$

One can easily show that the elasticity of marginal utility in consumption $\varepsilon = -\delta$. As before, the steady state value of $A$ can be obtained using (4.16) together with above cost and production functions. Explicitly, we get:

$$\left(A(1-\alpha) - (1+n)^{\eta}\right)^{-\delta} \eta - \beta\alpha^{1-\delta}(1+n)^{1-\eta-\delta} A^{1-\delta} = 0 \tag{4.36}$$

where $A$ is restricted to the positivity condition of first-period consumption (4.32). Additionally, the share of first-period consumption is obtained by (4.33).

Let us set $\alpha = 0.33$, $\delta = 4.44$, $n = 0.5175$ and $\beta = 0.3$. Therefore, $\varepsilon = -4.44 < -1.970\,6 \equiv \varepsilon^{sb}$. As a result, we get the following table:

| $\eta$ | 0.3 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|
| $\sigma_S^F$ | $2.604\,4 \times 10^{-6}$ | $3.575\,0 \times 10^{-2}$ | 0.052 33 | $6.127\,9 \times 10^{-2}$ |

Transaction costs influence the stability region through its effects on $\sigma_S^F$. Consequently, one can easily obtain that $\partial\sigma_S^F/\partial\eta > 0$ and from (4.28) we get $\partial J/\partial\sigma < 0$. Similar to previous explanation, transaction costs act as a destabilizing factor. For all values of $\eta$, the steady state value of $A$ (4.32) is verified as well as the SOCs.

## 4.9 Conclusion

This chapter analyzes the economic stability in an overlapping generations model with exogenous labor supply. We extend the standard one-dimension OLG by introducing transaction costs related to the amount of investment. Young agents consume and save according to the wage income, while in the next period, old agents who are retired consume all their saving returns. We mainly consider two different aspects with respect to household preferences. Initially, we focus on a general non-separable formulation of preferences then it is supposed to be

separable. It is shown that the presence of transaction costs with respect to saving promote cycles and it is proved that these costs act as a destabilizing factor. It is also demonstrated that under specific conditions, the steady state may be dynamically inefficient (or efficient) if there is an over-accumulation (or under-accumulation) of capital with respect to the Golden Rule, *i.e.* the net return of capital is higher (or lower) than the population growth. Comparing to the model of Diamond (1965) where he proposes a high substitutability between current and future consumptions, the main contribution of this chapter is the emergence of cycles of period two. This chapter generalizes the stability condition of a steady state equilibrium obtained by Diamond.

## 4.10 Appendix

### 4.10.1 (A) Sufficient conditions for utility maximization

Using the Lagrangian function (4.4), we calculate the associated Hessian matrix with respect to $(\lambda_t, \mu_t, c_t, d_{t+1}, s_t)$:[17]

$$H \equiv \begin{bmatrix} 0 & 0 & -1 & 0 & -\varphi' \\ 0 & 0 & 0 & -1 & r \\ -1 & 0 & u_{11} & u_{12} & 0 \\ 0 & -1 & u_{12} & u_{22} & 0 \\ -\varphi' & r & 0 & 0 & -\lambda_1 \varphi'' \end{bmatrix}$$

Household problem is considered as a maximization problem if and only if the determinant of the leading principal minors of above Hessian matrix changes its sign. If the determinant of $H$ has the same sign as $(-1)^n$ and the last $n-m$ diagonal principal minors have alternative signs. Here, the number of variables $n = 3$ and the number of constraints $m = 2$. Thus, the optimum is a local maximum only if $\det H < 0$. We need to find the conditions under which the matrix $H$ is negative definite (negative semi-definite) over the set of values satisfying the first-order conditions and the constraints. Therefore,

$$Det\ H = r^2 u_{22} - 2r\varphi' u_{12} - \varphi'' \lambda_1 + (\varphi')^2 u_{11} < 0$$

Using (4.8) and the FOCs (4.5), (4.6) and (4.7), we obtain a lower bound for the elasticity of transaction cost with respect to savings, that is,

$$\eta_1 \frac{1-\gamma}{\gamma} \varepsilon_{11} - 2\varepsilon_{12} + \varepsilon_{22} < \eta_2 \tag{4.37}$$

### 4.10.2 (B) Proof of Proposition 3

**Proof.** According to definitions $1-3$ and Assumptions 6 and 7, over-accumulation of capital gives $Af'(a^*)/\varphi'[(1+n)a^*] < 1+n$. This demonstration is based on previous work of De la

[17] For simplicity, we omit the arguments and the time subscripts related with the functions.

Croix and Michel (2002) and Drugeon *et al.* (2010). We have to prove that we can decrease capital stock and raise consumption at one date without reducing consumption at another date. In a neighborhood $(a^* - 2\varkappa, a^* + 2\varkappa)$ of $a^*$, we have $Af'(a)/\varphi'[(1+n)a] < 1+n$. After some date $t_0$, we have $a_t \in (a^* - \varkappa, a^* + \varkappa)$ with $Af'(a_t)/\varphi'[(1+n)a_{t+1}] < 1+n$ and $Af'(a_t - \varkappa)/\varphi'[(1+n)(a_{t+1} - \varkappa)] < 1+n$. The concavity of $f(.)$ and $\varphi[.]$ implies respectively:

$$Af(a - \varkappa) - Af(a) \geq -Af'(a - \varkappa)\varkappa$$

and

$$\varphi[(1+n)(a - \varkappa)] - \varphi[(1+n)a] \geq -\varphi'[(1+n)(a - \varkappa)](1+n)\varkappa$$

Let us decline capital stock by $\varkappa$ after date $t_0$ and forever. Investment $a_{t_0+1}$ is reduced by $\varkappa$ and consumption $\Sigma(a_{t_0}, a_{t_0+1})$ is increased by $\varphi'[(1+n)a_{t_0+1}](1+n)\varkappa$. At date $t > t_0$, the new consumption level is:

$$\begin{aligned}
& \Sigma(a_t - \varkappa, a_{t+1} - \varkappa) \\
= \; & Af(a_t - \varkappa) - \varphi[(1+n)(a_{t+1} - \varkappa)] \\
\geq \; & Af(a_t) - Af'(a_t - \varkappa)\varkappa - \left(\varphi[(1+n)a_{t+1}] - \varphi'[(1+n)(a_{t+1} - \varkappa)](1+n)\varkappa\right) \\
\geq \; & Af(a_t) - \varphi[(1+n)a_{t+1}] + \left[\varphi'[(1+n)(a_{t+1} - \varkappa)](1+n) - Af'(a_t - \varkappa)\right]\varkappa \\
> \; & Af(a_t) - \varphi[(1+n)a_{t+1}] = \Sigma(a_t, a_{t+1})
\end{aligned}$$

So, consumption can be increased for all future periods and the path is dynamically inefficient.

Now, we go forward to show that whenever $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ then there exists an under-accumulation of capital. To prove this, it is enough to show the impossibility of raising one period $t_1$ consumption without reducing other period's consumption. Moreover, $Af'(a^*)/\varphi'[(1+n)a^*] > 1+n$ gives that $Af'(a^*)/\varphi'[(1+n)a^*] > b(1+n)$ with some $b > 1$. Along an equilibrium path and for $t \geq t_0$, we have $Af'(a_t)/\varphi'[(1+n)a_t] > b(1+n)$. At any

date $t$, the difference from another feasible path $\tilde{a}_t$ satisfies:

$$\begin{aligned}\Delta C_t &= Af(\tilde{a}_t) - Af(a_t) - (\varphi[(1+n)\tilde{a}_{t+1}] - \varphi[(1+n)a_{t+1}]) \\ &\leq Af'(a_t)(\tilde{a}_t - a_t) - \varphi'[(1+n)a_{t+1}](1+n)(\tilde{a}_{t+1} - a_{t+1})\end{aligned}$$

Where $\Delta C_t$ is the difference of total consumption. This implies:

$$\varphi'[(1+n)a_{t+1}](1+n)(\tilde{a}_{t+1} - a_{t+1}) \leq Af'(a_t)(\tilde{a}_t - a_t) - \Delta C_t \tag{4.38}$$

Assume that consumption never decreases which means that capital never increases. Indeed, by induction if $\tilde{a}_t - a_t \leq 0$, which is true at $t = 0$, and if $\Delta C_t \geq 0$, then (4.38) implies $\tilde{a}_{t+1} - a_{t+1} \leq 0$. Moreover, suppose that consumption increases at time $t_1$: $\Delta C_{t_1} > 0$, then the previous argument gives that $\tilde{a}_t - a_t < 0$, for all $t > t_1$. This implies that for $t > t_2 = \max\{t_0, t_1\}$ :

$$\begin{aligned}(1+n)(\tilde{a}_{t+1} - a_{t+1}) &\leq \frac{Af'(a_t)}{\varphi'[(1+n)a_{t+1}]}(\tilde{a}_t - a_t) \\ &< b(1+n)(\tilde{a}_t - a_t)\end{aligned}$$

since $\tilde{a}_t - a_t < 0$ and $Af'(a_t)/\varphi'[(1+n)a_{t+1}] > b(1+n)$. Hence,

$$\tilde{a}_{t+1} - a_{t+1} < b(\tilde{a}_t - a_t)$$

and

$$\tilde{a}_{t+1} - a_{t+1} < b^{t-t_2}(\tilde{a}_{t_2} - a_{t_2}) < 0$$

As $b > 1$ and $a_{t+1}$ converges to the steady state, we have $\tilde{a}_t - a_t$ converges to $-\infty$ and $\tilde{a}_{t+1}$ becomes negative, which is impossible. ■

### 4.10.3 (C) Proof of Proposition 4

**Proof.** In this proof, we show the existence of flip bifurcation according to different configurations. As shown before the flip cycles appear at $\sigma = \sigma^F$ given by (4.23) and since its

denominator is positive, then the existence of flip bifurcation require a negative numerator, that is:

$$\eta_1 \left( \frac{\varepsilon_{11}}{\gamma} - \frac{1-\alpha}{\alpha} \left(1 - \varepsilon_{12} + \varepsilon_{22}\right) \right) > \frac{\varepsilon_{12}}{(1-\gamma)} \tag{4.39}$$

In order to simplify, let us take two different cases concerning the sign of $\varepsilon_{12}$.

*1.* $\varepsilon_{12} > 0$.

In (4.39), the sign of $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha$ is unknown. However, condition (4.39) can not hold whenever $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha < 0$. Thus, $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha > 0$ is a necessary condition in order for condition (4.39) to verify and thus $\sigma^F > 0$ is satisfied for $\eta_1 > \eta_1^*$.

Moreover, $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha > 0$ requires $\varepsilon_{22} < \varepsilon_{22}^*$ and can be written as $\gamma > \gamma^*$. Since $\gamma$ represents the consumption share out of wage, that is, $\gamma \in (0,1)$. Therefore, $\gamma^* < 1$ if and only if $\varepsilon_{11} > \varepsilon_{11}^*$.

As a result flip bifurcation at $\sigma = \sigma^F$ whenever $\varepsilon_{11} > \varepsilon_{11}^*$, $\gamma > \gamma^*$, $\varepsilon_{22} < \varepsilon_{22}^*$ and $\varepsilon_{12} > 0$ (condition $(1.i)$).

*2.* $\varepsilon_{12} < 0$.

We consider the following configurations:

**A.** $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha > 0$.

As before, $\sigma^F > 0$ for all $\eta_1 > 0$. Condition A requires $\varepsilon_{22} < \varepsilon_{22}^*$ and is equivalent to $\gamma > \gamma^*$. However, $\gamma^* < 1$ for $\varepsilon_{11} > \varepsilon_{11}^*$. This implies condition $(1.ii)$.

**B.** $\varepsilon_{11}/\gamma - (1-\alpha)\left(1 - \varepsilon_{12} + \varepsilon_{22}\right)/\alpha < 0$.

In this case, $\sigma^F > 0$ if and only if $\eta_1 < \eta_1^*$. For $\varepsilon_{22} > \varepsilon_{22}^*$, then condition (B) is verified for all $\gamma \in (0,1)$. However, for $\varepsilon_{22} < \varepsilon_{22}^*$, the condition (B) is equivalent to $\gamma < \gamma^*$. In this case, $\gamma^* < 1$ if and only if $\varepsilon_{11} > \varepsilon_{11}^*$. (conditions (2) and (3)). ■

# Chapter 5

# Progressive Income Taxes and Macroeconomic Instability

## 5.1 Introduction

Taxation is an unavoidable tool in contemporary economies used to finance public expenditures which in turn aim to achieve primary economic and social objectives. In particular, individual income taxes have been the largest source for U.S. government since the middle 20th century with an average of 8 percent of American GDP and about 45 percent of Federal Tax Revenue in 2008.[1]Most OECD countries apply income taxes with progressive features where tax rate increases as the taxable base amount gets higher. For instance, the average income tax rate in 2008 ranged from roughly 2.6 percent of income for the bottom half of tax returns to about 23.27 percent for the top 1 percent.[2]Therefore, I think that considering a progressive taxation policy is a suitable strategy to keep a reasonable gap between rich and poor agents.

Individual income taxes consist of both labor-income tax and capital-income tax. However, in this paper, we restrict our attention to a labor-income tax only with a progressive policy. This is consistent with U.S. tax code where labor-tax rate is more progressive that capital-tax

[1]Source: Congressional Budget Office, a preliminary analysis of the president's budget and an update of CBO's budget and economic outlook, March 2009.

[2]For more details, see "Summary of Latest Federal Individual Income Tax Data" 2010, Fiscal Facts No. 249, Tax Foundation.

rate.[3]

Our objective is to study the stability effect of labor-income tax progressivity, in other words, does tax progressivity prevent/promote the emergence of endogenous fluctuations due to self-fulfilling expectations?[4]

We propose an overlapping generation model (thereafter, OLG) with two-period consumption à la Diamond (1965) where in the first period, young agents allocate their after-tax wage income between consumption and saving. In the second period, old agents, who are retired, consume their entire income that comes from the returns of first-period saving. Government purchases which are financed through agent's labor income tax do not contribute to either production or agent's utility.

Since the seminal work of Diamond (1965) and Reichlin (1986), numerous studies have been extended to include fiscal policy. More precisely, Dromel and Pintus (2006) (thereafter D-P (2006)) focus on a non-linear progressive capital-income taxation rule in an OLG model à la Reichlin (1986) where they show that endogenous fluctuations are ruled out and saddle-point stability is ensured whenever tax progressivity is sufficiently high. Seegmuller (2003) proposes a constant tax policy together with public services in a similar OLG framework. However, Chen and Zhang (2009a, 2009b) and Gokan (2009a) consider an endogenous linear tax policy jointly with a two-period consumption OLG model. Particularly, Chen and Zhang (2009a) concentrate on labor-income taxation while Chen and Zhang (2009b) study the effect of capital-income tax rule on the appearance of endogenous cycles. They deduce that labor-income tax policy (resp. capital-income tax policy) operates as a destabilizing factor (resp. a stabilizing factor). These contradictory results are intuitively explained as labor income taxation makes consumption-saving ratio smaller for young agents which makes cycles more likely to emerge.

Comparing to above literature, considering a progressive labor-income tax in an OLG model is an extension of the work of Chen and Zhang (2009a). At the same time, this work allows to study the robustness of existing results obtained by D-P (2006) where they deduce that progressive fiscal policy has a stabilization power if taxed income finances all or at least most of consumptions generated by agents. Hence, first-period consumption might be a crucial element

[3] See Hall and Rabushka (1995).

[4] Initially, income taxes have been considered as a stabilizing instrument by Musgrave and Miller (1948).

for the stability effect of fiscal rules in an OLG economy.[5]

This paper shows that high labor-income tax progressivity performs as a destabilizing factor in the sense that it promotes endogenous cycles. These cycles arise due to the presence of interaction of two conflicting effects on savings that operate through wages and real interest rate. The intuition goes as follows: assume that current capital increases from its stationary level. This implies an increase in wages and so more capital accumulation. At the same time, agents expect that an increase in future capital reduces future interest rate. Using real interest rate and real wage equations, a lower in expected return induces them to supply less labor which in turn has a negative effect on capital accumulation. Cyclical paths emerge whenever the interest rate effect dominates the wage effect. This requires a sufficiently low elasticity of input substitution together with a high propensity to save out of after-tax wage income, a high tax progressivity and a high elasticity of marginal utility of consumption.

Therefore, while taxation policy acts as a destabilizing factor in our model, it has a stabilizing power in D-P (2006). Such a difference is due to that, in this paper, taxes are imposed on labor income and agents consume in both periods. On the contrary, D-P (2006) assume that taxes are imposed on capital income and agents consume only when they are old. These assumptions are crucial to give rise to different results from a stability point of view. In this study, indeterminacy requires high saving rate which means that a low fraction of consumptions is financed by labor income while in D-P (2006) consumptions are totally financed by capital income. This confirms their principle conclusion: if agent's consumptions are mostly financed by wage income (resp. capital income), then the endogenous fluctuations are ruled out if progressive taxes are applied to labor income (resp. capital income). Likewise, progressive labor-income tax is not helpful to stabilize both labor and consumptions since second-period consumption depends strictly on the interest rate.

The stability effects of taxation policies have been studied also in a one-sector infinite horizon representative agent model.[6]The pioneering paper of Schmitt-Grohé and Uribe (1997) considers a balanced-budget rule with a constant returns-to-scale technology where a constant

[5]See Cazzavillan and Pintus (2004) for the importance of first-period consumption in OLG model.

[6]The role of taxation progressivity is also treated in a heterogeneous agents' framework. For instance, in a segmented agents model à la Woodford (1986), see among others, Dromel and Pintus (2008) and Lloyd-Braga, Modesto and Seegmuller (2008), however, heterogeneity à la Becker (1980) is considered by Sarte (1997), Sorger (2002) and Bosi and Seegmuller (2010b).

government spending is financed by fixed income taxes. This countercyclical tax policy can generate indeterminate steady state and a continuum of sunspots equilibria. However, Guo and Harrison (2004) extend Schmitt-Grohé-Uribe model by allowing endogenous government expenditures financed by proportional tax rates on capital and labor incomes. They show that indeterminacy does no longer appear and the economy exhibits saddle-path stability. Moreover, Guo and Lansing (1998) find that introduction a progressive income tax in a model with increasing returns as in Benhabib and Farmer (1994) can prevent agents' self-fulfilling expectations and provides a unique equilibrium. More recently, Dromel and Pintus (2007) consider a linear income-tax policy into the model of Benhabib and Farmer (1994) where it is assumed that this tax is constant and applied to income only when the latter is higher than a threshold value. They demonstrate that taxes can prevent the occurrence of endogenous fluctuations. The stabilizing role of the progressive tax can be interpreted as follows: when agents are optimistic and expect a higher wage tomorrow and simultaneously they realize to face an increasing tax rate. This reduces their expected after-tax returns and prevents the occurrence of self-fulfilling expectation.

The outline of this paper is the following: Section 2 presents the model. Section 3 characterizes the intertemporal equilibrium. Section 4 demonstrates the existence of a steady state. The local dynamics is introduced in section 5 followed by a numerical example in section 6 and finally section 7 concludes.

## 5.2 The model

We consider a competitive, non monetary, two-period overlapping generations model with identical agents. In each period $t$, $N_t$ individuals are born and live only two periods "young and old". There exists a unique good which can be either consumed or saved. In the first period, agents allocate consumption and saving according to their after-tax income and in the next period they do not supply labor and consume the entire first-period saving returns. In addition, agents are imposed to pay a labor income tax which is supposed to be progressive.

Given the real wage $w_t$ and the gross real interest rate $R_{t+1}$, an agent born at time $t \geq 0$ choose labor $l_t$, saving $s_t$ and both period consumptions $(c_t, d_{t+1})$ to maximize the following

additive separable preferences:

$$\max_{c_t,\, d_{t+1},\, l_t} \left[u\left(c_t/B\right) + \beta u\left(d_{t+1}\right) - v\left(l_t\right)\right] \tag{5.1}$$

subject to

$$\begin{aligned} c_t + s_t &= \varphi\left(w_t l_t\right) && (5.2)\\ d_{t+1} &= R_{t+1} s_t && (5.3)\\ c_t &\geq 0,\, d_{t+1} \geq 0 \text{ for all } t \geq 0 \end{aligned}$$

with $\varphi\left(w_t l_t\right) = w_t l_t - \tau\left(w_t l_t\right)$ is the after-tax income and $\tau\left(w_t l_t\right)$ is the amount of tax payment, $\beta \in (0,1)$ is the discount factor and $B$ is a scaling parameter.

**Assumption 1** *$u(x)$ is continuous, increasing $u'(x) > 0$ and concave $u''(x) < 0$ for $x = c_t/B, d_{t+1}$, while $v(l)$ is continuous for $0 < l < \zeta$ and increasing $v'(l) > 0$ and convex $v''(l) > 0$. Additionally, $\lim_{x\to+\infty} u'(x) = 0$, $\lim_{x\to 0^+} u'(x) = +\infty$, $\lim_{l\to 0^+} v'(l) = 0$ and $\lim_{l\to\zeta} v'(l) = +\infty$.*

For future reference, we present the following necessary elasticities: the elasticity of marginal utility of current and future consumption $\varepsilon_{11} = u''(x)\, x/u'(x) < 0$ for $x = c_t/B, d_{t+1}$ and the elasticity of marginal dis-utility of labor $\varepsilon_v = v''(l_t)\, l_t/v'(l_t) > 0$.

**Assumption 2** *The function $\varphi\left(w_t l_t\right) \in C^2$ is positive with $\varphi(0) = 0$. Further, it satisfies $0 < \varphi'\left(w_t l_t\right) \leq 1$, $\varphi''\left(w_t l_t\right) < 0$ for all $w_t l_t > 0$. Moreover, the income tax is a progressive tax, that is, $\varphi\left(w_t l_t\right)/w_t l_t$ is non-increasing for $w_t l_t > 0$ or equivalently $\varphi'\left(w_t l_t\right) w_t l_t/\varphi\left(w_t l_t\right) \leq 1$.*[7]

The Lagrangian function for household problem is:

$$Lag = u\left(c_t/B\right) + \beta u\left(d_{t+1}\right) - v\left(l_t\right) + \lambda_t\left(\varphi\left(w_t l_t\right) - c_t - s_t\right) + \mu_t\left(R_{t+1} s_t - d_{t+1}\right) \tag{5.4}$$

[7] For details, see among others Dromel and Pintus (2006) and Bosi and Seegmuller (2010b).

The first-order conditions with respect to $c_t$, $d_{t+1}$, $l_t$ and $s_t$ imply:

$$Bv'(l_t) = u'(c_t/B)\, w_t \varphi'(w_t l_t) \tag{5.5}$$

$$v'(l_t) = \beta u'(d_{t+1})\, R_{t+1} w_t \varphi'(w_t l_t) \tag{5.6}$$

Income tax progressivity can be obtained whenever the marginal income tax rate is higher than the average tax rate. More explicitly, the marginal tax rate is $\tau'(w_t l_t) = 1 - \varphi'(w_t l_t)$ and the average tax rate is $\tau(w_t l_t)/w_t l_t = 1 - \varphi(w_t l_t)/w_t l_t$. Thus, progressive taxation requires marginal rate greater than average rate, *i.e.*, $\varphi'(w_t l_t) \leq \varphi(w_t l_t)/w_t l_t$. This fiscal policy generalizes Chen and Zhang (2009a) where they suppose an endogenous linear taxation policy. The corresponding elasticities of after-tax income are:

$$(\eta_1, \eta_2) \equiv \left( \frac{\varphi'(wl)\, wl}{\varphi(wl)}, \frac{\varphi''(wl)\, wl}{\varphi'(wl)} \right) \tag{5.7}$$

where $\eta_1 \in (0, 1]$ and $\eta_2 \leq 0$. Moreover, $\eta_2$ measures the degree of marginal progressivity on labor income taxation.[8]

### 5.2.1 Firms

On the production side, firms which are identical utilize capital $K_t$ and labor $L_t$ to produce final goods and to maximize the profit:

$$\pi_t = AF(K_t, L_t) - R_t K_t - w_t L_t$$

with $F(K_t, L_t)$ is the production function which has the following features.

**Assumption 3** *$F(K, L)$ is a continuous function defined on $[0, +\infty)$, homogeneous of degree one, strictly increasing in both arguments $(F_K(K, L) > 0, F_L(K, L) > 0)$ and strictly concave $(F_{KK}(K, L) < 0, F_{LL}(K, L) < 0)$.*[9] *Additionally, $F(0, 0) = 0$ and the boundary conditions $\lim_{k \to 0^+} f'(k) = +\infty$, $\lim_{k \to +\infty} f'(k) = 0^+$ are satisfied, where $f(k) \equiv F(k, 1)$ is the*

[8] In this paper, tax progressivity is defined in marginal terms as implemented by Musgrave and Thin (1948).

[9] *Notice that: $F_L(K, L) \equiv \partial F(K, L)/\partial L$ and $F_K(K, L) \equiv \partial F(K, L)/\partial K$. This notation holds across the paper.*

*production per labor and* $k \equiv K/L$ *is the capital-labor ratio.*

If we set $\rho(k) \equiv f'(k)$ and $\omega(k) = f(k) - kf'(k)$ we then obtain:

$$R(k) = A\rho(k) \text{ and } w(k) = A\omega(k) \tag{5.8}$$

As a result, the elasticity of interest rate $k\rho'(k)/\rho(k) = -(1-s)/\sigma < 0$ and the elasticity of real wage $\omega'(k)k/\omega(k) = s/\sigma > 0$, with $s \in (0,1)$ is the capital share in total income and $\sigma \in (0, +\infty)$ is the elasticity of capital-labor substitution.

### 5.2.2 Government

As in Schmitt-Grohé and Uribe (1997), Guo and Lansing (1998) and Bosi and Seegmuller (2010b) labor income taxes are used to finance government expenditure $G_t$. Hence, the instantaneous government budget constraint is:

$$G_t = N_t \tau(w_t l_t) \tag{5.9}$$

Notice that public spending does not contribute to either production or household utility.

## 5.3 Intertemporal equilibrium

The number of households at each generation grows at a constant rate $n > -1$ such that $1 + n = N_{t+1}/N_t$ where $N_t$ is the number of population born at time $t$. At equilibrium, all markets clears:

1. Capital market clears according to capital-accumulation equation: $N_t s_t = K_{t+1}$.
2. Labor market clears: $L_t = N_t l_t$.
3. Government spending $G_t$ is determined by the balanced budget rule (5.9).
4. By Walras' law, output market also clears: $N_t(c_t + s_t) + N_{t-1}d_t + G_t = AF(K_t, L_t)$.

From market clearing conditions, one can easily demonstrate that:

$$s_t = k_{t+1}(1+n)\, l_{t+1} \tag{5.10}$$

Substituting (5.10) and (5.8) together with conditions (5.2)-(5.3) in the first-order conditions (5.5) and (5.6) yields the following two-dimensional dynamic system of $k$ and $l$.

$$Bv'(l_t) = u'\left[\frac{\varphi(A\omega(k_t)\, l_t) - k_{t+1}(1+n)\, l_{t+1}}{B}\right] A\omega(k_t)\,\varphi'(A\omega(k_t)\, l_t) \tag{5.11}$$

$$v'(l_t) = \beta u'\left[A\rho(k_{t+1})\, k_{t+1}(1+n)\, l_{t+1}\right] A\rho(k_{t+1})\, A\omega(k_t)\,\varphi'(A\omega(k_t)\, l_t) \tag{5.12}$$

## 5.4 The steady state

The steady state of the dynamic system (5.11)-(5.12) is a solution $(k, l)$ of the system:

$$Bv'(l) = u'\left[\frac{\varphi(A\omega(k)\, l) - k(1+n)\, l}{B}\right] A\omega(k)\,\varphi'(A\omega(k)\, l) \tag{5.13}$$

$$v'(l) = \beta u'\left[A\rho(k)\, k(1+n)\, l\right] A\rho(k)\, A\omega(k)\,\varphi'(A\omega(k)\, l) \tag{5.14}$$

Obviously, there might exists one or multiple steady state for the system (5.13)-(5.14). However, to simplify the analysis, we follow Cazzavillan, Lloyd-Braga and Pintus (1998) by showing the existence of a normalized steady state such that $(k, l) = (1, 1)$ by selecting suitably the scaling parameters $A$, $B > 0$.[10]

**Assumption 4** $1 + \varepsilon_{11} > 0$.

This assumption has been made by, among others, Cazzavillan and Pintus (2004), Bosi and Seegmuller (2008) in order to ensure that consumptions in both periods and leisure are substitutable goods and saving function is increasing in the gross rate of return $R$.

**Proposition 13** *Let Assumptions* $1-4$ *be satisfied, there exists a steady state of dynamic system (5.11)-(5.12) such that* $k = 1$ *and* $l = 1$ *for one of these cases:*

[10] For simplicity, we concentrate on local dynamics around the normalized steady state without illustrating the possible existence of other steady state.

1. *If the following sufficient boundary conditions are satisfied:*

$$\lim_{A\to\underline{A}} \beta u'\left[A\rho\left(1\right)\left(1+n\right)\right]A\rho\left(1\right)A\omega\left(1\right)\varphi'\left(A\omega\left(1\right)\right) \quad < \quad v'\left(l\right) \tag{5.15}$$

$$\lim_{A\to+\infty} \beta u'\left[A\rho\left(1\right)\left(1+n\right)\right]A\rho\left(1\right)A\omega\left(1\right)\varphi'\left(A\omega\left(1\right)\right) \quad > \quad v'\left(l\right) \tag{5.16}$$

*then,* $\exists\, A^* > \underline{A} \equiv \varphi^{-1}\left(1+n\right)/\omega\left(1\right)$ *such that (5.14) is verified at* $(k,l)=(1,1)$*. However, if at* $(k,l)=(1,1)$*, we have* $1+\varepsilon_{11}+\eta_2+1>0$ *for all* $A$*, then* $A^*$ *is unique (because of the continuity of functions involved in (??), i.e., the functions* $\{v\,,u,F,\varphi\}\in C^2$*).*

2. *If the following sufficient boundary conditions are satisfied:*

$$\lim_{A\to\underline{A}} \beta u'\left[A\rho\left(1\right)\left(1+n\right)\right]A\rho\left(1\right)A\omega\left(1\right)\varphi'\left(A\omega\left(1\right)\right) \quad > \quad v'\left(l\right) \tag{5.17}$$

$$\lim_{A\to+\infty} \beta u'\left[A\rho\left(1\right)\left(1+n\right)\right]A\rho\left(1\right)A\omega\left(1\right)\varphi'\left(A\omega\left(1\right)\right) \quad < \quad v'\left(l\right) \tag{5.18}$$

*then,* $\exists\, A^* > \underline{A} \equiv \varphi^{-1}\left(1+n\right)/\omega\left(1\right)$ *such that (5.14) is verified at* $(k,l)=(1,1)$*. However, if at* $(k,l)=(1,1)$*, we have* $1+\varepsilon_{11}+\eta_2+1<0$ *for all* $A$*, then* $A^*$ *is unique (because of the continuity of functions involved in (5.14), i.e., the functions* $\{v\,,u,F,\varphi\}\in C^2$*).*

*Moreover, given the value of* $A^*$*, there exists* $B^*>0$ *which is a unique solution of (5.13) at* $(k,l)=(1,1)$*.*

**Proof.** See Appendix (A). ■

**Corollary 1** *Suppose that the after-tax income function* $\varphi\left(wl\right)$ *is locally isoelastic, then the boundary conditions (5.15)-(5.16) are sufficient for the existence of a unique* $A^* > \underline{A}$*.*

Simply, if $\varphi\left(wl\right)$ has an isoelastic formulation around the steady state, then one can show that $\eta_2=\eta_1-1$. As a result and according to Assumption 4 we have $1+\varepsilon_{11}+\eta_2+1>0$.

## 5.5 Local dynamics

In this section, we study the role of income taxation on the occurrence of endogenous fluctuations due to self-fulfilling prophecies. It is supposed that the after-tax income function $\varphi\left(w_t l_t\right)$ is

locally isoelastic which provides that $\eta_1 = 1 + \eta_2$. Linearizing the dynamic equation (5.11)-(5.12) around the normalized steady state $k = l = 1$ yields the following Jacobian matrix $J$:

$$J = \begin{bmatrix} \varepsilon_{11}\gamma/(1-\gamma) & \varepsilon_{11}\gamma/(1-\gamma) \\ \varepsilon_{11} - (1-s)(1+\varepsilon_{11})/\sigma & \varepsilon_{11} \end{bmatrix}^{-1} \begin{bmatrix} Z_2 & Z_1 \\ -s(1+\eta_2)/\sigma & \varepsilon_v - \eta_2 \end{bmatrix} \tag{5.19}$$

with $Z_1 \equiv \varepsilon_{11}(1+\eta_2)/(1-\gamma) + \eta_2 - \varepsilon_v$ and $Z_2 \equiv s(1+\eta_2)(\varepsilon_{11}/(1-\gamma)+1)/\sigma$. The characteristic polynomial of (5.19) is $P(\lambda) = \lambda^2 - T\lambda + D$, where the trace $T = \lambda_1 + \lambda_2$ and the determinant $D = \lambda_1\lambda_2$ are respectively given by:

$$T = \frac{X_1 + X_2 X_3}{\varepsilon_{11}\gamma(1-s)(1+\varepsilon_{11})} > 0 \tag{5.20}$$

$$D = \frac{s(1+\eta_2)(1+\varepsilon_v)}{\gamma(1-s)(1+\varepsilon_{11})} > 0 \tag{5.21}$$

with $X_1 \equiv \gamma\sigma\varepsilon_{11}(\varepsilon_v - \eta_2) + s\varepsilon_{11}(1+\eta_2)(1+\varepsilon_{11})$, $X_2 \equiv (1-s)(1+\varepsilon_{11}) - \sigma\varepsilon_{11}$ and $X_3 \equiv (\eta_2 - \varepsilon_v)(1-\gamma) + \varepsilon_{11}(\eta_2 + 1)$.

In this model, $l_t$ is an independently non-predetermined variable, this means that the steady state is locally indeterminate if and only if both eigenvalues are located within a unit circle, *i.e.*, $\lambda_1$,$\lambda_2 \in (-1, 1)$. Using the fact that the trace $T$ and the determinant $D$ are respectively the sum and the product of the eigenvalues, then local indeterminacy requires that $D < 1$ and $T < 1 + D$.

Therefore, we study analytically the conditions where the above inequalities are satisfied using the following parameters $\sigma$, $\varepsilon_{11}$, $\gamma$, $\eta_2$ and $\varepsilon_v$. As previously defined that $\sigma$ is the elasticity of capital-labor substitution, $\varepsilon_{11}$ is the elasticity of marginal utility of consumption, $\gamma$ is the propensity to save (saving over after-tax labor income), $\eta_2$ is the degree of marginal progressivity labor-income taxes, $\varepsilon_v$ is the elasticity of labor supply.

We present the critical values for $\eta_2$, $\varepsilon_{11}$, $\gamma$, $\sigma$ and $\varepsilon_v$ in Appendix (B). Our main results are summarized in the following proposition:

**Proposition 14** *Given that* $\gamma > \max\{\gamma^*, \gamma^D\}$ *and* $\sigma < \sigma^*$ *together with Assumptions* $1-4$,

*local indeterminacy emerges at the following conditions:*

1. $-1 < \varepsilon_{11} < \varepsilon_{11}^{D}$ *for all* $\varepsilon_v$ *and* $\eta_2 < \eta_2^{D}$.

2. $\varepsilon_{11}^{D} < \varepsilon_{11} < 0$ *with either* $\varepsilon_v < \varepsilon_v^{D}$, *all* $\eta_2$ *or* $\varepsilon_v > \varepsilon_v^{D}$ *and* $\eta_2 < \eta_2^{D}$.

**Proof.** See Appendix (C). ■

Mainly, Proposition (2) states that endogenous fluctuations require a sufficiently low elasticity of capital-labor substitution and high saving rates. Moreover, the presence of high progressive taxations ($\eta_2$ is sufficiently low) enforces the appearance of indeterminacy.

## 5.6 Discussions

In order to complete the characterization of economic stability, we provide the economic interpretations for local indeterminacy and show how the presence of progressive tax influences the appearance of endogenous cycles.

Before going through the economic intuition, we need to do the following computations: using (5.8) and the elasticity of interest rate $k\rho'(k)/\rho(k) = -(1-s)/\sigma$, we get:

$$\frac{\partial R_{t+1}}{R_{t+1}} = -\frac{1-s}{\sigma}\frac{\partial K_{t+1}}{K_{t+1}}, \frac{\partial R_{t+1}}{R_{t+1}} = \frac{1-s}{\sigma}\frac{\partial L_{t+1}}{L_{t+1}} \tag{5.22}$$

$$\frac{\partial l_t}{l_t} = \frac{1+\eta_2}{\varepsilon_v - \eta_2}\frac{\partial w_t}{w_t}, \frac{\partial l_t}{l_t} = \frac{1+\varepsilon_{11}}{\varepsilon_v - \eta_2}\frac{\partial R_{t+1}}{R_{t+1}} \tag{5.23}$$

The model of Cazzavillan and Pintus (2004) is recovered by setting zero labor income taxes, *i.e.*, $\eta_2 = 0$. The objective of this section is to show how a high tax progressivity promotes indeterminacy. Endogenous fluctuations arise due to the presence of interaction of two conflicting effects on savings that operate through wages and real interest rate.

Let us start by the Benchmark model where $\eta_2 = 0$. Assume at time $t$ an instantaneous increase in current capital stock $K_t$ from its steady state. This generates two opposite effects through wages and interest rate: an increase in $K_t$ implies a rise in the wage income, and according to (5.23), agents supply more labor. Given the budget constraint of young agents (5.2), a rise in wage income implies a rise in $K_{t+1}$.

The second effect is the anticipation effect that plays in the opposite direction in the sense that a higher $K_{t+1}$ is followed by a decline in the interest rate $R_{t+1}$. Given that $(\partial l_t/l_t)/(\partial R_{t+1}/R_{t+1}) = (1+\varepsilon_{11})/\varepsilon_v$, then a decrease in $R_{t+1}$ is followed by a decline in the labor supply and according to the budget constraint, a low accumulation of capital $K_{t+1}$ occurs. A cyclical path emerges whenever anticipation effect dominates current effect. This setting requires a sufficiently low elasticity of capital-labor substitution $\sigma$ as shown in (5.22) and a low elasticity of marginal dis-utility of labor $\varepsilon_v$ as shown in (5.23). Indeed, equality (5.23) shows that the interest rate has a strong influence on current labor supply for sufficiently high elasticity of intertemporal substitution in consumption.

In the presence of a progressive tax,[11]equation (5.23) shows that a higher progressivity ($\eta_2 \longrightarrow -1$) implies a lower elastic labor supply to both expected interest rate and real wage. In particular, when $\eta_2$ is close enough to $-1$, wages do not have any influence on labor supply (see (5.23)). As a result, this makes the anticipation effect more dominant than the current effect and so deterministic cycles are more likely to appear. Therefore, progressive taxes act as a destabilizing factor. This result is reasonable because introducing progressive taxes on labor income can stabilize wage income only. However, this is not enough to stabilize future consumptions since the latter depends on the capital real returns.

Above results are almost similar to those obtained by Chen and Zhang (2009a) with linear tax policy. Conversely, D-P (2006) consider a model where agents can consume only in the second period and save in the first period. They demonstrate that progressive taxation of capital income performs as a stabilizer factor. Intuitively, our results are in line with Dromel and Pintus in the sense that tax progressivity does not perform as a stabilizer if consumptions are partially financed by a non-taxed capital income.

Finally, in order to study the effect of the saving rate on steady state stability, consider the bifurcation value of $\sigma = \sigma^*$ given by (5.30) in the Appendix and differentiate it with respect to $\gamma$, we obtain

$$\frac{\partial \sigma^*}{\partial \gamma} = \frac{(1-s)(1+\varepsilon_{11})}{\varepsilon_{11}} \frac{\varepsilon_{11}+\eta_2-\varepsilon_v}{\varepsilon_v-\eta_2-\varepsilon_{11}(\eta_2+1)} > 0 \tag{5.24}$$

Therefore, a higher saving rate makes the emergence of local indeterminacy more likely.

[11]Notice that the presence of a progressive tax implies that $\eta_2$ declines from 0 to $-1$.

This result is analogous to Cazzavillan and Pintus (2004) after setting $\eta_2 = 0$. Therefore, in their model, the necessary condition for local indeterminacy becomes $\sigma < \sigma^*_{C,P}$ with

$$\sigma^*_{C,P} \equiv -\frac{s + (1+\varepsilon_{11})\left((1-s)\gamma - 1\right)}{\varepsilon_{11}}$$

The positivity of $\sigma^*_{C,P}$ requires a sufficiently high saving rate, that is, $\gamma > \gamma_{C,P}$ with $\gamma_{C,P} \equiv (1 + \varepsilon_{11} - s) / (1 + \varepsilon_{11})(1 - s)$. Additionally, one can easily find that $\sigma^*_{C,P} < s$ and $\partial\sigma^*_{C,P}/\partial\gamma > 0$. Thus, endogenous fluctuations necessitate a low elasticity of input substitution (complementary inputs) and a high propensity to save.

## 5.7 A numerical illustration

In this example, we present numerically the analytical results obtained in Proposition (2). To do that, consider the following explicit formulas for the production function, the preference and the after-tax income function respectively:

$$y = Af(k) = A\left[sk^{-\zeta} + (1-s)\right]^{-1/\zeta} \tag{5.25}$$

$$u(x) = \frac{x^{1-\varepsilon}}{1-\varepsilon} \text{ and } v(l) = \frac{l^{1+\varkappa}}{1+\varkappa} \text{ with } x = \frac{c}{B}, d \tag{5.26}$$

$$\varphi(w_t l_t) = (w_t l_t)^{1+\eta_2} \tag{5.27}$$

with $A > 0$ is the a scaling parameter, $s \in (0,1)$, $\eta_2 \in (-1,0)$, $\varepsilon \in (0,1)$, $\varkappa > 0$ and $\zeta > -1$ with $\zeta \neq 0$, $n = 0.5175$ and $\beta = 0.3$. Further, one can easy show that the elasticity of capital-labor substitution is $1/(1+\zeta)$, the elasticity of marginal utility of consumption is $\varepsilon_{11} = -\varepsilon < 0$ which is the inverse of the elasticity of intertemporal substitution in consumption $-1/\varepsilon$, the elasticity of marginal labor supply is simply $\varepsilon_v = \varkappa$ and the first-order elasticity of after-tax income is $1 + \eta_2 \in (0,1)$.

Given the normalized steady state $(k,l) = (1,1)$ then, capital share in total income does not depend on the elasticity of capital-labor substitution, thus $s \in (0,1)$. Let us set $s = 1/3$, then according to Proposition (2) one can show that the critical value $\varepsilon^D_{11} = -1/2$. Consistent with case (1) of Proposition (2), choose $\varepsilon = 0.55$ and $\varkappa = 1$ we obtain $\eta^D_2 = -0.55$. Therefore, consider $\eta_2 = -0.8$, we can easily compute the lower bound of the propensity to save values

$\gamma^D = 0.444\,44$ and $\gamma^* = 0.732\,15$. Since $\gamma > \max\{\gamma^D, \gamma^*\}$, then choose $\gamma = 0.74$, we get $\sigma^* = 5.267\,3 \times 10^{-3}$. This model has a one predetermined variable, so that, indeterminacy appears if and only if both eigenvalues of Jacobian matrix (5.19) belong to a unit cycle, *i.e.*, $\lambda_1, \lambda_2 \in (0,1)$. Therefore, setting $\sigma = 0.005$, we show that endogenous cycles emerge for above critical values with $\lambda_1 = 0.994\,19 \in (0,1)$ and $\lambda_2 = 0.604\,11 \in (0,1)$. Finally, given the above values, we compute the scaling parameter value $A = 12.134 > \underline{A} = 12.071$ which is necessary for normalizing the steady state.

Concerning the case (2) of Proposition (2), choose $\varepsilon = -0.4$ we obtain that $\varepsilon_v^D = 0.2$. Choose again $\varepsilon_v = 0.1$ and $\eta_2 = -0.5$, we can compute that $\gamma^D = 0.458\,33$ and $\gamma^* = 0.716\,67$. Thus, set $\gamma = 0.72$ we deduce the upper bound of the elasticity of capital-labor substitution $\sigma^* = 4.166\,7 \times 10^{-3}$, as a result, for $\sigma = 0.004$, endogenous cycles appear with eigenvalues: $\lambda_1 = 0.998\,72$ and $\lambda_2 = 0.637\,39$ and the scaling parameter $A = 6.214\,5 > 3.454\,2 = \underline{A}$.

The second conditions of case (2) are $\varepsilon_v > \varepsilon_v^D$ together with $\eta_2 < \eta_2^D$. Let us set $\varepsilon_v = 0.8$, then we get $\eta_2^D = -0.333\,33$. Therefore, choose $\eta_2 = -0.7$, then the saving rate values are $\gamma^D = 0.45$ and $\gamma^* = 0.789\,47$. If we consider $\gamma = 0.8$, then $\sigma^* = 1.234\,6 \times 10^{-2}$, so that, indeterminacy is obtained for $\sigma = 0.01$ with the eigenvalues: $\lambda_1 = 0.971\,81$ and $\lambda_2 = 0.578\,82$ with $A = 9.920\,4 > 6.023\,6 = \underline{A}$.

According to above numerical example, it is shown that local indeterminacy is obtained for non-plausible values of saving rate $\gamma$.[12]Moreover, one can observe that endogenous cycles require a sufficiently low elasticity of capital-labor substitution close to the Leontief case. However, for realistic values of consumption-wage ratio, around 65 percent, indeterminacy is no longer appear and the equilibria is locally unique and converges to the steady state.

## 5.8 Conclusion

We have studied the effect of a progressive labor-income tax policy on steady state properties in an OLG model where agents can consume in both periods (young and old). It is supposed separable preferences with respect to current-period consumption, future-period consumption and labor supply. Agents have to pay taxes related to their wage income. Further, the produc-

[12]The annual ratio of personal consumption expenditures over GDP have an average of 0.65 over the period (1959 - 2008) for US economy (see Economic Report of the President, 2008).

tion function exhibits a constant return-to-scale property. It is shown that progressive taxes act as a destabilizing factor where it makes endogenous fluctuation more likely. Similar to previous literature, indeterminacy occurs for sufficiently low elasticity of factor substitution close to Leontief and non-realistic high values of young's saving rate.

## 5.9 Appendix

### 5.9.1 (A) Proof of Proposition 1

**Proof.** Consider (5.13) and (5.14) at the normalized steady state $(1,1)$, we get:

$$v'(1) = \frac{1}{B}u'\left[\frac{\varphi(A\omega(1)) - (1+n)}{B}\right] A\omega(1)\varphi'(A\omega(1)) \tag{5.28}$$

$$v'(1) = \beta u'\left[A\rho(1)(1+n)\right] A\rho(1) A\omega(1)\varphi'(A\omega(1)) \tag{5.29}$$

Let us start with equality (5.29): LHS is a positive constant but the RHS is either increasing or decreasing in $A$. If we denote the RHS as $Q(A)$, then one can show that $\partial Q(A)/\partial A = 1 + \varepsilon_{11} + \eta_2 + 1$ which depends on $A$. Whenever $\partial Q(A)/\partial A > 0$ (resp. $< 0$), then the RHS of (5.29) is strictly increasing (resp. decreasing) in $A$. Moreover, the positivity of first period consumption requires $A > \underline{A} \equiv \varphi^{-1}(1+n)/\omega(1)$.

Let us firstly focus on the case where $\partial Q(A)/\partial A > 0$. In this case, the existence of a value $A^*$ that solves (5.29) requires the following boundary conditions: $\lim_{A\to\underline{A}} RHS < v'(l)$ and $\lim_{A\to+\infty} RHS > v'(l)$. Notice that if $\partial Q(A)/\partial A > 0$ for all $A$, then $A^*$ is unique.

However, when $\partial Q(A)/\partial A < 0$, the existence of $A^*$ which solves (5.29) necessitates initially $\lim_{A\to\underline{A}} RHS > v'(l)$ and further $\lim_{A\to+\infty} RHS < v'(l)$. If $\partial Q(A)/\partial A < 0$ for all $A$, then $A^*$ is unique.

Given $A = A^*$, then from (5.28), the LHS is a positive constant and the RHS is decreasing in $B$ according to Assumption 4. In addition, $\lim_{B\to 0^+} RHS = +\infty$ and $\lim_{B\to+\infty} RHS = 0^+$, therefore, there exists a unique $B^* > 0$ that solves (5.28).

Consequently, we have shown that there are unique $A^* > \underline{A}$ and $B^* > 0$ such that $(k,l) = (1,1)$ is a steady state of the system (5.11)-(5.12). ■

### 5.9.2 (B) The critical values

Here, we present the critical values of the model:

$$\sigma^* \equiv \frac{\varepsilon_{11}\left(1+\eta_2\right)\left[s\left(1+\varepsilon_v\right)-\left(1+\varepsilon_{11}\right)\right]+\left(1-s\right)\left(1+\varepsilon_{11}\right)\left[\left(\varepsilon_v-\eta_2\right)\left(1-\gamma\right)+\varepsilon_{11}\gamma\right]}{\varepsilon_{11}\left[\varepsilon_v-\eta_2-\varepsilon_{11}\left(\eta_2+1\right)\right]} \tag{5.30}$$

$$\gamma^* \equiv \frac{\varepsilon_{11}\left(1+\eta_2\right)\left(1+\varepsilon_{11}-s\left(1+\varepsilon_v\right)\right)-\left(1-s\right)\left(1+\varepsilon_{11}\right)\left(\varepsilon_v-\eta_2\right)}{\left(\varepsilon_{11}+\eta_2-\varepsilon_v\right)\left(1-s\right)\left(1+\varepsilon_{11}\right)} \tag{5.31}$$

$$\gamma^D \equiv \frac{s\left(1+\eta_2\right)\left(1+\varepsilon_v\right)}{\left(1-s\right)\left(1+\varepsilon_{11}\right)} \tag{5.32}$$

$$\eta_2^* \equiv -\frac{\varepsilon_{11}s\left(1+\varepsilon_v\right)+\left(1+\varepsilon_{11}\right)\left(\left(1-s\right)\varepsilon_v-\varepsilon_{11}\right)}{\varepsilon_{11}s\left(1+\varepsilon_v\right)-\left(1+\varepsilon_{11}\right)\left(1-s+\varepsilon_{11}\right)} \tag{5.33}$$

$$\eta_2^D \equiv \frac{\left(1-s\right)\left(1+\varepsilon_{11}\right)-s\left(1+\varepsilon_v\right)}{s\left(1+\varepsilon_v\right)} \tag{5.34}$$

$$\varepsilon_v^D \equiv \frac{\left(1-s\right)\left(1+\varepsilon_{11}\right)-s}{s} \tag{5.35}$$

$$\varepsilon_{11}^D \equiv \frac{2s-1}{1-s} \tag{5.36}$$

$$\varepsilon_{11}^{**} \equiv s-1 \tag{5.37}$$

### 5.9.3 (C) Proof of Proposition 2

**Proof.** In this proposition, we show that the existence of endogenous fluctuations requires $D<1$ and $T<1+D$. Let us begin with $D<1$.

$D<1$ if and only if $\gamma>\gamma^D$ with $\gamma^D>0$. However, $\gamma^D<1$ is met if and only if $\eta_2<\eta_2^D$ with $\eta_2^D>-1$. Since $\eta_2$ is the second-order elasticity, then it should always be negative, so $\eta_2^D<0$ if and only if $\varepsilon_v>\varepsilon_v^D$. One has to be sure that $\varepsilon_v^D$ is positive, a direct inspection implies that for $\varepsilon_v^D>0$ if and only if $\varepsilon_{11}>\varepsilon_{11}^D$.

As a result $D<1$ for (1) $\varepsilon_{11}>\varepsilon_{11}^D$, $\gamma>\gamma^D$ and either $\varepsilon_v>\varepsilon_v^D$ and $\eta_2<\eta_2^D$ or $\varepsilon_v<\varepsilon_v^D$, for all $\eta_2<0$. In addition, (2) $D<1$ for $\varepsilon_{11}<\varepsilon_{11}^D$, $\eta_2<\eta_2^D$ and $\gamma>\gamma^D$ for all $\varepsilon_v>0$.

Furthermore, $T<1+D$ if and only if $\sigma<\sigma^*$. The latter inequality requires $\sigma^*>0$ which is met for $\gamma>\gamma^*$ with $\gamma^*<1$. The positivity of $\gamma$ requires that $\gamma^*>0$ and this inequality is

satisfied if and only if $\eta_2 < \eta_2^*$. It is immediate to verify that $\eta_2^* > 0$ for $\varepsilon_{11} > \varepsilon_{11}^{**}$ and $\eta_2^* < 0$ for $\varepsilon_{11} < \varepsilon_{11}^{**}$. On the one hand, when $\eta_2^* > 0$, then $\gamma^* > 0$ for all $\eta_2 < 0$.

As a result, $T < 1 + D$ for $\sigma < \sigma^*$ and $\gamma > \gamma^*$ with $\varepsilon_{11} > \varepsilon_{11}^{**}$ for all $\eta_2 \in (-1, 0)$ and all values of $\varepsilon_v > 0$. On the other hand, whenever $\varepsilon_{11} < \varepsilon_{11}^{**}$ then $\eta_2^* \in (-1, 0)$ always. As a result, $T < 1 + D$ for $\sigma < \sigma^*$, $\varepsilon_{11} < \varepsilon_{11}^{**}$ and for all $\varepsilon_v > 0$ with *either* $\gamma > \gamma^*$ and $\eta_2 < \eta_2^*$ *or* $\gamma \in (0, 1)$ and $\eta_2 > \eta_2^*$.

If we do the intersection between the conditions of $D < 1$ and those of $T < 1 + D$ and considering that $\varepsilon_{11}^{**} < \varepsilon_{11}^{D}$ and $\eta_2^D < \eta_2^*$ for $\varepsilon_{11} < \varepsilon_{11}^{**}$, we get the local indeterminacy conditions summarized in Proposition (2). ■

# Chapter 6

# Conclusion

One of the main objectives of modern macroeconomic theory is to explain the cyclical characteristics of economic evolution. There are two principal theories trying to clarify the origins and the characteristics of business cycles. The real business cycles theory states that cyclical fluctuations arise due to the presence of exogenous productivity shocks. However, endogenous business cycles theory states that economic fluctuations, which emerge through the existence of local indeterminacy and sunspot equilibria, occur as a result of agents' self-fulfilling expectations with no effects concerning the fundamentals (resources, technology and preference).

This doctoral thesis is related to the endogenous cycle theory and is interested in macroeconomic instability and fluctuation issues. Mainly, we consider the infinite-horizon Ramsey model with heterogeneous agents in a standard overlapping generation framework.

Particularly, in chapter two, we introduce consumption externalities in a heterogeneous Ramsey model with endogenous labor supply and borrowing constraints. Heterogeneity stems from different initial capital, labor supply and preference. Moreover, it is proposed that a representative agent's preference depends on his own consumption in addition to other agents' levels of consumption. This model is based on the famous work of Becker (1980) and Becker and Foias (1987) and represents a bridge between the model of Bosi and Seegmuller (2010a) and Barbar and Barinci (2010) where in the latter they extend the model of Becker-Foias by considering consumption externalities while the former proposes an endogenous labor supply.

This chapter shows that there are two steady states concerning labor supply: the first where both agents supply labor and the second one consists of only impatient agents labor

supply while the patient agents enjoy leisure. As in a standard model, only the patient agent holds the entire capital stock at the steady state whereas the impatient agent consumes his wage-income only. The second result is that the steady state can change its stability through cycles of period two. The main contribution of this chapter consists on the existence of a room for local indeterminacy since the presence of endogenous labor supply and consumption externalities gives rise to an intratemporal effect, generating endogenous fluctuations due to self-fulfilling expectations. Finally, we show that, under specific parameterization values, these endogenous fluctuations can be ruled out even when the elasticity of input substitution is close to zero.

Chapter three tries to explain the U.S. subprime crisis, which has affected the global economy in the last few years, by studying the role of agents' collaterals in the emergence of endogenous fluctuations. Therefore, we consider a credit market imperfection in a heterogeneous Ramsey model with money and constant labor supply. Money is introduced via a CIA constraint. Heterogeneity among agents is similar to that of chapter two. We assume that collaterals arise in a partial cash-in-advance constraint in the sense that a fraction of consumption goods is paid by cash and the remainder is paid by credit. Payment by credit depends on the amount of collaterals held by agents where collaterals are represented by agent's current capital stock. More collateral is held by households, more consumption goods are financed by credit payments.

Two configurations are considered: the first where patient agents do not hold balances (since they hold all the capital in the market) while in the second they hold a segment portfolio of money and capital.

In both regimes, monetary expansions increase patient agents' welfare and reduce that of impatient agents. However, from an economic growth point of view, money growth has no effects on economic growth in the first regime while in the second regime, a high money growth implies a high economic growth.

In the second regime, a high money growth encourages the occurrence of endogenous fluctuations due to self-fulfilling expectations. Conversely, high collaterals held by agents act as a stabilizing effect in the sense that it reduces the possibility of economic fluctuation emergence.

Chapter Four introduces variable transaction costs into a simple OLG model à la Diamond (1965) with constant labor supply. Contrary to the literature, which consider financial envi-

ronments and aim to study the influences of transaction costs on stock and asset pricing and trading, our objective is to focus on the effect of these costs on the steady state stability and on agents' capital accumulation. It is supposed that cost function is increasing with respect to capital accumulation and its marginal increasing declines. It is shown that the presence of transaction costs makes the appearance of cycles of period two more likely. Moreover, the steady state is locally determinate in the sense that there exists a unique trajectory converging to the stationary equilibrium. Comparing to standard Diamond model, the steady state associated with transaction costs exhibits under-accumulation of capital according to standard Golden Rule.

Finally, in the last chapter, we consider a two-period consumption OLG model (for instance, Cazzavillan and Pintus (2004)) augmented to include progressive labor-income taxation policy. The objective is to study the effects of progressive fiscal rule on steady state stability. It is found that progressive tax policy performs as a destabilizing effect in the sense that it makes local indeterminacy more likely to appear. This result is analogous to that of Chen and Zhang (2009a) who propose an endogenous linear labor-income tax policy in a similar OLG framework. Furthermore, this outcome is related to the work of Dromel and Pintus (2006) where they introduce a progressive capital-income taxation policy into an OLG model à la Reichlin (1986) and deduce that this tax acts as a stabilizing effect in the sense that it makes endogenous fluctuations less likely. Their argument is that: if agents' consumptions are mostly financed by capital income (resp. labor income), then if the government is interested in economic stability, it should apply a progressive taxation on capital income (resp. labor income). In our model, consumptions are financed by both wage and capital income while taxation rule is applied only to wage income. In addition, local indeterminacy arises if and only if the first-period saving rate is sufficiently high which means that most of agents' consumptions are carried out by a non-taxed income (capital income). This explains why the fiscal policy applied in this model acts as a destabilizing factor.

For future research, a possible extension in chapter four is to consider an endogenous labor supply. As shown, in this chapter, that the presence of variable transaction costs promotes agents to save less (to accumulate lower capital). However, in a standard OLG model with endogenous labor supply, Cazzavillan and Pintus (2004) show that local indeterminacy appears

if and only if young agents' saving rate is unrealistically high which is a non-plausible value. Hence, we expect that introducing variable transaction costs will minimize the lower bound value of saving rates associated with endogenous fluctuations and sunspot equilibria.

However, chapter five allows to two possible extensions: the first is by considering an additional progressive taxation policy on capital income and hence both period consumptions are financed by a taxed income. According to Dromel and Pintus (2006), it is expected that taxation both capital and wage income makes the self-fulfilling expectations less likely to appear. Another extension is to consider an OLG model where agents can work even in the second period of their life and pay income taxes. Using Bureau of Labor Statistics Data (December 2009), we remark that agents who are 55 years old or older represent almost 11 percent of total labor force in 1998 while this percentage increases in 2008 to around 18 percent and expected to arrive to 23 percent in 2018. Therefore, considering an endogenous old-agents labor supply would not be considered as unrealistic assumption. We expect that steady state stability effects require a very high tax progressivity for second-period labor income. Additionally, since agents work in both periods, it is expected that young agents saving rate declines comparing to standard model and therefore local indeterminacy might emerge with a lower saving rate.

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