The 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019)

Paris, France

August 28^{th}-31^{th}, 2019

Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM2019)

Editors

Yilmaz SIMSEK, Abdelmejid BAYAD, Mustafa ALKAN, Irem KUCUKOGLU and Ortaç ÖNEŞ

ISBN 978-2-491766-00-9

http://micopam.akdeniz.edu.tr/

Conference Venue: Université d'Evry-Val-d'Essonne, 23 Boulevard François Mitterrand, 91000 Évry, Paris-FRANCE

TITLE

Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM2019)

EDITORS

Yilmaz SIMSEK

Abdelmejid BAYAD

Mustafa ALKAN

Irem KUCUKOGLU

Ortaç ÖNEŞ

PUBLISHER

Université d'Evry / Université Paris-Saclay

ISBN

978-2-491766-00-9

EDITION AND YEAR

First Edition, 2019

OPENING CEREMONY TALK of MICOPAM 2019

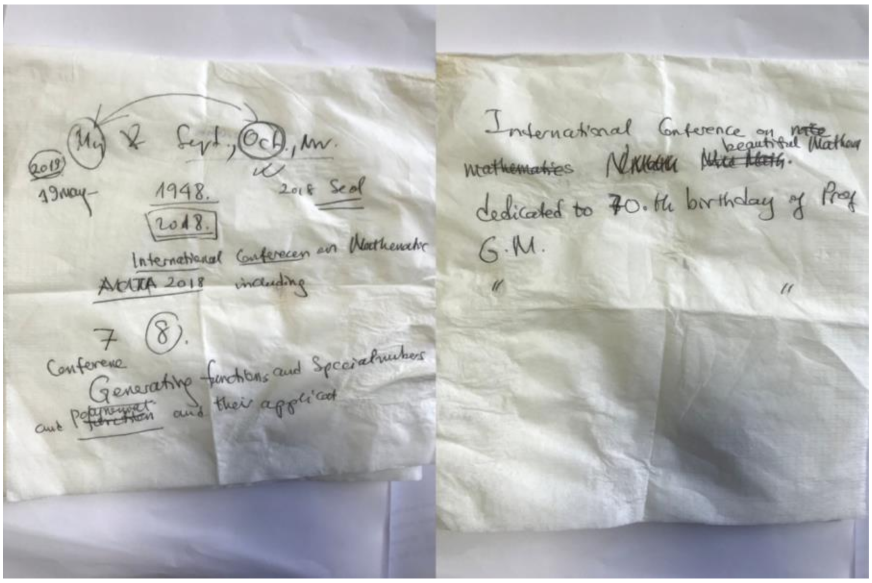
Dear distinguished participants of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019), held in Paris-France, on August 28–31, 2019.

On behalf of the Scientific and Organizing Committees, I would like to say that "Welcome to Paris, which is not only the capital of France, but also one of the most popular cities in Europe."

Now, this city, where art meets architecture, hosts to our conference MICOPAM 2019.

As you may already know, the first of MICOPAM conference (MICOPAM 2018) was held in Turkey's pretty and historical Mediterranean resort town Antalya on October 26–29, 2018. Note that the conference MICOPAM 2018 was dedicated to the 70th birthday of the renowned mathematician Professor Gradimir V. Milovanović, born in Zorunovac, Serbia, 2 January 1948, is one of the world leading scientists in the field of numerical analysis, approximation theory and special functions, a longtime professor at the University of Niš, Serbia, and a member of the Serbian Academy of Sciences and Arts.

The idea of organizing the MICOPAM conference has been firstly appeared in 2017 in Belgrade, Serbia, while speaking with Professor Gradimir V. Milovanović.



Brainstorming for conference name on napkin with Professor Gradimir V. Milovanović in 2017 at Belgrade, Serbia.

Our dreams came true last year and MICOPAM 2018 was successfully held over four days, with presentations made by not only researchers coming from the international communities, but also distinguished keynote speakers.

With the same passion, this year too it is aimed to continue this conference series started last year in Antalya and therefore, it is decided to organize the second of MICOPAM at Université d'Evry Val d'Essonne, Évry in Paris, France by the Professor

Abdelmejid Bayad's great cooperation with me and Professor Mustafa Alkan from Antalya Akdeniz University, Turkey.

Today, we are very happy to make the opening ceremony of the second of the conference MICOPAM together. Thus, dear distinguished participants, you have given honor to us by attending our conference: MICOPAM 2019.

I would like to remind you that MICOPAM conference will be held regularly every year. In particular, "The 3rd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas" is scheduled to be held again in Antalya, Turkey next year. By the way Antalya, which is one of our historical cities, has been a source of inspiration for many empires and civilizations. I hope that next year you will visit some part of this pretty and historical city of Turkey with Antalya near the cost of the Mediterranean Sea which connects almost all the countries, seas and oceans.

From Left to Right: Professor Yilmaz Simsek, Professor Walter Gautschi, Professor Gradimir V. Milovanović

I would like to thank to the following my colleagues and students who helped me at every stage of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019): Local Organizing Committee: (including especially Prof. Dr. Mustafa Alkan, Prof. Dr. Abdelmejid Bayad, Conference Secretary Asst. Prof. Dr. Irem Kucukoglu, Asst. Prof. Dr. Ortaç Öneş, Dr. Neslihan Kilar, Dr. Busra Al, Dr. Buket Simsek); my precious family: (my wife Saniye, my daughters Burcin and Buket); and also other friends whose names that I did not mention here.

Also, thanks to the following all invited speakers:

- Isabel Cação, (University of Aveiro, Portugal)
- Marc-Antoine Coppo, (Université Côte d'Azur, France)
- Alan Filipin, (Zagreb University, Croatia)
- Yoshinori Hamahata, (Okayama University of Science, Japan)
- Satish Iyengar, (University of Pittsburgh, USA)
- Daeyeoul Kim, (Chonbuk National University, South Korea)

- Branko Malešević, (University of Belgrade, Serbia)
- Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
- Hussein Mourtada, (University Université Paris 7, France)
- Ayoub Otmani, (University of Rouen Normandy, France)
- Mehmet Ali Özarslan, (Eastern Mediterranean University, North Cyprus, Turkey)
- Olivier Ruatta, (Université de Limoges, France)

Thanks also to Editors (Prof. Dr. Abdelmejid Bayad, Prof. Dr. Mustafa Alkan, Asst. Prof. Dr. Irem Kucukoglu, and Asst. Prof. Dr. Ortaç Öneş) for their most valuable contribution on preparing both Abstract Booklet and Proceedings Book of MICOPAM 2019.

As for mathematics; Mathematics is the common heritage of everyone; Mathematics is the common language of the world that is always passed from generation to generation by refreshing.

It would be appropriate to say the following:

In addition to the poetic and artistic aspect of mathematics, mathematics has such a spiritual, magical and logical power, all-natural science and social science cannot breathe and survive without mathematics.

Mathematics is such a branch of science that other sciences cannot develop without it. Therefore, Mathematics, which is the oldest of Science, has contributed fundamentally to the development of our world civilizations. So, we can enter into the science and technology centers using the power of mathematics and its branches. So, mathematics and its branches create the possibility of bridgework and communication between the Natural Sciences and the Engineering Sciences as well as the Economic and also Social Science.

The aim of the conference MICOPAM 2019 is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other. In fact, the main purpose of this conference is to bring to the fore the best of research and applications that will help our world humanity and society. Due to the valuable idea of the MICOPAM, this conference welcomes speakers whose talk or poster contents are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

To summarize my speech, this conference has provided a novel opportunity to our distinguished participants to meet each other and share their scientific works and friendships in the above areas.

I am delighted to note that all participants have free and active involvement and meaningful discussion with other participants during the conference at Université d'Evry Val d'Essonne, Évry in Paris, France.

It is my great pleasure to thank Professor Abdelmejid Bayad because he gives opportunity to organize MICOPAM in Paris, France. It is my great pleasure to thank again local organizing committee. Consequently, I send my thanks to all distinguished invited speakers, and all valuable participants with their accompanying persons.

On behalf of the Organizing Committee of MICOPAM 2019

PROF. DR. YILMAZ SIMSEK

Department of Mathematics, Faculty of Science,

Akdeniz University, TR-07058 ANTALYA-TURKEY, Tel: +90 242 310 23 43,

Email : ysimsek@akdeniz.edu.tr, ysimsek63@gmail.com

FOREWORD

Why we call the name of the conference as "The 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019)". Because by the Mediterranean Sea, almost all the countries, the sea and the oceans are connected. For this reason, just like Mediterranean Sea, the main aim of MICOPAM 2019 is to give connection between many areas of sciences including physical mathematics and engineering, especially all branches of mathematics. A few of these areas can be given as follows: Pure and Computational and Applied Mathematics, Statistics, Mathematical Physics (related to p-adic Analysis, Umbral Algebra and Their Applications). Another important purpose of MICOPAM 2019 is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other in the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

A brief description about the contents of “Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019)” is given as follows:

The first chapter of the Proceedings Book of MICOPAM 2019 includes opening ceremony talk, foreword and some information about MICOPAM including the meaning of its name and the name of conference committee members. Its second chapter includes full-text Proceedings (at least 3 pages and at most 5 pages long), of participants who signed “COPYRIGHT RELEASE FORM FOR MICOPAM 2019 of CONFERENCE PROCEEDINGS” sent after acceptance followed by the evaluation made by Scientific Committee. In addition to the Proceedings Book of MICOPAM 2019, Abstract Booklet of MICOPAM 2019 has been published. As for the scope of the Abstract Booklet of MICOPAM 2019, it consists of foreword about the MICOPAM 2019 including conference committees, and short abstracts of both oral and poster presentations that of some participants who did not submit full-text Proceedings to MICOPAM.

Consequently, the Proceedings Book and the Abstract Booklet of MICOPAM 2019 are resources for researchers who works in above mentioned branches of science (Mathematics, Physics, Engineering) and other relevant areas.

In this regard, we would like to thank to all invited speakers and participants for their contributions. Finally, we express our sincere thanks to all members of the scientific committee and all members of the organizing committee because of their efforts to the success of the conference.

Editors of MICOPAM 2019

ABOUT CONFERENCE

The 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019) has been held in Paris, France for four days from August 28 to August 31, 2019. MICOPAM 2019 was organized by Professors Abdelmejid Bayad (Université d'Evry Val d'Essonne, France), Yilmaz Simsek (Akdeniz University, Turkey) and Mustafa Alkan (Akdeniz University, Turkey).

During the dates of MICOPAM 2019, a great number of excellent oral and poster presentations was made by 65 participants from 16 different countries (Algeria, Canada, Croatia, France, India, Jordan, Nigeria, Portugal, Russia, Saudi Arabia, Serbia, Spain, South Africa, South Korea, Thailand, Turkey).

Contents of oral and poster presentations are mainly related to the following areas: Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

Further details about MICOPAM2019 is given as follows:

COMMITTEES

Scientific Committee

- Mustafa Alkan, Turkey,
- Ravi Agarwal, USA,
- Abdelmejid Bayad, France,
- Nenad Cakić, Serbia,
- Ismail Naci Cangul, Turkey,
- Abdullah Cavus, Turkey,
- Ahmet Sinan Cevik, Turkey,
- Junesang Choi, South Korea,
- Dragan Djordjević, Serbia,
- Allal Guessab, France,
- Mohand Ouamar Hernane, Algeria,
- Satish Iyengar, USA,
- Taekyun Kim, South Korea,
- Miljan Knežević, Serbia,
- Veli Kurt, Turkey,
- Branko Malešević, Serbia,
- Francisco Marcellán, Spain,
- Giuseppe Mastroianni, Italy,
- Gradimir V. Milovanović, Serbia,
- Tibor Poganj, Croatia,
- Abdalah Rababah, Jordan,
- Themistocles Rassias, Greece,
- Lothar Reichel, USA,
- Ekrem Savas, Turkey,
- Yilmaz Simsek, Turkey,
- Burcin Simsek, USA
- Miodrag Spalević, Serbia,
- Hari M. Srivastava, Canada,
- Marija Stanić, Serbia

Organizing Committees

Heads of the Organizing Committees

- Prof. Dr. Abdelmejid Bayad, (Université d'Evry Val d'Essonne, France)
- Prof. Dr. Yilmaz Simsek, (Akdeniz University, Turkey)
- Prof. Dr. Mustafa Alkan, (Akdeniz University, Turkey)

MICOPAM 2019 Conference Secretary

- Asst. Prof. Dr. Irem Kucukoglu, (Alanya Alaaddin Keykubat University, Turkey)

International Organizing Committee

- Chandrashekar Adiga, (University of Mysore, India)
- Mustafa Alkan, (Akdeniz University, Turkey)
- Ayman Badawi, (American University of Sharjah, United Arab Emirates (UAE))
- Abdelmejid Bayad, (Université d'Evry Val d'Essonne, France)
- Satish Iyengar, (University of Pittsburgh, USA)
- Daeyeoul Kim, (Chonbuk National University, South Korea)
- Taekyun Kim, (Kwangwoon University, South Korea)
- Irem Kucukoglu, (Alanya Alaaddin Keykubat University, Turkey)
- Veera Lokesha, (Vijayanagara Sri Krishnadevaraya University, India)
- Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
- Ortaç Öneş, (Akdeniz University, Turkey)
- Yilmaz Simsek, (Akdeniz University, Turkey)

Local Organizing Committee

- Busra Al
- Mustafa Alkan
- Abdelmejid Bayad
- Neslihan Kilar
- Irem Kucukoglu
- Ortaç Öneş
- Buket Simsek
- Yilmaz Simsek

INVITED SPEAKERS

- Isabel Cação, (University of Aveiro, Portugal)
- Marc-Antoine Coppo, (Université Côte d'Azur, France)
- Alan Filipin, (Zagreb University, Croatia)
- Yoshinori Hamahata, (Okayama University of Science, Japan)
- Satish Iyengar, (University of Pittsburgh, USA)
- Daeyeoul Kim, (Chonbuk National University, South Korea)
- Branko Malešević, (University of Belgrade, Serbia)
- Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
- Hussein Mourtada, (University Université Paris 7, France)
- Ayoub Otmani, (University of Rouen Normandy, France)
- Mehmet Ali Özarslan, (Eastern Mediterranean University, North Cyprus, Turkey)
- Olivier Ruatta, (Université de Limoges, France)

MICOPAM 2019

The 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas

Paris-France, August 28-31, 2019

At Université d’Evry Val d’Essonne

Invited Speakers

Isabel Cação, (University of Aveiro, Portugal)
Marc-Antoine Coppo, (Université Côte d’Azur, France)
Alan Filipin, (Zagreb University, Croatia)
Yoshinori Hamahata, (Okayama University of Science, Japan)
Satish Iyengar, (University of Pittsburgh, USA)
Daeyeoul Kim, (Chonbuk National University, South Korea)
Branko Malešević, (University of Belgrade, Serbia)
Gradimir V. Milovanović, (Serbian Academy of Sciences and Arts, Serbia)
Hussein Mourtada, (University Université Paris 7, France)
Ayoub Otmani, (University of Rouen Normandy, France)
Mehmet Ali Özarslan, (Eastern Mediterranean University, North Cyprus, Turkey)
Olivier Ruatta, (Université de Limoges, France)

Aim and Scopus

The aim of the conference is to bring together leading scientists of the pure and applied mathematics and related areas to present their researches, to exchange new ideas, to discuss challenging issues, to foster future collaborations and to interact with each other. This conference welcomes speakers whose talk or poster contents are mainly related to the following areas:

Mathematical Analysis, Algebra and Analytic Number Theory, Combinatorics and Probability, Pure and Applied Mathematics & Statistics, Recent Advances in General Inequalities on Pure & Applied Mathematics and Related Areas, Mathematical Physics, Fractional Calculus and Its Applications, Polynomials and Orthogonal systems, Special numbers and Special functions, Q-theory and Its Applications, Approximation Theory and Optimization, Extremal problems and Inequalities, Integral Transformations, Equations and Operational Calculus, Partial Differential Equations, Geometry and Its Applications, Numerical Methods and Algorithms, Scientific Computation, Mathematical Methods and computation in Engineering, Mathematical Geosciences.

www.univ-evry.fr

http://micopam.akdeniz.edu.tr/

Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas 2019 (MICOPAM 2019)

Contents

Divisor functions and Polygon Shape Numbers

Sang-Hoon Park[1] , *Abdelmejid Bayad*[2] , *Daeyeoul Kim*[1] *

Abstract

This note describes some polygon shape numbers derived from iteration of restricted divisor functions. These numbers are closely related to Mersenne primes. In this note, we state some questions related to derived iterated stable numbers and amicable pairs from iterated divisor functions.

2010 Mathematics Subject Classifications : 11M36, 11F11, 11F30.
Keywords: Restricted divisor function, divisor functions, Mersenne prime.

Introduction

The divisor function is an arithmetic function known from ancient times in the number theory. It was studied by Euler, Lambert, Waring, Jacobi, Dirichlet, Stern, Bouniakowsky, Liouville, Hermite, Glaisher, Ramanujan, Williams, etc. There is a great deal of literature concerning the iteration of the aliquot sum and odd divisor function, much of it concerned with whether the iterated values eventually terminate at zero (cycle) or become unbounded, depending on the value of n. Iterations of several restricted divisor functions can show curious results. It is also interesting to note that iterating the sum of divisor function and their applications have been investigated by Cohen and Riele. We give to interpret and understand these results in a number-theoretical way.

Observation of iterated numbers

The pair (m, n) is an *a*micable pair if $s(m) = n$ and $s(n) = m$ with $m \neq n$. And a collection of numbers n_1, $n_2, \cdots, n_r$ is called *s*ociable if $s(n_1) = n_2$, $s(n_2) = n_3$, $\cdots$, $s(n_{r-1}) = n_r$, $s(n_r) = n_1$. We will define the iterated amicable pair and the iterated stable number with respect to the restricted divisor functions $D(\bullet; t)$. Similarly, we define an ordered pair (m, n) is an *i*terated amicable pair of $D(\bullet; t)$ if $D(n; t) = m$ and $D(m; t) = n$, where m and n are distinct positive integers. A collection of numbers n_1, $n_2, \cdots, n_r$ is called *i*terated sociable of $D(\bullet; t)$ if $D(n_1; t) = n_2$, $D(n_2; t) = n_3, \cdots, D(n_r; t) = n_1$. In particular, if $n_1 = n_2 = ... = n_r$ then we call n an *i*terated stable number of $D(\bullet; t)$, that is, $D_u(n; t) = D_{u+l}(n; t)$ with $l \geq 0$. Since 1 is a trivial number, we define that 1 is not an iterated stable number of $D(\bullet; t)$. We will give the numerical data obtained by using the Mathematica 11.2, and suggest some questions which we consider the existence of iterated amicable pairs and stable numbers. Consider the cases $D(n, 2)$, we obtain stable and amicable pairs.

An amicable pair (m, n) derived from σ satisfies $\sigma(m) = \sigma(n) = s(m) + s(n) = m + n$, where $s(y) = \sigma(y) - y$. Similarly, an iterated amicable pair (m, n) derived from D satisfy $D(m) + D(n) = m + n$ but $D(m) \neq s(n) + s(m)$. The above two are a little different.

A perfect number is a positive integer that is equal to the sum of its proper positive divisors, that is, $\sigma(n) = 2n$ [8]. If $D(n) = 2n$, then we call n an iterated perfect number derived from D. A natural number 6 is a unique iterated perfect number derived from D.

Results of Observation

In [1], we see the invariants(order, m-gonal shape number, type, convexity, area) derived from iterated odd divisor functions $S_m(n)$. Similarly, we examine the properties of the restricted divisor function $D_m(n)$. Consider a sequence $\{D_m(n)\}$, $m = 0, 1, 2, 3, ...$, where $D_0(n) := n$, $D(n) := D_1(n; 2)$, and $D_m(n) := D(D_{m-1}(n))$ with $m \geq 1$. First, we introduce three sets

$$A_0(2) := \{1\},\ A_1(2) := \{3, 4\},\ A_2(2) := \{12\}.$$

Using Mathematica 11.2, we compute values of $D_m(n)$ with $n \leq 2^{20}$. By fixing n, we calculate the value of $D_m(n)$ while increasing m, and stop the calculation when $D_m(n) \in A_i(2)$. That is, when $D_m(n) \in A_i(2)$, we make an algorithm to stop the calculation. Here, $i \in \{0, 1, 2\}$. In fact, we could see that $D_m(n) \in A_i(2)$ with $n \leq 2^{20}$ for some integer m.

Amicable pairs was studied by Pythagoras, Thabit ibn Kurrah, Fermat, Descartes, Mersenne, Euler, Dickson, Legendre, Escott, etc. In 1986, the list of all 1427 amicable pairs which are smaller than 10^{10} was published by Riele. So, we want to know which iteration list of amicable pairs will be the same. Let us examine the iterated amicable pairs by using the well-known formulas.

Acknowledgements

This research is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(NRF-2018R1D1A1B07041132).

References

[1] A. Bayad and D. Kim, *Polygon Numbers Associated with the Sum of Odd Divisors Function*. Exp. Math. 26 (2017), no 3, 287–297.

[2] G. L. Cohen, H. J. J. te Riele, *Iterating the sum-of-divisors function*. Experiment. Math. **5** (1996), no. 2, 91–100.

[3] P. Erdös *Some remarks on the iterates of the ϕ and σ functions*, Colloq. Math. **17** (1967), 195–202.

[4] E. B. Escott, *Amicable numbers*. Scripta Math. **12** (1946), pp. 61–72.

[5] D. Kim, U. Sharp, S. Ikikardes, *Certain Combinatoric Convolution Sums arising from Bernoulli and Euler polynomials*. Miskolc Mathematical Notes, **23** no 1 (2019), 311–330.

[6] G. Lord, *Even perfect and superperfect numbers*, Elem. Math. **30** (1975), 87–88.

[7] J. H. Silverman, *A Friendly Introduction to Number Theory*. Pearson Education, Inc., 2012.

[8] D. E. Smith, *The History of Mathematics: Volume II.* Dover, 1958.

[9] GIMPS Project Discovers Largest Known Prime Number: $2^{282,589,933} - 1$, Mersenne Research, Inc. 21 December 2018. Retrieved 21 December 2018.

[10] László Tóth, *A survay of the alternating sum-of-divisors function.* ACTA Univ. SAPIENTIAE, MATHEMATICA, **5**, 1 (2013), 93-107.

[11] D. Suryanarayana, *Super perfect numbers*, Elem. Math., **24** (1969), 16-17.

[12] The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[1]Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896, South Korea

[2]Université d'Evry Val d'Essonne, Département de mathématiques, Bâtiment I.B.G.B.I., 3ème étage, 23 Boulevard de France, 91037 Evry cedex, France

E-mail : decent@jbnu.ac.kr, abayad@maths.univ-evry.fr, kdaeyeoul@jbnu.ac.kr

On hypercomplex orthogonal polynomials

Isabel Cação[1]

Abstract

Hypercomplex function theory (or Clifford analysis) generalizes the theory of holomorphic functions of one complex variable to higher dimensions in the framework of Clifford algebras. In this context, we present different representations of a general orthogonal Appell sequence that is solution of a second order differential equation and that satisfies a three-term recurrence formula, similar to the case of orthogonal polynomials of one real variable. Each polynomial of the sequence is the product of an arbitrarily chosen generalized holomorphic polynomial and a certain homogeneous polynomial that turns out to be obtained by a simple shift on the coefficients of the hypercomplex generalization of the holomorphic powers done in 2007 by Falcão/Malonek. Moreover, the study of that coefficients shows interesting new insights into rational number sequences closely related to important results of Vietoris on the positivity of certain trigonometric sums (1958).

2010 Mathematics Subject Classifications : 30G35, 32A05, 11B83
Keywords: hypercomplex function theory, orthogonal polynomials, Appell polynomials, Vietoris' sequences.

Introduction

Hypercomplex function theory provides an alternative way to deal with arbitrary dimensions in the framework of Clifford algebras, generalizing the one-variable complex function theory. We begin by providing the essential notation and definitions (for details see [2]).

For $n \in \mathbb{N}$, the orthonormal basis $\{e_1, \ldots, e_n\}$ of the Euclidean vector space $\mathbb{R}^n$, subject to the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j},\ i, j = 1, \ldots, n$, generates the associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ whose elements are of the form $\sum_A x_A e_A,\ x_A \in \mathbb{R}$, where $e_A = e_{i_1} \cdots e_{i_k}$ for $A = \{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$, with $1 \leq i_1 < \cdots < i_k \leq n$ and if $A = \emptyset$, $e_\emptyset := 1$ is the identity. In particular, the element $x = x_0 + x_1e_1 + \cdots + x_ne_n := x_0 + \underline{x} \in \mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$ is called paravector and the vector space $\mathbb{R}^{n+1}$ is embedded in the Clifford algebra $\mathcal{C}\ell_{0,n}$ by the identification of the vector $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the paravector x. The conjugate and the norm of x are given by $\bar{x} = x_0 - \underline{x}$ and $|x| = (x\bar{x})^{\frac{1}{2}}$, respectively. The complex numbers constitute the only commutative Clifford algebra corresponding to $n = 1$.

In what follows we consider generalized holomorphic functions, i.e., real differentiable functions $f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$ that are in the kernel of the generalized Cauchy-Riemann operator $\bar{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$, and that are usually called *monogenic*. An important tool in hypercomplex function theory is the Cauchy-Kovalevskaya extension that generates a unique monogenic function from its restriction to a real codimension one subspace.

Theorem 1. *[2] If $\tilde{f}(x_1,\ldots,x_n)$ is a real analytic function defined in an open set $\tilde{\Omega}$ of $\mathbb{R}^n$, then there exists an open neighbourhood Ω of $\tilde{\Omega}$ in $\mathbb{R}^{n+1}$ and a unique monogenic function f defined in Ω such that its restriction to $\tilde{\Omega}$ is precisely $\tilde{f}$. If moreover $\tilde{\Omega}$ contains the origin, then in an open neighbourhood of the origin this extension f is given by*

$$CK[\tilde{f}(\underline{x})] := f(x_0,\underline{x}) = exp(-x_0\,\partial_{\underline{x}})\tilde{f}(\underline{x}).$$

Like in the complex case, the conjugate generalized Cauchy-Riemann operator, $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ plays the role of a hypercomplex derivative operator within the class of monogenic functions (cf. [9]). This permits to extend the concept of Appell sequences[1] to the hypercomplex case as $\mathcal{C}\ell_{0,n}$- valued polynomial sequences $(\mathcal{F}_k)_{k\geq 0}$ of exact degree (of homogeneity) k, for $k = 0, 1, \ldots$ and such that $\partial\mathcal{F}_k(x) = k\mathcal{F}_{k-1}(x),\ x \in \mathcal{A}_n,\ k = 1, 2, \ldots$.

A structural monogenic Appell sequence

In the complex case, the monomials are the simplest example of a holomorphic Appell sequence. The hypercomplex counterpart cannot be $\{x^k\}_{k\in\mathbb{N}_0}$, $x \in \mathcal{A}_n$, since these powers do not belong to the class of monogenic functions. For instance, $\bar{\partial}x = \frac{1}{2}(1-n)$ is zero only in the case $n = 1$ (the complex case). This fact causes problems and poses challenges on the construction of Appell sequences in $\mathbb{R}^{n+1}$ in the hypercomplex context, even for simple cases. The first example of such a sequence appeared for the first time in [7], where the authors constructed a special monogenic Appell sequence combining the variable x and its conjugate $\bar{x}$. For each $k \in \mathbb{N}_0$, among different representations of the constructed sequence, we consider here the Appell monogenic polynomials given by the binomial-type formula

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \quad x \in \mathcal{A}_n, \tag{1}$$

where

$$c_0(n) = 1 \ \text{ and } \ c_s(n) = \frac{\left(\frac{1}{2}\right)_{\left\lfloor \frac{s+1}{2} \right\rfloor}}{\left(\frac{n}{2}\right)_{\left\lfloor \frac{s+1}{2} \right\rfloor}}, \quad s > 0, \tag{2}$$

with $(a)_m$ the Pochhammer symbol. These monogenic polynomials generalize to higher dimensions the usual holomorphic powers z^k, $z \in \mathbb{C}$. In fact, as the complex case corresponds to consider $n = 1$, it is clear that $c_s(1) = 1$, $s = 0, \ldots, k$, and from (1), $\mathcal{P}_k^1(x) = (x_0 + x_1 e_1)^k \simeq z^k$. The real powers are included as well in the formula (1) by setting $\underline{x} \equiv 0$. The importance of the polynomials (1) was recognized during the last decade by several authors, both in theoretical and in practical applications (see for instance [1, 3]).

Recently, a detailed study on the coefficients (2) was carried out. The rational numbers (2) form a n-parameter generalization of the sequence

$$1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{5}{16},\ \tfrac{35}{128},\ \tfrac{35}{128},\ \tfrac{63}{256},\ \tfrac{63}{256},\ \tfrac{231}{1024},\ \tfrac{231}{1024},\ldots, \tag{3}$$

that appeared for the first time in [11] connected with the positivity of certain trigonometric sums. In fact the Vietoris numbers (3) correspond to consider $n = 2$ in (2). For more details, see [5].

In the next section we will see how the Appell sequence $\{\mathcal{P}_k^n\}_{k\in\mathbb{N}_0}$ is structural in the construction of homogeneous polynomials that constitute more general orthogonal monogenic polynomial sequences.

[1] In the classical sense, Appell sequences are sequences of polynomials p_k of exact degree k ($k \in \mathbb{N}_0$) and such that $\frac{d}{dz}p_k(z) = k\,p_{k-1}(z)$, $k = 1, 2, \ldots$

General orthogonal monogenic Appell sequences

More general monogenic Appell sequences in $\mathbb{R}^{n+1}$ can be obtained from normalized homogeneous polynomials in $\mathbb{R}^n$ by the Cauchy-Kovalevskaya extension. More precisely, for $k \in \mathbb{N}_0$ and $j = 0, \ldots, k$, the orthogonal[2] monogenic Appell sequence $\left\{\widetilde{X}^{(k)}_{n+1,j}\right\}$, defined in $\mathcal{A}_n$ and with values in $\mathcal{C}\ell_{0,n}$, is given by

$$\widetilde{X}^{(k)}_{n+1,j}(x) := X^{(k-j)}_{n+1,j}(x) P_j(\underline{x}) = CK[c_{k-j}(n) \binom{k}{j} \underline{x}^{k-j} \, P_j(\underline{x})],$$

where $c_{k-j}(n)$ are the real constants (2) and P_j are arbitrarily chosen monogenic polynomials in $\mathbb{R}^n$ (see [4],[8]). The factors $X^{(k-j)}_{n+1,j}$ $(j = 0, \ldots, k)$ are $\mathcal{A}_n$-valued homogeneous polynomials of degree $k-j$ constructed with the help of the Gegenbauer polynomials,

$$X^{(k-j)}_{n+1,j}(x) = F^{(k-j)}_{n+1,j}(x) + \tfrac{j+1}{n+2j} F^{(k-j-1)}_{n+1,j+1}(x)\, \underline{x}, \quad x \in \mathcal{A}_n, \tag{4}$$

where $F^{(k-j)}_{n+1,j}(x) = \frac{(j+1)_{k-j}}{(n+2j-1)_{k-j}} |x|^{k-j} \, C^{\frac{n-1}{2}+j}_{k-j}\left(\frac{x_0}{|x|}\right)$, with C^{λ}_{k-j} the Gegenbauer polynomial of degree $k-j$ and parameter $\lambda = \frac{n-1}{2} + j$.

Remark 1. *The polynomials (1) are included in the sequence* $\left\{\widetilde{X}^{(k)}_{n+1,j} : j = 0, \ldots, k\right\}_{k\in\mathbb{N}_0}$ *when* $j = 0$ *and* $P_0(\underline{x}) = 1$.

A remarkable feature of the polynomials (1) is their structural nature in the sense that they generate the (in general) non monogenic building blocks $X^{(k-j)}_{n+1,j}$ $(j = 0, \ldots, k)$ by a simple shift of their coefficients (2), from the parameter n to the parameter $n + 2j$.

Theorem 2. *For each* $n \in \mathbb{N}$, *each* $k \in \mathbb{N}_0$ *and* $j = 0, \ldots, k$, *let* $X^{(k-j)}_{n+1,j}$ *be defined by (4). Then it holds*

$$X^{(k-j)}_{n+1,j}(x) = \binom{k}{j} \mathcal{P}^{n+2j}_{k-j}(x), \quad x \in \mathcal{A}_n,$$

where $\mathcal{P}^{n+2j}_{k-j}(x)$ *are defined by (1)-(2).*

Proof. The proof is based on a representation of the polynomials (1) in terms of specific Gegenbauer polynomials and their derivatives (for details, see [4]). □

As a direct consequence of this result, the polynomials $X^{(k-j)}_{n+1,j}$ $(j = 0, \ldots, k)$ can be written in a easier way as the following binomial-type formula

$$X^{(k-j)}_{n+1,j}(x) = \binom{k}{j} \sum_{s=0}^{k-j} \binom{k-j}{s} c_s(n+2j)\, x_0^{k-j-s}\, \underline{x}^s, \; x \in \mathcal{A}_n.$$

Remark 2. *The homogeneous polynomials (4) appear as building blocks of an iterative process for the construction of an orthogonal basis for the space of homogeneous polynomials of degree k, that are null solutions of the Dirac operator* $\partial_{\underline{x}}$ *(see [10]). The reformulation of this building process for the case of the generalized Cauchy-Riemann operator leads to a basis formed by the monogenic polynomials of the form* $f_{k,\mu} = X^{(k-k_n)}_{n+1,k_n} X^{(k_n-k_{n-1})}_{n,k_{n-1}} \cdots X^{(k_3-k_2)}_{3,k_2} \zeta^{k_2}$, *where* $\{\zeta^{k_2}\}$ *is an orthogonal basis of the*

[2] With respect to the $\mathcal{C}\ell_{0,n}$-valued inner product $(f,g)_{\mathcal{C}\ell_{0,n}} = \int_{B^{n+1}} \bar{f}\, g \, d\lambda^{n+1}$, where λ^{n+1} is the Lebesgue measure in $\mathbb{R}^{n+1}$.

space of monogenic polynomials of degree k_2 in $\mathbb{R}^2 \cong \mathcal{A}_1 \cong \mathbb{C}$, μ is an arbitrary sequence of integers $(k_{n+1}, k_n, \ldots, k_3, k_2)$ such that $k = k_{n+1} \geq k_n \geq \ldots \geq k_3 \geq k_2 \geq 0$ and the building blocks $X_{l,j}^{(i-j)}$ $(l = 3, \ldots, n+1)$ are the polynomials (4).

It is well known that real and complex orthogonal polynomials satisfy a three-term recurrence formula and are solutions of a certain second order differential equation. Analogous properties have the orthogonal polynomials $\widetilde{X}_{n+1,j}^{(k)}$, $j = 0, \ldots, k$.

Theorem 3. *[4] For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k)}$ satisfy the three-term type recurrence*

$$(n+k+1+j)(k+2-j)\widetilde{X}_{n+1,j}^{(k+2)} - \left[(n+2k+2)x_0 + \underline{x}\right](k+2)\widetilde{X}_{n+1,j}^{(k+1)} + (k+2)(k+1)\,|x|^2\,\widetilde{X}_{n+1,j}^{(k)} = 0,$$

$$\widetilde{X}_{n+1,j}^{(j)} = P_j(\underline{x}), \qquad \widetilde{X}_{n+1,j}^{(j+1)} = (j+1)(x_0 + \frac{1}{n+2j}\underline{x})P_j(\underline{x}).$$

Theorem 4. *[4] For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k)}$ are solutions of the the second order differential equation*

$$|x|^2\partial^2 y(x) - ((n+2k-2)x_0 + \underline{x})\partial y(x) + (n+k+j-1)(k-j)y(x) = 0.$$

Ladder operators for the homogeneous monogenic sequence $\left\{\widetilde{X}_{n+1,j}^{(k)}\right\}_{k\in\mathbb{N}_0}$ are now easy to obtain. Its nature as an Appell sequence,

$$\partial\widetilde{X}_{n+1,j}^{(k)}(x) = k\,\widetilde{X}_{n+1,j}^{(k-1)}(x),\ x \in \mathcal{A}_n,\ k \in \mathbb{N}, \tag{5}$$

provides naturally the lower operator ∂. A raising operator can be obtained with the help of the Euler operator $\mathbb{E} := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ since any homogeneous function is an eigenfunction of this operator. Setting $M := \mathbb{E}\left[(2\mathbb{E}+n-2)x_0 + \underline{x} - |x|^2\partial\right]$, from Theorem 3 we get

$$M\widetilde{X}_{n+1,j}^{(k)} = (n+k+j)(k+1-j)\widetilde{X}_{n+1,j}^{(k+1)},\ n \in \mathbb{N},\ k \in \mathbb{N}_0 \text{ and } j = 0, \ldots, k. \tag{6}$$

Combining (5) and (6) the next result follows immediately.

Theorem 5. *[4] For all $k \in \mathbb{N}_0$ and each fixed j $(j = 0, \ldots, k)$, the monogenic polynomials $\widetilde{X}_{n+1,j}^{(k)}$ are eigenvectors of the operator $L := M\partial$ with corresponding eigenvalues $\alpha_{k,j}^n := k(n+k+j-1)(k-j)$.*

A matrix representation

Multivariate polynomials are frequently represented in matrix form because it provides an elegant and easier way for handling these polynomials. The approach proposed in [6] for the monogenic Appell sequence $\left\{\widetilde{X}_{n+1,j}^{(l+j)}\right\}_{l\in\mathbb{N}_0}$ (considering $l = k-j$ in the previous notation) extends previous results obtained in [1] and highlights the role of two matrices with a simple structure, namely the *creation matrix* H defined by

$$(H)_{rs} = \begin{cases} r, & r = s+1 \\ 0, & \text{otherwise} \end{cases}$$

and the so-called *shift matrix* J, defined by

$$(J)_{rs} = \begin{cases} 1, & r = s+1 \\ 0, & \text{otherwise}, \end{cases} \qquad r, s = 0, 1, \ldots, m.$$

For each fixed j, denoting by $\widetilde{\mathbf{X}}_j(x_0,\underline{x})$ the vector containing the $m+1$ first polynomials of the sequence, the Appell property of those polynomials implies the following differential relation in matrix form

$$\partial_0 \widetilde{\mathbf{X}}_j(x_0,\underline{x}) = (H + jJ)\,\widetilde{\mathbf{X}}_j(x_0,\underline{x}).$$

Consequently,

$$\begin{aligned}\widetilde{\mathbf{X}}_j(x_0,\underline{x}) &= e^{(H+jJ)x_0}\,\widetilde{\mathbf{X}}_j(0,\underline{x})\\ &= \mathrm{S}_j(x_0)\widetilde{\mathbf{X}}_j(0,\underline{x}),\end{aligned}$$

where $\mathrm{S}_j(x_0) = e^{(H+jJ)x_0}$ is the shifted generalized Pascal matrix

$$(\mathrm{S}_j(x_0))_{il} = \begin{cases} \binom{r+j}{s+j} {x_0}^{r-s}, & r \geq s \\ 0, & \text{otherwise}, \quad r,s = 0,1,\ldots,m. \end{cases}$$

Conclusion

The paper gives an overview on different representations and properties of general orthogonal monogenic Appell sequences and stresses the key role of a special monogenic Appell sequence generalizing the holomorphic powers to $\mathbb{R}^{n+1}$ in the hypercomplex context.

Acknowledgements

The author was supported by Fundação para a Ciência e a Tecnologia (FCT), within project UID/MAT/04106/2019 (CIDMA).

References

[1] L. Aceto, H.R. Malonek, and G.Tomaz, Matrix approach to hypercomplex Appell polynomials, Appl. Numer. Math. 116 (2017), 2–9.

[2] F. Brackx, R. Delanghe, and F. Sommen, Clifford Analysis, Boston-London-Melbourn: Pitman; 1982.

[3] S. Bock, On Monogenic Series Expansions with Applications to Linear Elasticity. Adv. Appl. Clifford Algebras, 24 no. 4 (2014), 931–943.

[4] I. Cação, M. I. Falcão, and H. R. Malonek, Three-Term Recurrence Relations for Systems of Clifford Algebra-Valued Orthogonal Polynomials, Adv. Appl. Clifford Algebras, 27 no. 1 (2017), 71–85.

[5] I. Cação, M. I. Falcão, and H. R. Malonek, On generalized Vietoris' number sequences, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.10.017.

[6] I. Cação, H. R. Malonek, and G. Tomaz, Shifted Generalized Pascal Matrices in the Context of Clifford Algebra-Valued Polynomial Sequences, In B. Murgante et al. (eds.) Lecture Notes in Computer Science, Vol. 10405, Springer-Verlag Berlin Heidelberg, (2017), 409–421.

[7] M. I. Falcão, and H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions, In Theodore E. Simos, George Psihoyios, and Ch. Tsitouras (eds.) AIP Conference Proceedings, vol. 936, pp. 738741 (2007)

[8] Delanghe, R., and Sommen, F., Soucek, V., Clifford Algebra and Spinor-Valued Functions. Kluwer Academic Publishers, Dordrecht (1992)

[9] K. Gürlebeck, and H.R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{m+1}$ and its applications, Complex Variables 39 (1999), 199–228.

[10] R. Lávička, Complete orthogonal Appell systems for spherical monogenics, Complex Anal. Oper. Theory 6 (2012), 477480

[11] Vietoris, L, Über das Vorzeichen gewisser trigonometrischer Summen. Sitzungsber. sterr. Akad. Wiss 167, 125135 (1958)

[1] CIDMA - Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail : isabel.cacao@ua.pt

Combinatorial Numbers Arising From Two Parametric Apostol-Type Polynomials

Yilmaz Simsek[1]

Abstract

In "Some classes of generating functions for generalized Hermite- and Chebyshev-type polynomials: Analysis of Euler's formula, arXiv: 1907.03640v1 (2019)", Kilar and Simsek introduced and investigated some families of generating functions for Hermite-based r-parametric Milne-Thomson-type polynomials which unify two parametric Apostol-type polynomials. In this study, by using functional equations of some generating functions with those of two parametric Apostol-type polynomials, we study and investigate some properties of these polynomials. Moreover, relations between two parametric Apostol-type polynomials and combinatorial numbers are given.

2010 Mathematics Subject Classifications : 05A15, 11B73, 26C05.
Keywords: Parametric kinds of Apostol-type polynomials, Generating function, Functional equation, Combinatorial numbers.

Introduction

Let $i^2 = -1$. The elegant Euler's formula, which is given by

$$e^{iz} = \cos(z) + i\sin(z),$$

has had a wide variety of applications to date not only in mathematics especially in complex analysis, but also in physics and engineerings. By using this formula, recently some interesting generating functions for new families of polynomials have been constructed (*cf.* [1]-[10]). In this presentation, by using these functions, we give some interesting formulas and relations including λ-Stirling numbers, the numbers $y_1(n,k;\lambda)$ and two parametric Apostol-type polynomials.

Throughout this paper, we consider the following notations and definitions:

Let $\mathbb{N} = \{1,2,3,...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. Additionally, let

$$(\alpha)^{\underline{n}} = \alpha(\alpha-1)...(\alpha-n+1) \quad (n \in \mathbb{N}),$$

(*cf.* [1]-[10]; see also the references cited therein).

The generating function for the λ-Stirling numbers is given by

$$F_{S2}(t,k;\lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}, \tag{1}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [4], [6], [7]).

The generating function for the numbers $y_1(n,k;\lambda)$ is given by

$$F_{y_1}(t,k;\lambda) = \frac{(\lambda e^t + 1)^k}{k!} = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!} \tag{2}$$

(*cf.* [7]).

The generating functions for the polynomials $C_n(x,y)$ and the polynomials $S_n(x,y)$ are, respectively, given by

$$F_C(t,x,y) = e^{xt}\cos(yt) = \sum_{n=0}^{\infty} C_n(x,y)\frac{t^n}{n!} \tag{3}$$

and

$$F_S(t,x,y) = e^{xt}\sin(yt) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} \tag{4}$$

(*cf.* [3], [5], [10]).

Two parametric kinds of Apostol-Bernoulli polynomials of order k, two parametric kinds of Apostol-Euler polynomials of order k and two parametric kinds of Apostol-Genocchi polynomials of order k are defined by Srivastava and Kizilates [9], respectively, as follows:

$$F_{BC}(t,x,y;\lambda,k) = \left(\frac{t}{\lambda e^t - 1}\right)^k e^{xt}\cos(yt) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(C,k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{5}$$

$$F_{BS}(t,x,y;\lambda,k) = \left(\frac{t}{\lambda e^t - 1}\right)^k e^{xt}\sin(yt) = \sum_{n=0}^{\infty} \mathcal{B}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{6}$$

$$F_{EC}(t,x,y;\lambda,k) = \left(\frac{2}{\lambda e^t + 1}\right)^k e^{xt}\cos(yt) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(C,k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{7}$$

$$F_{ES}(t,x,y;\lambda,k) = \left(\frac{2}{\lambda e^t + 1}\right)^k e^{xt}\sin(yt) = \sum_{n=0}^{\infty} \mathcal{E}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!} \tag{8}$$

and

$$F_{GC}(t,x,y;\lambda,k) = \left(\frac{2t}{\lambda e^t + 1}\right)^k e^{xt}\cos(yt) = \sum_{n=0}^{\infty} \mathcal{G}_n^{(C,k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{9}$$

$$F_{GS}(t,x,y;\lambda,k) = \left(\frac{2t}{\lambda e^t + 1}\right)^k e^{xt}\sin(yt) = \sum_{n=0}^{\infty} \mathcal{G}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!} \tag{10}$$

(see, for details, [9]).

Identities and Relations

In this section, by using the methods of generating functions and their functional equations, some identities and formulas including two parametric Apostol-type polynomials of order k, the λ-Stirling numbers and the numbers $y_1(n,k;\lambda)$ are given.

Theorem 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_n(x,y) = \frac{k!}{2^k}\sum_{j=0}^{n}\binom{n}{j} y_1(j,k;\lambda)\,\mathcal{E}_{n-j}^{(S,k)}(x,y;\lambda). \tag{11}$$

Proof. By the help of the equations (2), (4) and (8), we get the following functional equation:

$$2^k F_S(t,x,y) = k! F_{y_1}(t,k;\lambda)\, F_{ES}(t,x,y;\lambda,k)$$

which yields

$$\sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} = \frac{k!}{2^k}\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{E}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

By the Cauchy product rule in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} = \frac{k!}{2^k}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} y_1(j,k;\lambda)\,\mathcal{E}_{n-j}^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_{n-k}(x,y) = \frac{k!}{2^k (n)^{\underline{k}}}\sum_{j=0}^{n}\binom{n}{j} y_1(j,k;\lambda)\,\mathcal{G}_{n-j}^{(S,k)}(x,y;\lambda). \tag{12}$$

Proof. By the help of the equations (2), (4) and (10), we get the following functional equation:

$$(2t)^k F_S(t,x,y) = k! F_{y_1}(t,k;\lambda)\, F_{GS}(t,x,y;\lambda,k)$$

which yields

$$t^k\sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} = \frac{k!}{2^k}\sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

By the Cauchy product rule in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty}(n)^{\underline{k}} S_{n-k}(x,y)\frac{t^n}{n!} = \frac{k!}{2^k}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} y_1(j,k;\lambda)\,\mathcal{G}_{n-j}^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 3. *Let $n \in \mathbb{N}_0$. Then we have*

$$S_{n-k}(x,y) = \frac{k!}{(n)^{\underline{k}}}\sum_{j=0}^{n}\binom{n}{j} S_2(j,k;\lambda)\,\mathcal{B}_{n-j}^{(S,k)}(x,y;\lambda). \tag{13}$$

Proof. By the help of the equations (1), (4) and (6) we get the following functional equation:

$$t^k F_S(t,x,y) = k! F_{S2}(t,k;\lambda)\, F_{BS}(t,x,y;\lambda,k).$$

which yields

$$t^k\sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!} = k!\sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{B}_n^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

By the Cauchy product rule in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty}(n)^{\underline{k}} S_{n-k}(x,y)\frac{t^n}{n!} = k!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} S_2(j,k;\lambda)\,\mathcal{B}_{n-j}^{(S,k)}(x,y;\lambda)\frac{t^n}{n!}.$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Acknowledgements

The paper was supported by Scientific Research Project Administration of Akdeniz University with Project Number: FKA-2019-4385.

References

[1] N. Kilar and Y. Simsek, Relations on Bernoulli and Euler polynomials related to trigonometric functions, Adv. Stud. Contemp. Math. 29(2) (2019), 191–198.

[2] N. Kilar and Y. Simsek, Some classes of generating functions for generalized Hermite- and Chebyshev-type polynomials: Analysis of Euler's formula, arXiv:1907.03640v1 (2019).

[3] T. Kim and C. S Ryoo, Some identities for Euler and Bernoulli polynomials and their zeros, Axioms 7(3), 56 (2018), 1–19.

[4] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput. 217 (2011), 5702–5728.

[5] M. Masjed-Jamei and W. Koepf, Symbolic computation of some power-trigonometric series, J. Symb. Comput. 80(2) (2017), 273–284.

[6] Y. Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications, Fixed Point Theory Appl. 2013(87) (2013), 1–28.

[7] Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, Appl. Anal. Discrete Math. 12 (2018), 1–35.

[8] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inform. Sci. 5 (2011), 390–444.

[9] H. M. Srivastava and C. Kızılateş, A parametric kind of the Fubini-type polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. RACSAM (2019), https://doi.org/10.1007/s13398-019-00687-4.

[10] H. M. Srivastava, M. Masjed-Jamei and M.R. Beyki, A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, Appl. Math. Inf. Sci. 12(5) (2018), 907-916.

[1]Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya, Turkey

E-mail : ysimsek@akdeniz.edu.tr

Some Relations Between Partitions and Fibonacci Numbers

Busra Al[1], *Mustafa Alkan*[2]

Abstract

In this note, we define sets by using both commutative and noncommutative partitions of an integer and obtain algebraic and combinatorial equations on these sets. By doing these calculations, we relate the partition to Fibonacci numbers.

2010 MATHEMATICS SUBJECT CLASSIFICATIONS : $03E02, 05A17, 05A18, 11P81, 11B39$.
KEYWORDS: Fibonacci Numbers, Number of Partition, Partition Theory.

Introduction

For a positive integer n, the partition function to be studied is the number of ways n can be written as a sum of positive integers n. The summands are called parts.

Partitions are divided into two as commutative and noncommutative. The displacement of summands in commutative partitions are not important.

Many reasearcher (cf, [3], [6], [8], [9], [13], [15]) study the number of the commutative partition to find more useful formula for $p(n)$.

Example 1. *The set of commutative partition of* 3 *is*

$$\{3, (1+2), (1+1+1)\}$$

and 3 *has* 3 *partitions.*

The set of commutative partition of 4 *is*

$$\{4, \ (1+3), (1+1+2), (2+2), (1+1+1+1)\}$$

and 4 *has* 5 *partitions.*

$p(1) = 1, \ p(3) = 3, \ p(4) = 5, \ p(5) = 7, \ p(10) = 4, \ p(100) = 190569292, \ p(200) = 3972999029388, ...$

Euler investigated the generating function of the number of commutative partitions of an integer n, as follows

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)} = \sum_{n=0}^{\infty} p(n)x^n$$

where $0 < x < 1$ and $p(0) = 1$ [15, p.306, Theorem(14.2)].

Theorem 2. *[15, p.315, Theorem 14.4] Let $p(0) = 1$ and define $p(n)$ to be 0 if $n < 0$. Then for $n \geq 1$,*

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - ... = 0$$

or, what amounts to the same thing,

$$p(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \{p(n - w(k)) + p(n - w(-k))\}$$

where $w(k) = \frac{k(3k-1)}{2}$.

The displacement of summands in noncommutative partitions are important but unlike the number of the commutative partition of an integer n, the number of non commutative partition of an integer n is 2^{n-1} (cf. [1]).

Example 3. *The set of non commutative partition of* 3 *is*

$$\{3, (1+2), (2+1), (1+1+1)\}$$

and 3 *has* $2^{3-1} = 4$ *partitions.*

The set of of 4 *is*

$$\{4, (1+3), (3+1), (1+1+2), (1+2+1), (2+1+1), (2+2), (1+1+1+1)\}$$

and 4 *has* $2^{4-1} = 8$ *partitions.*

Let us recall basic properties of the Fibonacci numbers. The Fibonacci numbers, commonly denoted F_n form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is,

$$F_0 = 0, \qquad F_1 = 1$$

and

$$F_n = F_{n-1} + F_{n-2}, \qquad \text{for} \qquad n > 1.$$

Thus the sequance

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

From [14], generating function of Fibonacci Number, as follows

$$f(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F(n)x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + ...$$

From [14, p.72], the finite sum of even Fibonacci numbers is well know:

Theorem 4. *Let* n *be a positive integer. We have*

$$f_{2(n+1)} = 1 + \sum_{i=0}^{n} (f_{2i}) + f_{2n}. \tag{1}$$

Main Results

In this notes, we focus on the part of non commutative partitions and study the relations between Fibonacci numbers and non commutative partitions. The set of non commutative partition of 3 is $\{3, 1+1+1, 2+1, 2+1\}$ and 3 has $2^{3-1} = 4$ partitions. Then the summand of the partiton $2+1$ are 1, 2 and their product is 2.

By using the non commutative partition of an integer n, we define the non commutative partition set;

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

In fact, the element of P_n is a partition of integer n and so P_n is the set of a non commutative partition of an integer n.

By using the non commutative partition set of an integer n, we define a sequance on integers

$$T_n = T(P_n) = \sum_{(a_1,a_2,\ldots,a_t)\in P_n} a_1.a_2.a_3...a_t$$

and we may assume that $T_0 = 1$.

Now we want to construct P_{n+1} by using P_n. Let us define two operations with the element $a \in P_n$.

Let $a = (a_1, a_2, ..., a_t) \in P_n$ and define

$$(1 \odot a) = (1, a_1, a_2, ..., a_t),$$

$$(1 \oplus a) = (a_1 + 1, a_2, ..., a_t).$$

Then $1 \oplus a, 1 \odot a \in P_{n+1}$ and now define sets $1 \oplus P_n = \{1 \oplus a : a \in P_n\}$, $1 \odot P_n = \{1 \odot a : a \in P_n\}$.

Theorem 5. *For a positive integer n, we have*

$$P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n).$$

Proof. If $a \in P_n$ then $b = 1 \oplus a$, $b = 1 \odot a \in P_{n+1}$ and so $(1 \oplus P_n) \cup (1 \odot P_n) \subseteq P_{n+1}$.

Let $b = (b_1, b_2, ..., b_k) \in P_{n+1}$ and so $b_1 + b_2 + ... + b_k = n + 1$.

If $b_1 \neq 1$ then $c_1 = b_1 - 1$, $c_i = b_i$, where $i \in \{2, ..., k\}$ and so $c_1 + c_2 + ... + c_k = n$. It follows that $c = (c_1, c_2, ..., c_k) \in P_n$ and $b = 1 \oplus c \in (1 \oplus P_n)$.

If $b_1 = 1$ then $c_{i-1} = b_i$, where $i \in \{2, ..., k\}$ and so $c_1 + c_2 + ... + c_{k-1} = n$. It follows that $c = (c_1, c_2, ..., c_{k-1}) \in P_n$ and $b = 1 \odot c \in (1 \odot P_n)$. Theorefore, we have that $P_{n+1} = (1 \oplus P_n) \cup (1 \odot P_n)$ and $(1 \oplus P_n) \cap (1 \odot P_n) = \varnothing$. □

Example 6. $P_1 = \{(1)\}$ *and* $T_1 = 1$.

$P_2 = \{(2), (1, 1)\}$ *and* $T_2 = 3$.

$P_3 = \{(3), (1, 1, 1), (1, 2), (2, 1)\}$ *and* $T_3 = 8$.

$$1 \odot P_3 = \{(1, 3), (1, 1, 1, 1), (1, 1, 2), (1, 2, 1)\}$$

$$1 \oplus P_3 = \{(4), (2, 1, 1), (2, 2), (3, 1)\}$$

and $P_4 = (1 \odot P_3) \cup (1 \oplus P_3)$

$T(1 \odot P_3) = 3 + 1 + 2 + 2 = 8 = T_3$

$T(1 \oplus P_3) = 4 + 2 + 4 + 3 = 13$. *Hence* $T_4 = 8 + 13 = 21$.

Conclusion

By using Equation 1 and definition of T_n, we get the following results;

Theorem 7. *Let n be a positive integer and*

$$P_n = \left\{(a_1, a_2, ..., a_t) : \ a_1 + a_2 + ... + a_t = n, \quad a_i, t \in \mathbb{Z}^+\right\}.$$

Then we have

$$T_n = \sum_{(a_1,a_2,\ldots,a_t)\in P_n} (a_1.a_2.....a_t) = f_{2n}. \qquad (2)$$

References

[1] A. V. Sılls, Compositions, partitions, and Fibonacci numbers, Fibonacci Quarterly, 49, 4, (2011), 348-354.

[2] B. Al, and M. Alkan, On the relations for the number of partitions, Filomat; 2019(Accepted).

[3] B. C. Bernedt, Lecture Notes on the Theory of Partitions.

[4] G.H. Hardy and E.M. Wright, An Introduction to the Theory of Numbers, 4th ed., *Clarendon Press,* Oxford, 1960.

[5] G.N. Watson, Two tables of partitions, Proc. London Math. Soc. 42,(1937), 550-556.

[6] J. A. Ewell, Recurrences for the partition function and its relatives, Rocky Mountaın Journal Of Mathematıcs, Springer, 2, (2004), 34.

[7] L. Euler, Introduction to analysis of the infinite, vol. 1, Springer-Verlag, 1988 (translation by J.D. Blanton).

[8] M. Merca, Fast computation of the partition function, Journal of Number Theory 164, (2016), 405-416.

[9] M. Merca, New relations for the number of partitions with distinct even parts, Journal of Number Theory 176 (2017), 1-12.

[10] M. Merca, On the number of partitions into parts of k different, Discrete Mathematics 340, (2017), 644-648.

[11] P. A. MacMahon, Combinatory Analysis, vol. 1, Cambridge: at the University Press, 1915.

[12] P.A. MacMahon, Note on the parity of the number which enumerates the partitions of a number, Proc. Cambridge Philos. Soc. 20, (1921), 281-283.

[13] S. C. Chen, On the number of partitions with distinct even parts. Discrete math. 12, 311 (2011), 940-943.

[14] T. Koshy, Fibonacci and Lucas Numbers with Applications, Canada: Wiley-Interscience Publication; (2001), 6-38.

[15] T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag; 1976.

[1,2]Department of Mathematics, Faculty of Science University of Akdeniz, Antalya TR-07058, Turkey.

E-mail : busraal0707@gmail.com, alkan@akdeniz.edu.tr

Relations Between Apostol-Type Polynomials and Combinatorial Numbers

Neslihan Kilar [1] , *Yilmaz Simsek*[2]

Abstract

The aim of this presentation is to give not only recent development on some well-known special numbers and polynomials including two parametric kinds of Apostol-type polynomials, the λ-Stirling numbers of the second kind and the combinatorial numbers, but also identities and relations associated with these polynomials. Moreover, we give some remarks and observations on these special polynomials and their generating functions. At the end of this presentation, current comments on these functions, special numbers and polynomials are also given.

2010 Mathematics Subject Classifications : 05A15, 11B73, 26C05.
Keywords: Parametric kinds of Apostol-type polynomials, Generating function, Functional equation, Combinatorial numbers.

Introduction

Throughout this paper the following notations and definitions are used.

$\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. For $n \in \mathbb{N}$, $(\alpha)^{\underline{n}} = \alpha(\alpha - 1) ... (\alpha - n + 1)$.

The λ-Stirling numbers of the second kind are defined by the following generating function:

$$F_{S2}(t, k; \lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!}, \tag{1}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [4], [6], [7], [8]).

The numbers $y_1(n, k; \lambda)$ are defined by means of the following generating function:

$$F_{y_1}(t, k; \lambda) = \frac{(\lambda e^t + 1)^k}{k!} = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!} \tag{2}$$

(*cf.* [7]).

The polynomials $C_n(x, y)$ and $S_n(x, y)$ are defined by means of the following generating functions, respectively:

$$F_C(t, x, y) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!}, \tag{3}$$

$$F_S(t, x, y) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!} \tag{4}$$

(*cf.* [3], [5], [10]).

In [9], Srivastava and Kizilates defined the following two parametric kinds of Apostol-Bernoulli polynomials of order k, two parametric kinds of Apostol-Euler polynomials of order k, two parametric kinds of Apostol-Genocchi polynomials of order k, respectively:

$$F_{BC}\left(t,x,y;\lambda,k\right)=\left(\frac{t}{\lambda e^{t}-1}\right)^{k}e^{xt}\cos\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{B}_{n}^{(C,k)}(x,y;\lambda)\frac{t^{n}}{n!}, \tag{5}$$

$$F_{BS}\left(t,x,y;\lambda,k\right)=\left(\frac{t}{\lambda e^{t}-1}\right)^{k}e^{xt}\sin\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{B}_{n}^{(S,k)}(x,y;\lambda)\frac{t^{n}}{n!}, \tag{6}$$

$$F_{EC}\left(t,x,y;\lambda,k\right)=\left(\frac{2}{\lambda e^{t}+1}\right)^{k}e^{xt}\cos\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(C,k)}(x,y;\lambda)\frac{t^{n}}{n!}, \tag{7}$$

$$F_{ES}\left(t,x,y;\lambda,k\right)=\left(\frac{2}{\lambda e^{t}+1}\right)^{k}e^{xt}\sin\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(S,k)}(x,y;\lambda)\frac{t^{n}}{n!} \tag{8}$$

and

$$F_{GC}\left(t,x,y;\lambda,k\right)=\left(\frac{2t}{\lambda e^{t}+1}\right)^{k}e^{xt}\cos\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(C,k)}(x,y;\lambda)\frac{t^{n}}{n!}, \tag{9}$$

$$F_{GS}\left(t,x,y;\lambda,k\right)=\left(\frac{2t}{\lambda e^{t}+1}\right)^{k}e^{xt}\sin\left(yt\right)=\sum_{n=0}^{\infty}\mathcal{G}_{n}^{(S,k)}(x,y;\lambda)\frac{t^{n}}{n!} \tag{10}$$

(*cf.* [9]).

Identities and Relations

In this section, we give some relations and identities which are related to two parametric Apostol-type polynomials of order k, the λ-Stirling numbers of the second kind and the numbers $y_1\left(n,k;\lambda\right)$.

Theorem 1. *Let $n\in\mathbb{N}_0$. Then we have*

$$2^{k}C_{n}\left(x,y\right)=k!\sum_{j=0}^{n}\binom{n}{j}y_{1}\left(j,k;\lambda\right)\mathcal{E}_{n-j}^{(C,k)}(x,y;\lambda). \tag{11}$$

Proof. By using (2), (3) and (7), we derive the following functional equation:

$$2^{k}F_{C}\left(t,x,y\right)=k!F_{y_{1}}\left(t,k;\lambda\right)F_{EC}\left(t,x,y;\lambda,k\right).$$

From the above equation, we have

$$2^{k}\sum_{n=0}^{\infty}C_{n}\left(x,y\right)\frac{t^{n}}{n!}=k!\sum_{n=0}^{\infty}y_{1}\left(n,k;\lambda\right)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}\mathcal{E}_{n}^{(C,k)}(x,y;\lambda)\frac{t^{n}}{n!}.$$

Therefore

$$2^{k}\sum_{n=0}^{\infty}C_{n}\left(x,y\right)\frac{t^{n}}{n!}=k!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}y_{1}\left(j,k;\lambda\right)\mathcal{E}_{n-j}^{(C,k)}(x,y;\lambda)\frac{t^{n}}{n!}.$$

Comparing the coefficients of $\frac{t^{n}}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$2^k (n)^{\underline{k}} C_{n-k}(x,y) = k! \sum_{j=0}^{n} \binom{n}{j} y_1(j,k;\lambda)\, \mathcal{G}_{n-j}^{(C,k)}(x,y;\lambda). \tag{12}$$

Proof. By using (2), (3) and (9), we derive the following functional equation:

$$(2t)^k F_C(t,x,y) = k! F_{y_1}(t,k;\lambda)\, F_{GC}(t,x,y;\lambda,k).$$

From the above equation, we get

$$(2t)^k \sum_{n=0}^{\infty} C_n(x,y) \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} y_1(n,k;\lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{G}_n^{(C,k)}(x,y;\lambda) \frac{t^n}{n!}.$$

Therefore

$$2^k \sum_{n=0}^{\infty} (n)^{\underline{k}} C_{n-k}(x,y) \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} y_1(j,k;\lambda)\, \mathcal{G}_{n-j}^{(C,k)}(x,y;\lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 3. *Let $n \in \mathbb{N}_0$. Then we have*

$$(n)^{\underline{k}} C_{n-k}(x,y) = k! \sum_{j=0}^{n} \binom{n}{j} S_2(j,k;\lambda)\, \mathcal{B}_{n-j}^{(C,k)}(x,y;\lambda). \tag{13}$$

Proof. By using (1), (3) and (5), we derive the following functional equation:

$$t^k F_C(t,x,y) = k! F_{S2}(t,k;\lambda)\, F_{BC}(t,x,y;\lambda,k).$$

From the above equation, we obtain

$$t^k \sum_{n=0}^{\infty} C_n(x,y) \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} S_2(n,k;\lambda) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{B}_n^{(C,k)}(x,y;\lambda) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} (n)^{\underline{k}} C_{n-k}(x,y) \frac{t^n}{n!} = k! \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S_2(j,k;\lambda)\, \mathcal{B}_{n-j}^{(C,k)}(x,y;\lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Conclusion

It is well-known that special numbers and polynomials and their generating functions have very important role in mathematics, mathematical physics and other science. For instance, applications of these polynomials and their generating functions are widely used for solve different real world problems and complicated problems in the sciences. Also, recently, many researchers have been written widespread works of parametric kinds of special polynomials (see for detail [1], [2], [3], [9], [10]). For this reason, by using these polynomials with the help of generating functions and their functional equations, various identities and formulas including, two parametric kinds of Apostol-type polynomials of order k, the λ-Stirling numbers of the second kind and the numbers $y_1(n, k; \lambda)$, which are related to the combinatorial numbers, are obtained.

In the future research, we would like to continue to study on parametric special polynomials related to the Euler's formula and trigonometric functions, and to investigate their applications in both analytic number theory, mathematical physics, engineering and other related sciences.

Acknowledgements

The paper was supported by Scientific Research Project Administration of Akdeniz University with Project Number: FKA-2019-4385.

References

[1] N. Kilar and Y. Simsek, Relations on Bernoulli and Euler polynomials related to trigonometric functions, Adv. Stud. Contemp. Math. 29(2) (2019), 191–198.

[2] N. Kilar and Y. Simsek, Some classes of generating functions for generalized Hermite- and Chebyshev-type polynomials: Analysis of Eulers formula, https://arxiv.org/abs/1907.03640, (2019).

[3] T. Kim and C. S Ryoo, Some identities for Euler and Bernoulli polynomials and their zeros, Axioms 7(3), 56 (2018), 1–19.

[4] Q.-M. Luo and H. M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput. 217 (2011), 5702–5728.

[5] M. Masjed-Jamei and W. Koepf, Symbolic computation of some power-trigonometric series, J. Symb. Comput. 80(2) (2017), 273–284.

[6] Y. Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications, Fixed Point Theory Appl. 2013(87) (2013), 1–28.

[7] Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, Appl. Anal. Discrete Math. 12 (2018), 1–35.

[8] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inform. Sci. 5 (2011), 390–444.

[9] H. M. Srivastava and C. Kızılateş, A parametric kind of the Fubini-type polynomials, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM, https://doi.org/10.1007/s13398-019-00687-4.

[10] H. M. Srivastava, M. Masjed-Jamei and M.R. Beyki, A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, Appl. Math. Inf. Sci. 12(5) (2018), 907-916.

[1,2]Department of Mathematics, Faculty of Science University of Akdeniz TR-07058 Antalya, Turkey

E-mail : neslihankilar@gmail.com, ysimsek@akdeniz.edu.tr

Investigation of Air Quality Index by Spatial Data Analysis: a Case Study on Turkey

Fusun Yalcin[1] , *Ahmet Tepe*[2] , *Güray Doğan*[2] , *Nurfer Cizmeci*[1]

Abstract

Natural and anthropogenic sources have adverse effects on air quality. If industrialization, which has an important place among anthropogenic sources, is maintained in adverse conditions, air pollution and waste problem occurs. In this study, air quality index in overall Turkey and the factors affecting this index has been determined. In this context, data suitable for the analyzes were compiled and prepared for analysis. ArcGIS and GeoDa were used to create maps from these data, MATLAB and SPSS were used for the calculations. Spatial interactions were determined with these maps. In addition, a mathematical modeling is proposed for the spatial analysis.

2010 Mathematics Subject Classifications : 62H11, 91D25, 91B72, 62P12
Keywords: Statistical Data Analysis, Exploratory Spatial Data Analysis (ESDA), Mathematical Modelling, Air Quality Index(AQI)

Introduction

Increasing environmental pollution due to the rapid increase in population and industrialization causes the deterioration of ecosystem and fundamental sources and this situation is becoming a serious problem. The determination of anthropogenic and natural sources that cause the deterioration of the natural balance and taking measures for these sources is one of the most important fields of study for humanity. The Air Quality Index (AQI) is used to get information pollution level in the ambient atmosphere. In the study, conducted by Tepe and Doğan [9] focused on the determination of the locations of long-distance domestic and foreign sources that affect the sulfur concentration in the air especially in the summer months of Antalya. Exploratory spatial data analysis is one of the multivariate statistical methods used in the literature to examine many subjects in many fields. The research entitled as "The spatial analysis of factors affecting the room price proposed by hotels in Antalya city" was also a reference for the beginning of this study (*cf.* [3], [4]). There are studies in the literature to determine the sources of various pollutants in different cities of Turkey, but none of these studies have been investigated the sources with the same variables and the same sampling period. The aim of this study is to determine the variables that cause the decrease of air quality in Turkey. Spatial maps of air pollution and spatial model with descriptive spatial data analysis methods, which is one of the multivariate statistical methods, were formed by editing the data obtained from the study.

Methodology And Data

In this study, the existence of spatial interaction with exploratory spatial data analysis method was investigated and a mathematical model explaining AQI was proposed. For this method, which is an exploratory statistical method, firstly the variables of air pollution, which is the subject of the research, have been determined. In this study, three different datasets from 2014 were used. The data was transformed into spatial data by adding the coordinates of the cities to which they belong. After that, spatial distribution maps were created. In addition to spatial distribution maps, Moran I statistics which measures of spatial dependence were used. Spatial weight matrices were formed according to the state of spatial dependence to be determined. These determined weights measures spread or interaction within parameters. The appropriate spatial model was determined with these weight matrices. Spatial dependence generally means non-zero covariance between random variables for neighborhoods.

$$cov(y_i, y_j) = E(y_i, y_j) - E(y_i).E(y_j) \neq 0 \forall i \neq j$$

where, i and j are the spatial locations (*cf.* [8]). Spatial weights matrix (i.e., $W_i j$) generated to determine the spatial effect is simply defined as Wij = 1 if neighborhood and $W_{ij} = 0$ if not (*cf.* [1], [7]). This value ranges from -1 to 1. The value 1 means perfect positive spatial dependence, while -1 stand for perfect negative spatial dependence. Global Morans I index can be expressed as:

$$I = \frac{N}{S_0} \frac{\Sigma_{i=1}^n \Sigma_{j=1}^n w_{ij} Z_i Z_j}{\Sigma_{i=1}^n Z_i^2}, \ S_0 = \Sigma_{i=1}^n \Sigma_{j=1}^n w_{ij}$$

Where, W_{ij} is a spatial weights matrix, Z_i denotes the deviation from the average of the samples and N is the number of observations. Moran I values of AQI and other variables, were calculated, and global spatial relationship maps were created. The most general linear model that does not include spatial relations defined as follows: $Y = \alpha I_n + X\beta + \varepsilon$ Elhorst, expressed the most general spatial regression model as: $Y = \rho WY + \alpha I_n + X\beta + WX\theta + u, \quad u = \lambda W_u + \varepsilon$ (*cf.* [6]). Where; WY; The endogenous interaction effects between dependent variables pWY; The endogenous effects between independent variables WX; The exogenous interaction effects between dependent variables Wu; shows the spatially dependence disturbance (*cf.* [5], [6]).

Result

The first component of Exploratory Spatial Data Analysis (ESDA) is to examine whether there is a neighborhood relationship in the percentage maps of the variables. Therefore, the first study is the creation of percentage maps. The second component of the ESDA analysis is the calculation of the global Moran I index. Moran I values of AQI and other variables, were calculated. The Arc-distance neighborhood matrix was chosen as the neighboring matrix. An explanatory map of this relationship is shown in Figure1-(a)For the dependent variable AQI, independent variables such as the number of cars per person and the average annual humidity have the highest Moran I values as 0.239, 0.781 and 0.584, respectively. Figure 1-(b) shows that the AQI is heavily influenced by neighboring provinces. The number of cars per person is an independent variable with the highest Moran values. Since cars emits lots of SO_2 and PM, such relation seems meaningful. Figure 1-(c) Annual average humidity, which is an independent variable, has the second highest Moran I value. Figure 1-(d) shows a parallel neighborhood relationship.

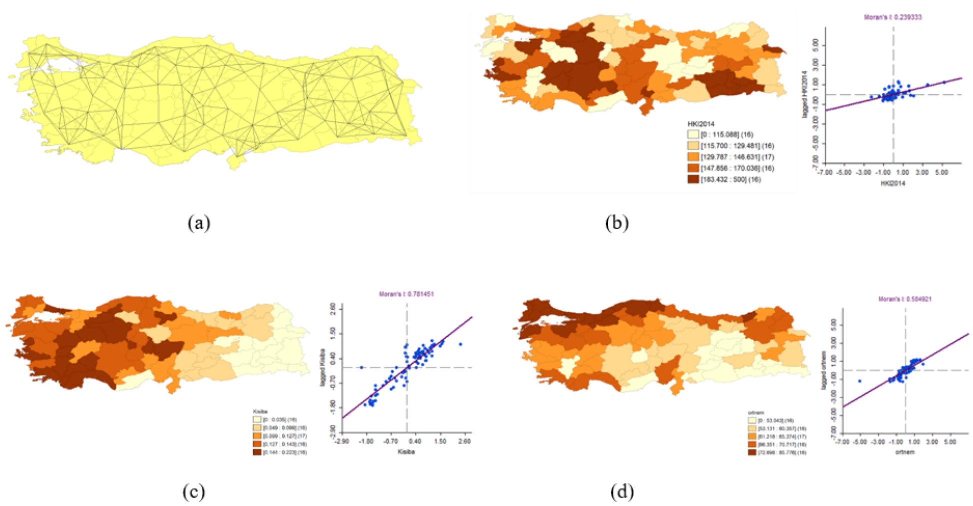


Figure 1: (a)Linkage map indicating the neighborhood relationship, (b)The global moran i graph and the cluster map for AQI in 2014, (c)The global moran i graph and the cluster map for the number of cars per person in 2014, (d)The global moran i graph and the cluster map for the annual average humidity

There are studies that take the AQI as a dependent variable. The primary aim of this study is to determine the independent variables that can explain the AQI. For this reason, it was aimed to propose a model based on the global distribution maps. Both the cluster maps and Moran I graphs have proved that the AQI and other variables have a spatial distribution. Table 1 provides a brief description of the variables included in the model.

Table 1: Explanatory variables

Explanatory variable	Unit	Brief description
AQI		Air quality index
Wind speed	m/s	Mean wind speed
Atmospheric pressure	kPa	Annual mean atmospheric pressure
RH	%	Relative humidity
Elevation	m	Average elevation of a city
Sunshine duration	hour	Annual average sunshine duration of a city
Precipitation	mm	Total precipitation
Temperature	C^o	Annual mean temperature
Cars per person	pcs	Number of private cars per person
GDP	TL	Gross domestic product of a city
Per_GDP	TL	Per capita GDP
Total_pop	people	Total population
Pop_density	/sq km	Population density in persons per square kilometer

In the literature, it was stated that spatial regression models explain AQI better than classical regression model (*cf.* [2]). For this study, both classical regression model with least squares method and spatial model were used together. Calculations were performed in MATLAB and Stata 14 programs.

When selecting spatial model, it was examined whether spatial effect is caused by the error term or the lag. For 2014, Robust LM (error) probability value was found as 0.349, while Robust LM (lag) probability value was 0.889. In this case the smaller one was chosen. The results of Least Square Regression model (LSRM) and Spatial Error Model (SEM) results are given in Table 2. For 2014, the R-squared obtained by LSRM was found as 0.2102 and the R-squared for SEM was found as 0.2478. This shows that spatial regression models are better than LSRM. In addition, the value in the model was found as 0.680 and it means that the model is statistically significant.

Table 2: Parameter summary of LSRM and SEM for AQI in 2014

Variables	LSRM	SEM	Spatial dependence	Value	Probability value
Constant	2,80362**	24,69**			
Wind speed	12,0799***	13,7585***	Moran's I(error)	6,8405	0.000
Atmospheric Pressure	-0,130911	0,0295	Log-Likelihood	-444,959	
RH	0,789265*	-0,4439**	LM (lag)	13,1418	0.000
Elevation	0,0365429	0,0307	Robust LM (lag)	0,0192	0,889
Sunshine duration	3,280082	2,7526	LM (error)	13,9996	0,000
Precipitation	0,00171795	0,0206*	Robust LM (error)	0,8770	0,349
Temperature	10,5184*	5,3407	LM (SARMA)	14,0188	0,000
Cars per person	-0,052*	-0,002*			
GDP	-0,0004*	0,0031*			
Per_GDP	0,00180305	-0,0002			
Total_Pop	0,02162	0,00323**			
Pop_density	-0,0216	-0,0023			
W=ρ		0,680***			
AIC	923,918				
SIC	964,624				
R-squared	0,2102				
Pseudo R-squared		0,2478			

Conclusion

With this study the spatial distribution of Turkey's AQI of 2014 has been determined. The results showed that air pollution in a city also affects neighboring provinces. For this study, the proposed mathematical model determined as SEM.

Acknowledgements

This study was supported by Akdeniz University the Scientific Research Projects Coordination Unit with Project No: FBA-2018-3410.

References

[1] R. J. G. M Florax, L. Anselin, & S. J. Rey, (Eds.), Advances in Spatial Econometrics: Methodology, Tools and Applications, Berlin: Springer-Verlag; 2004, 513 pages.

[2] H. Liu, C. Fang, X. Zhang, Z. Wang, C. Bao & F. Li, The effect of natural and anthropogenic factors on haze pollution in Chinese cities: A spatial econometrics approach, Journal of cleaner production, 165 (2017), 323–333.

[3] F. Yalcin, Spatial analysis of the factors affecting the room price offered by the hotels in Antalya city (in Turkish), Unpublished Doctorate Thesis, Akdeniz University, Graduate School of Social Sciences, Antalya, Turkey, 2016.

[4] F. Yalcin, M. Mert, Determination of Hedonic Hotel Room Prices with spatial Effect in Antalya Economa, Sociedad y Territorio, 18(58) (2018), 697–734.

[5] S. E. Tuzcu, Mekansal Ekonometri ve Sosyal Bilimlerde Kullanim Alanlari, Ankara niversitesi SBF Dergisi, 71(2) (2016), 401–436.

[6] J. P. Elhorst, Applied Spatial Econometrics: Raising the Bar, Spatial Economic Analysis, 5(1) (2010), 9–28.

[7] J. Pinkse, Asymptotics of the Moran test and a test for spatial correlation in probit models, Department of Economics Working Paper, 1999.

[8] J. L. Anselin, & A. K. Bera, Introduction to spatial econometrics. Handbook of applied economic statistics, 1998, 237–289.

[9] A. M. Tepe, G. Doğan, Antalya ortam havasindaki PM2,5 ve PM2,5-10 kaynaklarinin belirlenmesi, VII. ULUSAL HAVA KRLL VE KONTROL SEMPOZYUMU, 2017, 203–219.

[1]Akdeniz University, Faculty of Science, Department of Mathematics, 07058 Antalya, Turkey

[2]Akdeniz University, Faculty of Engineering, Department of Environment Sciences, 07058 Antalya, Turkey

E-mail : fusunyalcin@akdeniz.edu.tr, ahmettepe@akdeniz.edu.tr, gdogan@akdeniz.edu.tr, ncizmeci17@gmail.com

Comparison of Physico-Mechanical Properties of Clova and Lyca Marbles in Akcay (Antalya) Region by Using Independent-Samples T-test Statistics

Orhan Kemal Tarinc[1] , *Ozge Ozer*[2] , *Burcu Aydin*[3] , *M. Gurhan Yalcin*[4]

Abstract

Marbles which are classified as metamorphic rocks go through a set of metamorphism process. It is known that the differentiation of physico-mechanical properties changes the quality and use purpose of the marble. In this study, it is aimed to compare the physico-mechanical values of 2 different marbles (Clova and Lyca marbles). Since the measures of shape of distribution (skewness and kurtosis values), which are descriptive statistics of physico-mechanical data, are determined to be in the range between +1 and -1, Independent-Samples T-test statistical method, which is used for parametric data, was applied in the study by using SPSS23 software package and then interpretations were made after the test. The p-values (significance) calculated by the Levene's test for equality of variances for Hk, A, and Ap values, which are some of the physico-mechanical properties, have been determined to be less than 0.05 (there is a difference between the variances, which means that they are not equal); they have been greater than 0.05 for Pt and Po values (there is not a difference between the variances, which means that they are equal). According to t-test statistics, depending on whether the difference between variances is equal or not, the p-values calculated in t-test showed that there is a statistical difference in terms of some physico-mechanical properties of these two marbles in Akcay region, and this difference was statistically significant. The statistical methodology helped researchers to understand this difference and understanding the differences in the marble quarries in terms of sustainable quality-use is of great importance.

2010 Mathematics Subject Classifications : 62H30, 62P30, 62J02
Keywords: Statistics, Independent sample t-test, SPSS, P-value

Introduction

Marble samples that are classified as natural stones have been used as construction materials in the construction sector for thousands of years [1]. The use of marbles as construction materials depends mainly on their physico-mechanical properties and chemical contents [2]. Some of the studies on this subject are as follows; in a study carried out in Lyca beige quarry in Akcay region by [3], various physico-mechanical tests were performed on 56 samples taken from 8 different locations. They analyzed the obtained data using multivariate statistical analysis methods. The result of the correlation analysis showed that there was a negative high correlation relationship between two physico-mechanical properties (abrasion resistance - Knoop hardness determination); their regression value can be considered as significant (R^2=0.761). In a study by [4] on the statistical evaluation of the chemical analysis results of Clova marbles, another marble quarry in Akcay region, since the chemical data did not meet the assumption of normality, the effect of the highest correlation on the formation process of the marble quarry was discussed by applying Spearman correlation. In a study by [5], the reddish marble samples taken from the marble quarry in Afyon/Iscehisar region were subjected to both chemical and physico-mechanical tests. The researchers examined the relationship between the physical and mechanical properties of the marble and its impact strength by making statistical analyses. The results obtained from the regression analysis showed that the impact strength increased at the same rate as the ratio of physico-mechanical properties such as density, porosity, water absorption, compressive strength, etc. increased. In the study carried out by [6] on marble samples brought from different regions, the relationships between uniaxial compressive strength, modulus of elasticity, stress strength, and Knoop hardness determination, which are some of the physico-mechanical properties of marbles, were examined. In the conclusion of the study, they proposed a linear model with a correlation coefficient of 0.74. Akcay region is located near the Antalya-Fethiye highway. Marble samples taken from different quarries in the same region were not evaluated statistically by using hypothesis tests.

Materials and Method

The most important of these parameters are effective porosity, homogeneity of the environment, and the marbles physico-mechanical properties which develop depending on them (Figure 1). it is necessary to understand the parameters that play an active role in the formation mechanism such as the effective porosity of the marble (Equation 1, [7]), and its homogeneity [8].

$$EffectivePorosity = n_e = \frac{V_b}{V_t} = \frac{PoreVol.}{TotalBulkVol.}$$

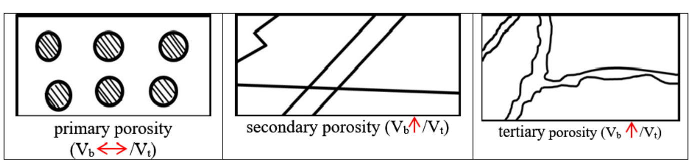


Figure 1. Schematic diagram of the formation of the heterogeneous condition

Parametric hypothesis test (independent t-test)

The hypothesis testing is an analysis method that allows to examine and compare the sample data set in a statistically significant way for the population parameters. There is a hypothesis theory tested in hypothesis tests [9]. In this study, the null hypothesis to be tested is as follows;

• Ha: There is no difference between the two marble quarries in terms of their physico-mechanical properties. The alternative hypothesis for the null hypothesis is designed as follows;

• Ho: There is a difference between the two marble quarries in terms of their physico-mechanical properties.

Results

Before the application of independent sample t-test, the assumption of normality was tested for the hypothesis tests (Table 1)[10].

Table 1. The results of the normality tests for two marble quarries in Akcay region

Test of Normality							
		Kolmogorov-Smirnova			Shapiro-Wilk		
		Statistic	df	Sig.	Statistic	df	Sig.
Hk	Clova Marble	250	8	150	855	8	107
	Lyca Beige Marble	276	8	073	808	8	035
Pt	Clova Marble	201	8	200*	911	8	363
	Lyca Beige Marble	256	8	131	810	8	036
A	Clova Marble	143	8	200*	979	8	959
	Lyca Beige Marble	165	8	200*	949	8	702
Po	Clova Marble	360	8	003	773	8	015
	Lyca Beige Marble	193	8	200*	870	8	151
Ap	Clova Marble	195	8	200*	896	8	263
	Lyca Beige Marble	196	8	200*	858	8	114

*. This is a lower bound of the true significance.
a. Lilliefors Significance Correction

Independent-samples t-test compares the mean values of the two sample groups. There should not be any common members in them [9] (Table 2).

Table 2. The results of the independent sample t-test for marble quarries

Independent Samples Test								
		Levene's test		t-test for equality of means				
		F	Sig.	t	df	Sig.	Mean Differ	Std. Error Differ
Hk	Equal variances assumed	70,888	,000	-6,317	14	,000	-260,87250	41,29869
	Equal variances not assumed			-6,317	7,374	,000	-260,87250	41,29869
Pt	Equal variances assumed	3,154	,097	24,516	14	,000	13,19375	,53817
	Equal variances not assumed			24,516	9,300	,000	13,19375	,53817
A	Equal variances assumed	5,592	,033	3,782	14	,002	1,96500	,51961
	Equal variances not assumed			3,782	8,080	,005	1,96500	,51961
Po	Equal variances assumed	2,767	,118	14,642	14	,000	5,16750	,35293
	Equal variances not assumed			14,642	7,448	,000	5,16750	,35293
Ap	Equal variances assumed	12,898	,003	-23,833	14	,000	-,35375	,01484
	Equal variances not assumed			-23,833	8,695	,000	-,35375	,01484

Discussion and Conclusion

Before interpreting independent-samples t-test, the assumption of normality for the data was tested using Kolmogorov-Smirnov and Shapiro-Wilk tests; the results of these tests were examined and the Significance (sig.) value was determined to be sig.>0.05 for the confidence interval of 95%. For interpreting independent-samples t-test, each physico-mechanical property was evaluated separately; therefore, the values obtained from Levene's test for equality of variances were primarily examined. For the physico-mechanical properties of Hk (Knoop Hardness Determination), Abrasion Value (A), Apparent Density (Ap), the significance value was determined to be Sig.<0.05 using Levene's test for equality of variances. In this case, there seems a difference between the variances. The first line of Table 2 shows that the significance value calculated by t-test for equality of means is Sig.<0.05. Since the Sig. value is less than 0.05, it means that the interpretation is statistically significant. For the physico-mechanical properties of Total porosity (Pt) and Open porosity (Po), the significance value was determined to be Sig.>0.05 using Levene's test for equality of variances. In this case, there seems no difference between the variances, i.e. the variances are equal. When the second line of Table 2 is examined, it is seen that the significance value calculated by t-test for equality of means is Sig.<0.05. Since the Sig. value is less than 0.05 means that the interpretation is statistically significant. That the Clova and Lyca marbles in Akcay region have different physico-mechanical properties was proved statistically significantly by performing hypothesis tests (independent-samples t-test). In this case, hypothesis Ha, which is the alternative of the null (absence, Ho) hypothesis, is supported. It is argued that all factors affecting the effective porosity process (the presence of soluble rock, solvent water, and heterogeneous condition) are the reason why marbles have different physico-mechanical properties (Figure 1) and this erosion phenomenon has its own morphogeological effects.

References

[1] Akesson. U., Lindqvist. J. E., Schouenborg. B., Grelk. B. (2006). Relationship between microstructure and bowing properties of calcite marble claddings Bulletin of Engineering Geology and the Environment,65(1), 73.

[2] Hossaini. S., Mirzaie. M., Nazari, H., Asadi, H., (2019). Mechanical and Petrographic Properties of Afghan White Marble Stone; Famous Marble in Afghanistan

[3] Aydin. B., Yalcin. F., Ozer, O., Yalcin, M. G., (2019) Regression Analysis and Statistical Examination of Knoop Hardness on Abrasion Resistance in Lyca Beige Marbles

[4] Tarinc. O. K., Ozer. O., Yalcin. F., Yalcin, M. G., (2019) Akcay Ky (Elmal, Antalya) Blgesi Mermerlerinin Kimyasal zellikleri Kullanlarak statistiksel Deerlendirmesi, International Symposium on Advanced Engineering Technologies (ISADET 2019)

[5] Sariisik. G. (2012). Determining performance of marble finished products on their usage areas by a new impact-resistance test method. Journal of Testing and Evaluation, 40(6), 945-951.

[6] Kaklis. K., Agioutantis. Z., Alevizos. G., Maurigiannakis. S. Mcrohardness Testng Of Marbles And Correlaton Wth The Strength Propertes.

[7] Glover. P. (2001). Formation Evaluation MSC course notes. Aberdeen University, 43-53.

[8] Espinal. L. (2002). Porosity and its measurement. Characterization of Materials, 1-10.

[9] S. Kalayci.(2009) SPSS Applied Multivariate Statistical Techniques, Ankara: Asil Publishing.

[10] Mert, M. (2016). Yatay Kesit Veri Analizi Bilgisayar Uygulamalar.Ankara: Detay Yayincilik.

[1,2,3,4]Akdeniz University, Department of Geological Engineering, 07058, Antalya
E-mail : orhankemaltarinc@icloud.com, oozer@akdeniz.edu.tr, burcuaydin.geo@gmail.com, gurhanyalcin@akdeniz.edu.tr

Appraisal of Metal Accumulation in Beach Sand Using Contamination Indices and Multivariate Statistical Analysis

Ozge Ozer[1] , *Fusun Yalcin*[2] , *Daniel G. Nyamsari*[1] , *M. Gurhan Yalcin*[1]

Abstract

It is a known fact that coastlines are always exposed to pollutant resulting from human and/or natural activities. It is possible to understand whether pollution levels of metal elements do exist in the sand sediments, and their posible sources through chemical tests and subsequent interpretation of the results. Knowledge of the level and means of accumulation of these pollutants is important to ensure sustainable protection and use of coastlines. In this study, the geochemical contents of 10 beach sand samples obtained from different locations on the Kemer-Sarsu coastline were analyzed. Igeo and PLI indices, as well as PCA were evaluated for metallic elements. Igeo revealed extreme contamination levels for Cr, stronglyly contaminated for Co and uncontaminated to moderately contaminated accumulation levels Ni and Mn. Highest values of PLI (> 2) was observed for samples (BS 6, 7, 9 and 10) with Igeo indicating at least a moderate to strongly contaminated levels for both Co and Cr. PCA also identified substential amount of these elements may have resulted from natural and unnatural origin.

2010 Mathematics Subject Classifications : 62H30, 62P30, 62J02
Keywords: I_{geo} index, PLI index, Principal component matrix, Correlations

Introduction

It is important for researchers to carefully monitor and evaluate pollution level on coastlines open for public use [1],[2] without restricting the studies to a single discipline, to provide scientists with a multidisciplinary platform for researcher [3]. In 2019, [1] evaluated results of chemical analysis of 47 beach sand sampled stations in the eastern region of Antalya using Pollution Index (PI), Enrichment Factor (EF) and Potential Ecological Risk Index (RI) guidelines. [4] studied chemical data of 42 sampled stations from Dden and Gksu rivers using heavy metal pollution index (HPI) and multivariate statistical analysis. Lots of researches on heavy metal accumulation in beach sands along the Mediterranean have been published ([5],[6], [7],[8],[9]). This study is aimed at finding out the similarities in the results of two risk assessment indices, geoaccumulation index (Igeo) and pollution load index (PLI) guidelines, to that of multivariate statistical analysis, to determine possible levels of metal elements concentration.

Materials and Method

Samples and chemical properties

10 samples from different locations on Kemer - Sarsu coastline of SW Antalya, were tested elementally by XRF (X-ray fluorescence) analysis.

Risk Assessment Indices

I_{geo} was used to measure the metal elements accumulation levels with repect to the background value for the element using eqution 1 [10]. PLI [6] was used to measure the comprehensive possible contamination levels of all the metal elements in a samples, by applying equation 2.

Equation − −1	**Equation − −2**
$I_{geo} = \log_2 \left(\frac{C_{Sample}}{1.5 C_{Background}} \right)$	$PLI = (CF_1 \times CF_2 \times CF_3 \times ... \times CF_n)^{\frac{1}{n}}$

Where CF is concentration ratio of element in sample to their background value Csample/Cbackground PLI<1 implies no pollution, whereas PLI>1 implies pollution by metals [1].

Result and Discussion

Element abundance shows Mn>Cr>Co>V>Ni>Zn (Table 1). Elements were unevenly distributed in the samples. Co, Zn and V were not present in all the samples. Mn, Cr, Ni and V are right skewed with extreme greater than their respective mean (Mn in BS4, Cr in BS 8 and 9, and V in BS8).

Table 1. Concentration of metal elements within the samples

Sample No	Mn	Cr	Co	Sr	Ni	Zn	V	Sn
BS1	1022.29	175.84	0.00	421.10	128.88	0.00	0.00	0.00
BS2	1223.65	0.00	0.00	0.00	129.66	82.75	0.00	0.00
BS3	1277.86	0.00	0.00	621.51	0.00	0.00	0.00	0.00
BS4	1881.94	126.58	168.42	504.82	87.23	83.55	74.50	0.00
BS5	1386.28	184.73	0.00	733.13	83.30	81.14	0.00	82.70
BS6	1223.65	1197.35	138.57	624.89	92.73	0.00	0.00	0.00
BS7	1277.86	821.04	184.05	657.87	0.00	82.75	0.00	101.61
BS8	1347.56	0.00	116.54	681.55	89.59	0.00	957.89	0.00
BS9	1533.43	7184.10	255.11	774.56	105.30	126.13	103.07	94.52
BS10	1053.27	2196.28	184.05	731.44	0.00	0.00	0.00	81.13
BS11	1122.97	718.41	108.72	741.58	94.30	81.94	0.00	0.00
Summary	**Mn**	**Cr**	**Co**	**Sr**	**Ni**	**Zn**	**V**	**Sn**
Mean	1304.61	1145.85	105.04	590.22	73.73	48.93	103.22	32.72
Skewness	1.38	2.77	0.00	2.15	0.80	0.07	3.22	0.71

Igeo for all elements in four of the samples (BS1, BS2, BS3 and BS5), indicated uncontaminated. Co showed moderate to strongly contaminated levels (BS4, and 6 to 10); Cr was of moderately to strongly contaminated levels (BS6, 7, 10 and 11) and of extremely contaminated levels (BS9); and V was of slightly contaminated and moderately contaminated levels (BS4 and BS8). Averagely, Igeo values for elements (Mn, Cr, Ni, Zn and V) was at uncontaminated to moderately contaminated level, except Co at strongly contaminated levels (Table 2).

Table 2. Geoaccumulation ndex value for elements in samples

Igeo	Mn	Cr	Co	Ni	Zn	V
BS1	0.3187	0.3813	-	0.3374	-	-
BS2	0.0593	-	-	0.3462	0.7842	-
BS3	0.0032	-	-	-	-	-
BS4	0.5617	0.0929	2.5630	0.2257	0.7702	1.3881
BS5	0.1207	0.4525	-	0.2922	0.8125	-
BS6	0.0593	3.1488	2.2816	0.1375	-	-
BS7	0.0032	2.6045	2.6911	-	0.7842	-
BS8	0.0799	-	2.0318	0.1872	-	2.2964
BS9	0.2663	5.7338	3.1621	0.0460	0.1761	0.9198
BS10	0.2756	4.0240	2.6911	-	-	-
BS11	0.1832	2.4118	1.9316	0.1132	0.7983	-

PLI values ranged from 17 in all the samples, indicating slight levels of pollution due to high concentrations of the metal (Table 3).

Table 3. PLI index results

Sample	BS-1	BS-2	BS-3	BS-4	BS-5	BS-6	BS-7	BS-8	BS-9	BS-10	BS-11
PLI	1.645	1.337	1.503	1.617	1.368	3.715	3.280	3.117	3.829	6.641	2.353

Pearson correlation; The strongest significant relationship between the elements is of at a strong leve between Cr and Co (r=0.684*). Relationship between Zn and Mn and Cr is at a moderate level (r=0.40.59). Factor and Principal Component Analysis: Three load factor analysis explains 78.224% of the variances of the elements. This high value indicated reliability of the results. Components 1, 2 and 3 accounts for 41.43%, 22.127% and 14.665% respectively of the total variance (Table 4). Majority of Cr and Co are thought to have originated from a natural origin while a substantial amount of these elements can be attributed to both natural and unnatural sources; as we as most of Mn and Zn. Majority of Ni was identified to be anthropogenic. No identifiable source could be attributed to V.

Table 4. Total variance explained and Component matrix

Total Variance Explained			
	Extraction Sums of Squared Loadings		
Component	**Total**	**% of Variance**	**Cumulative %**
1	7.458	41.432	41.432
2	3.983	22.127	63.559
3	2.640	14.665	78.224
Extraction Method:Principal Component Analysis.			

Component Matrix	Component		
	1	**2**	**3**
Mn	,243	,806	,435
Cr	,689	,585	,257
Co	,603	,571	,008
Ni	,342	,190	,439
Zn	,105	,748	,004
V	,104	,164	,032

Average Igeo level for Co was moderately to strongly contaminated and Cr was moderately. Samples BS (6, 7, 9 and 10) with PLI>3 also contained very high concentrations of Cr and Co; BS-8 contained very high concentrations of V as well as BS9, which has also has high concentrations of Zn. These elements also indicated high values of Igeo. Pearson correlation indicates a strong possible of Cr and Co originating from same source. Zn possibly originated from same source as some Mn and Cr, and a less likely possibility of same origin as Co, Ni, and Zn. PCA indicates Majority of Cr and Co are from the natural source while a substantial amount is undistinguished between a natural and anthropogenic source, as well as most of Mn and Zn. Slight contribution of Mn and Ni may be attributed to unnatural source. The high level of concentrations in most of these elements occurred mostly in samples BS (6, 7, 8, 9, 10 and 11) and may have been influenced by the rock types in the area.

Acknowledgements

This study is financed by department of scientific research project at Akdeniz University (BAP, FBA-2018-3484) to which we are grateful.

References

[1] Yalcin. F. 2019 Application of Multivariate Statistic and Pollution Index Techniques to Determine Beach Sand Element Distribution. East of Antalya City (Accepted-Filomat)

[2] Yalcin, M. G, Coskun, B, Nyamsari, D. G, Yalcin, F. (2019). Geomedical, ecological risk, and statistical assessment of hazardous elements in shore sediments of the Iskenderun Gulf, Eastern Mediterranean, Turkey. Environmental Earth Sciences, 78(15), 438. DOI 10.1007/s12665-019-8435-5

[3] Alshahri. F. 2017 Heavy Metal Contamination n Sand And Sediments Nearto Disposal Site Of Reject Brine From Desalination Plant. Arabian Gulf: Assessment Of Environmental Pollution. DOI 10.1007/s11356-016-7961-x

[4] Leventeli. Y, Yalcin. F, Kilic. M. (2019) An Investigation About Heavy Metal Pollution Of Duden And Goksu Streams (Antalya. Turkey). Applied Ecology And Environmental Research 17(2). 2423-2436.

[5] Ozer. O, Dyansamri. D. G, Yalcin. F, Yalcin M. G. 2019 Kemer-Sarisu (Antalya) Sahil eridindeki Plaj Kumlarinin statistiksel Anomalileri le nsan Salg Acisindan Deerlendirilmesi 72nd Geological Congress of Turkey with international participation

[6] Tomlinson. DC, Wilson. JG, Harris. CR, Jeffrey DW. (1980) Problems in the assessment of heavy metal levels in estuaries and the formation of a pollution index. Helgol. Wiss. Meeresunters.33. 566-575.

[7] Yalcin. F, Nyamsari. D. G, Paksu. E, Yalcin. M. G.(2015) Statistical Assessment of Heavy Metal Distribution and Contamination of Beach Sands of Antalya-Turkey: an Approach to the Multivariate Analysis Techniques DOI 10.2298/FIL1604945Y

[8] Hammoud. H. A, Rabee. A. M. (2017). Assessment of Heavy Metals Pollution in Sediment of Shatt Al-Hilla by Using Ecological Indices. Iraqi Journal of Science. 58(3C). 1609-1616.

[9] Yalcin F, Ozer O, Nyamsari D. G, Yalcin M. G. (2019) Statistical Evaluation Of The Geochemical Content Of Beach Sand Along The Sarisu - Kemer Coastline Of Antalya, Turkey, International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2018) page 433-436c AIP Conference Proceedings 2116, 100005 (2019) doi.org/10.1063/1.5114081

[10] Muller, G. (1969) Index of Geoaccumulation in sediments of the Rhine riverRiver. Geol J. 2. 108118.

[1,3,4]Akdeniz University, Department of Geological Engineering, 07058, Antalya

[2]Akdeniz University, Department of Mathematics, 07058, Antalya, Turkey

E-mail : oozer@akdeniz.edu.tr, fusunyalcin@akdeniz.edu.tr, dnyamsari@yahoo.com, gurhanyalcin@akdeniz.edu.tr

Analysis of the Students' Attitudes Towards Math and Geometry Courses by Using Exploratory Factor Analysis

Nurfer Cizmeci[1] , *Fusun Yalcin*[1]

Abstract

This study aims to determine students' attitudes towards mathematics and geometry courses. In this context, two schools in Kepez district of Antalya province were selected and a survey was carried out for high school students. 301 students participated in this survey. The reliability and validity analyses were performed for the research and Cronbachs Alpha value was determined to be 0.936 for geometry attitude scale and 0.959 for mathematics attitude scale. Students' attitudes towards these courses were examined by using factor analysis.

2010 Mathematics Subject Classifications : 62H25, 62-02, 62N05
Keywords: Exploratory Factor Analysis, Statistics, Mathematics Attitude Scale, Geometry Attitude Scale, Reliability and Validity Analyses

Introduction

The studies using exploratory factor analysis, which is one of the multivariate statistical methods, have been increasing and gaining importance each passing day. The main objective of this analysis is to obtain a small number of significant independent factors from a large number of related variables. In other words, both the number of variables is reduced and the variables are classified (*cf.* [3]). Factors are formed by classifying variables that have significant correlation among themselves (*cf.* [1]). A Scale for Attitudes Toward Geometry was developed by Bulut and oth. [5] to determine the attitudes of individuals towards geometry. The reliability and validity analyses of the scale they developed were performed and the factor analysis method was used in this study. When the studies in this area are examined in the literature, it has been seen that attitude scale towards mathematics or geometry courses has been developed or multivariate statistical analyses were carried out by using existing scales. As a result of the literature review, it has been seen that no such study was carried out in Antalya. For this reason, the purpose of this study is to examine the attitude scale of the students towards Mathematics and Geometry courses considering entire Antalya. The sample of this study consists of 301 high school students selected by simple random sampling method from 2 schools in Kepez district of Antalya Province in 2018-2019 academic year. 172 of the students were female and 129 were male. This study is the first in that the mathematics attitude scale and geometry attitude scale were applied together at the same time and the survey was carried out in Antalya province. Also, attitudes of secondary school students towards mathematics and geometry courses were examined by using exploratory factor analysis method.

Methods

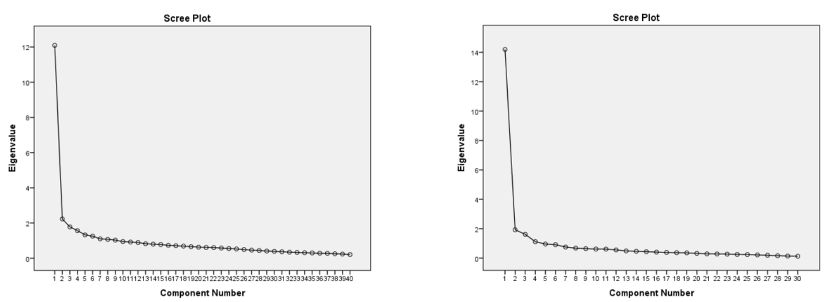
In exploratory factor analysis, it is aimed to obtain independent and less number of new variables (factors) from many interrelated variables by using the covariance and correlation matrix of the data. Let X be a random data matrix with p variables and N subjects, Σ be the covariance matrix of X, and μ be the mean vector. Two types of factor models, orthogonal and oblique factor models, can be established between the observation vector X and the unobservable factors. The orthogonal factor model will be discussed below. Also, the factor model refers to the linear factor model[3]. The factor analysis model is written as follows:

$$X_1 - \mu_1 = l_{11}.F_1 + l_{12}.F_2 + ... + l_{1k}.F_k + \varepsilon_1$$
$$X_2 - \mu_2 = l_{21}.F_1 + l_{22}.F_2 + ... + l_{2k}.F_k + \varepsilon_2$$
$$\vdots$$
$$X_p - \mu_p = l_{p1}.F_1 + l_{p2}.F_2 + ... + l_{pk}.F_k + \varepsilon_p$$

where l_{ij}, the factor loading, specifies the factor loading of the i^{th} variable on the j^{th} factor. The matrix form of the factor analysis model is written as follows: $X - \mu = LF + \varepsilon$ where, $X - \mu$: difference vector with the dimension of (px1), L: factor loadings matrix with the dimension of (pxk), F: factor vector with the dimension of (kx1) and ε : error vector with the dimension of (px1) (*cf.* [3]). The factor analysis method is carried out in four basic steps.First step is Assessing the suitability of the data set for factor analysis, second step is Obtaining the factors, third step is Rotation of the factors and forth step is Giving a name for each factor (*cf.* [2]). When investigating the suitability of a data set for factor analysis, firstly the correlation coefficients between the variables are examined. When the correlation coefficient between the variables is higher, the probability of getting co-factor increases. Then Bartlett's Test of Sphericity and Kaiser-Meyer-Olkin Test(KMO) are performed. Bartlett's Test of Sphericity is the requirement to test whether the correlation matrix between the variables is a unit matrix. For the data set to be suitable for factor analysis, the null hypothesis "The correlation matrix is a unit matrix "must be tested using Bartlett's Test of Sphericity and this hypothesis must be rejected. The KMO measure of sampling adequacy is the ratio of the correlation coefficients between the variables to the partial correlation coefficients. This value ranges between zero and one. If this value is less than 0.5, the data set is not suitable for the factor analysis. There are multiple methods for obtaining the factors and the most commonly used methods are eigenvalue statistics and scree plot (*cf.* [3]).

Results

A survey of demographic information, geometry attitude scale of 40 items (*cf.* [7]) and mathematics attitude scale of 30 items (*cf.* [6]) were applied to two high schools randomly selected in Kepez district of Antalya. These scales were compiled by adding some questions. 301 students were reached in the 2018-2019 academic year. Reliability analysis of the data collected by the survey was performed. The results of the analyses showed that the internal consistency coefficient(Cronbachs Alpha) was 0.936 for the geometry attitude scale and 0.959 for the mathematics attitude scale. These values are quite high. Sinne the size of our data set is over 300, it is assumed that it is distributed normally according to the central limit theorem. The Kaiser-Mayer-Olkin(KMO) and Bartlett tests were used to test the suitability of the geometry attitude scale data for factor analysis. For the data set to be suitable for factor analysis, Mert (*cf.* [1]) states that the data set is not suitable for factor analysis if the KMO value is less than 0.5 and very convenient if it is close to 1. The KMO value for the geometry attitude scale was 0.921 and 0.952 for the mathematics attitude scale. Since the KMO values for both scales are close to 1, our data sets are suitable for factor analysis. The test statistics for Bartletts Test of Sphericity for the geometry attitude scale was calculated as 4922.589 and the P value(Sig.) was calculated as 0.00. The test statistics for Bartletts Test of Sphericity for the mathematics attitude scale was calculated as 6291.662 and the P value(Sig.) was calculated as 0.00. Since P <0.01 for both scales, there is a relationship between the variables, which means that the result is significant. Eigenvalue statistics and scree plot, which are the most commonly used methods, were used to determine the number of factors. According to the eigenvalue statistic, the number of factors was determined by the number of eigenvalues which are greater than 1. Accordingly, 8 factors were determined for the geometry attitude scale and 4 factors were determined for the mathematics attitude scale. Determined factors explain 63% of the total variance for the geometry attitude scale and 59% for the mathematics attitude scale. Considering the scree plot, it is seen that the slope of the curve levels off significantly after the 8^{th} factor at the scree plot of the geometry attitude scale, and it levels off significantly after the 4^{th} factor at the scree plot of the mathematics attitude scale. Figure 1 supports our factor numbers.



(a) (b)

Figure 1: Scree plot for geometry attitude scale (a) scree plot for mathematics attitude scale (b)

The factor names we determined are given in Table 1-(a) and Table 1-(b).

Table 1: (a): Factor names for geometry attitude scale,where Factor 1: Perceived academic achievement towards geometry course, Factor 2: The place of geometry in daily life, Factor 3: Difficulty in perceiving geometry, Factor 4: Enjoying geometry, Factor 5: Feeling the need for geometry, Factor 6: 3D Visual Intelligence, Factor 7: Relationship between geometric shapes and nature, Factor 8: Geometric Shape Drawing Skills. (b): Factor names for mathematics attitude scale, where Factor 1: Perceived academic achievement towards mathematics course, Factor 2: The place of mathematics in daily life, Factor 3: Enjoying mathematics, Factor 4: Interest in the history of mathematics.

(a)

Rotated Component Matrixa

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7	Factor 8
G30	0.652							
G6	0.613							
G34	0.612							
G32	0.533							
G33	0.51							
G19	0.506							
G21	0.506							
G3	0.48							
G14	0.456							
G27		0.709						
G17		0.652						
G9		0.599						
G16		0.559						
G13		0.551						
G7		0.543						
G38		0.441						
G28			0.69					
G24			0.684					
G26			0.61					
G12			0.521					
G10			0.435					
G37				0.72				
G35				0.633				
G39				0.525				
G31				0.477				
G40				0.424				
G25					0.761			
G5					0.56			
G1					0.465			
G8						0.688		
G18						0.658		
G36							0.687	
G20							0.599	
G11								0.545
G23								0.511
G15								0.472
Extraction Method: Principal Component Analysis.								
Rotation Method: Varimax with Kaiser Normalization.								
a. Rotation converged in 15 iterations.								

(b)

Rotated Component Matrixa

	Factor 1	Factor 2	Factor 3	Factor 4
M15	0.754			
M16	0.735			
M9	0.731			
M14	0.73			
M10	0.719			
M20	0.706			
M11	0.706			
M5	0.699			
M13	0.686			
M6	0.685			
M18	0.683			
M2	0.659			
M3	0.652			
M4	0.652			
M8	0.648			
M17	0.642			
M12	0.632			
M1	0.619			
M7	0.59			
M19	0.529			
M21	0.443			
M26		0.695		
M25		0.618		
M27		0.594		
M29		0.517		
M22			0.735	
M23			0.677	
M30			0.576	
M24				0.918
M28				0.917
Extraction Method: Principal Component Analysis.				
Rotation Method: Varimax with Kaiser Normalization.				
a. Rotation converged in 10 iterations.				

Conclusion

In this study, a pilot study was carried out by applying a survey on 301 high school students to measure their attitudes towards mathematics and geometry courses. According to the results, 8 factors were determined from the geometry attitude scale and 4 factors were determined from the mathematics attitude scale. In line with these findings, the first factor was determined as perceived academic achievement towards geometry for geometry course and perceived academic achievement towards mathematics for the mathematics course. The results of this study correspond to the results of the other studies in the literature.

Acknowledgements

We would like to thank the Mediterranean University BAP Unit for their contribution (Project NO: FYL-2019-4384).

References

[1] M. Mert, Yatay Kesit Veri Analizi Bilgisayar Uygulamalari, Ankara: Detay Yayincilik; 2016, pp. 113-118.

[2] S. Kalayci, SPSS Uygulamali Cok Degiskenli Istatistik Teknikleri, Ankara, Turkey: Asil Yayin Dagitim; 2010, pp. 321–331.

[3] K. Ozdamar, Paket Programlar ile Istatiksel Veri Analizi: SPSS-MINITAB, Eskisehir: Kaan Kitabevi; 2002, pp. 235–338.

[4] A. Duatepe, & S. Cilesiz, Matematik tutum olcegi gelistirilmesi, Hacettepe Universitesi Egitim Fakultesi Dergisi, 16(16) (1999), 45–52.

[5] S. Bulut, C. Ekici, A. I. Iseri, & E. Helvaci, Geometriye yonelik bir tutum olcegi, Egitim ve Bilim, 27(125) (2002), 3–7.

[6] P. Askar, Matematik dersine yonelik tutumu olcen likert tipi bir olcegin gelistirilmesi, Egitim ve Bilim, 11(62) (1986), 31–36.

[7] R. Bindak, Geometri tutum olcegi guvenirlik gecerlik calismasi ve bir uygulama, Unpublished doctoral dissertation, University of Dicle, Diyarbakir, Turkey, 2004.

[1]Akdeniz University, Faculty of Science, Department of Mathematics, 07058 Antalya, Turkey

E-mail : ncizmeci17@gmail.com, fusunyalcin@akdeniz.edu.tr

Combinatorial sums arising from integral representations for certain families of combinatorial numbers and polynomials

Irem Kucukoglu[1]

Abstract

The aim of this study is to provide some integral representations for certain families of combinatorial numbers and polynomials. By using these integral representations and their relationships with some well-known special functions such as the beta function and the Euler gamma function, we obtain not only some combinatorial sums including the binomial coefficients, but also their computation formulas. Finally, we investigate relations between Gauss hypergeometric function and these integral representations.

2010 Mathematics Subject Classifications : 05A15, 05A19, 11B83, 11S23, 33B15, 33C05.

Keywords: Generating functions, Combinatorial numbers and polynomials, Combinatorial sums, Binomial coefficients, Beta function, Euler gamma function, Integral representation, Hypergeometric functions.

Introduction

Up to the present, combinatorial sums containing binomial coefficients have been served as auxiliary tools for solving many problems in the fields of mathematics, mathematical physics and the other related ones. Due to the requirements of explicit formulas for some commonly used combinatorial sums in the literature, we are here motivated to derive computation formulas and identities for some combinatorial sums containing binomial coefficients. By using a slightly similar techniques that of the works [3], [4] and [7], it has been reached to the goals of this study. In order to provide some formulas for the aforementioned combinatorial sums, we start to recall the following notations and definitions.

Let k be a natural number and λ be a real or complex number. In this study, we consider to investigate certain families of the combinatorial numbers and polynomials given by the following generating functions:

$$\mathcal{G}\left(t,k;\lambda\right)=2^{-k}\left(\lambda\left(1+\lambda t\right)-1\right)^{k}=\sum_{n=0}^{\infty}Y_{n}^{(-k)}\left(\lambda\right)\frac{t^{n}}{n!}, \tag{1}$$

and

$$\mathcal{G}\left(t,x,k;\lambda\right)=\mathcal{G}\left(t,k;\lambda\right)\left(1+\lambda t\right)^{x}=\sum_{n=0}^{\infty}Q_{n}\left(x;\lambda,k\right)\frac{t^{n}}{n!} \tag{2}$$

which were introduced by Kucukoglu et al. in their recent paper [1].

Explicit formula for the numbers $Y_n^{(-k)}(\lambda)$ is given by (*cf.* [1]):

$$Y_n^{(-k)}(\lambda) = \begin{cases} 2^{-k} n! \binom{k}{n} \lambda^{2n} (\lambda-1)^{k-n} & \text{if} \quad n \leq k \\ 0 & \text{if} \quad n > k. \end{cases} \tag{3}$$

Relation between the numbers $Y_n^{(-k)}(\lambda)$ and the polynomials $Q_n(x;\lambda,k)$ is given by (*cf.* [1]):

$$Q_n(x;\lambda,k) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} Y_j^{(-k)}(\lambda) (x)_{n-j} \tag{4}$$

where $(x)_n = x(x-1)(x-2)\dots(x-n+1)$.

The beta function $B(\alpha,\beta)$ is defined by (*cf.* [2], [5, p. 9, Eq-(60)], [6]):

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t, \qquad (\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0), \tag{5}$$

so that the relation between the beta function $B(\alpha,\beta)$ and the Euler gamma function $\Gamma(\alpha)$ is given by

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{6}$$

and note that if we replace α and β by natural numbers n and m respectively, then we have

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!}, \tag{7}$$

(*cf.* [2], [5], [6]).

Integral representations

In this section, we provide some formulas for integral representations of the combinatorial numbers $Y_n^{(-k)}(\lambda)$ and the combinatorial polynomials $Q_n(x;\lambda,k)$. By using these formulas, we obtain some combinatorial sums including binomial coefficients and falling factorial.

Theorem 1. *Let $n \leq k$. Then we have*

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n} 2^{-k} n! \binom{k}{n} \sum_{j=0}^{k-n} (-1)^j \binom{k-n}{j} \frac{1}{2n+j+1}. \tag{8}$$

Proof. By integrating both sides of the equation (3) for its special case when $n \leq k$, with respect to the parameter λ, from 0 to 1, we get

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = 2^{-k} n! \binom{k}{n} \int_0^1 \lambda^{2n}(\lambda-1)^{k-n}\,\mathrm{d}\lambda. \tag{9}$$

Then, by applying the binomial theorem to the right-hand side of the above equation, we have

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n} 2^{-k} n! \binom{k}{n} \sum_{j=0}^{k-n} (-1)^j \binom{k-n}{j} \int_0^1 \lambda^{2n+j}\mathrm{d}\lambda$$

which yields the desired result. □

Theorem 2. *Let $n \leq k$. Then we have*

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}\,2^{-k} n! \binom{k}{n} \frac{\Gamma(2n+1)\,\Gamma(k-n+1)}{\Gamma(k+n+2)}. \tag{10}$$

Proof. If we modify the right-hand side of (9), then we have

$$\int_0^1 Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}\,2^{-k} n! \binom{k}{n} \int_0^1 \lambda^{2n}(1-\lambda)^{k-n}\,\mathrm{d}\lambda.$$

By combining the above equation with (5) and (6), we thus arrive at the desired result. ☐

Theorem 3. *The following formula holds true:*

$$\sum_{j=0}^{k-n} (-1)^j \binom{k-n}{j} \frac{1}{2n+j+1} = \frac{(2n)!\,(k-n)!}{(k+n+1)!}. \tag{11}$$

Proof. If we compare the right-hand sides of equations (8) and (10) by using (7), then we arrive at the desired result. ☐

Now, we provide a relation between the Gauss hypergeometric function and the integral representation of the combinatorial numbers $Y_n^{(-k)}(\lambda)$ by the following theorem:

Theorem 4. *Let $n \leq k$. Then we have*

$$\int_0^z Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}\,2^{-k}\,(k)_n\,z^{2n+1}}{2n+1}\ {}_2F_1(n-k, 2n+1; 2n+2; z),$$

where ${}_2F_1$ denotes the Gauss hypergeometric function.

Proof. Integrating both sides of (3) for its special case when $n \leq k$, with respect to λ, from 0 to z, we have

$$\int_0^z Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}\,2^{-k}\,(k)_n \sum_{m=0}^{\infty} (-1)^m \binom{k-n}{m} \int_0^z \lambda^{2n+m} d\lambda. \tag{12}$$

We thus arrive at

$$\int_0^z Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = (-1)^{k-n}\,2^{-k}\,(k)_n\,z^{2n+1} \sum_{m=0}^{\infty} (-1)^m \frac{(k-n)_m}{2n+m+1} \frac{z^m}{m!} \tag{13}$$

which is equivalent to

$$\int_0^z Y_n^{(-k)}(\lambda)\,\mathrm{d}\lambda = \frac{(-1)^{k-n}\,2^{-k}\,(k)_n\,z^{2n+1}}{2n+1} \sum_{m=0}^{\infty} \frac{(n-k)^{(m)}\,(2n+1)^{(m)}}{(2n+2)^{(m)}} \frac{z^m}{m!},$$

in which, $(x)^{(m)} = x(x+1)(x+2)\dots(x+m-1)$ denotes the well-known Pochhammer symbol. Hence we arrive at the desired result. ☐

Theorem 5.

$$\int_0^1 Q_n\left(x;\lambda,k\right)\mathrm{d}\lambda = \frac{2^{-k}k!}{(k+n+1)!}\sum_{j=0}^{n}\left(-1\right)^{k-j}\binom{n}{j}\left(x\right)_{n-j}\left(n+j\right)!. \tag{14}$$

Proof. By integrating both sides of the equation (4) with respect to the parameter λ, from 0 to 1, we get

$$\int_0^1 Q_n\left(x;\lambda,k\right)\mathrm{d}\lambda = \sum_{j=0}^{n}\binom{n}{j}\left(x\right)_{n-j}\int_0^1 \lambda^{n-j}Y_j^{(-k)}\left(\lambda\right)\mathrm{d}\lambda.$$

By (3), for $n \leq k$ we have

$$\int_0^1 Q_n\left(x;\lambda,k\right)\mathrm{d}\lambda = 2^{-k}\sum_{j=0}^{n}\binom{n}{j}\left(k\right)_j\left(x\right)_{n-j}\int_0^1 \lambda^{n+j}\left(\lambda-1\right)^{k-j}\mathrm{d}\lambda.$$

By combining the above equation with (5) and (7), we get a combinatorial sum for integral representations of the combinatorial polynomials $Q_n\left(x;\lambda,k\right)$ which is the assertion (14) of theorem. □

Acknowledgements

This paper has been presented at **"The 2nd Mediterranean International Conference of Pure&Applied Mathematics and Related Areas"**, Paris-France, August 28-31, 2019.

References

[1] I. Kucukoglu, B. Simsek and Y. Simsek, New families of combinatorial numbers and polynomials: approach to Poisson-Charlier polynomials and probability distribution function, *preprint*.

[2] E. D. Rainville, Special functions, New York: The Macmillan Company; 1960.

[3] Y. Simsek, A new class of polynomials associated with Bernstein and beta polynomials, Math. Meth. Appl. Sci. 37 (2014), 676–685.

[4] Y. Simsek, Analysis of the Bernstein basis functions: an approach to combinatorial sums involving binomial coefficients and Catalan numbers, Math. Meth. Appl. Sci. 38 (2015), 3007–3021.

[5] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Dordrecht, Boston and London: Kluwer Acedemic Publishers; 2001.

[6] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Amsterdam, London and New York: Elsevier Science Publishers; 2012.

[7] H. M. Srivastava, I. Kucukoglu and Y. Simsek, Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials, J. Number Theory 181 (2017), 117–146.

[1]Department of Engineering Fundamental Sciences, Faculty of Engineering, Alanya Alaaddin Keykubat University TR-07425 Antalya, Turkey.

E-mail : irem.kucukoglu@alanya.edu.tr

On q-Bernstein-Schurer Polynomials

Erkan Agyuz[1]

Abstract

The main object of this paper is to construct some new useful properties for q-Bernstein-Schurer polynomials. Furthermore, we give recurrence relations and derivative formula for these polynomials under ordinary and q-calculus.

2010 Mathematics Subject Classifications : 12D10, 26C05, 05A15, 11B65, 11B83

Keywords: Bernstein polynomials, Bernstein-Schurer polynomials, q- Bernstein-Schurer polynomials, generating function, special functions

Introduction

In recent years, q-calculus has been studied by many researchers not only mathematics but also statistics, engineering and applied sciences. Oruc and Phillips defined q-Bernstein polynomials. Kim suggested a new type q-Bernstein polynomials which is differ from q-Bernstein polynomials of Oruc and Phillips. Simsek and Acikgoz constructed a new generating function of the q-Bernstein type polynomials and obtained some elementary properties of this function. Simsek gave a new generating function for the generalized q-Bernstein-type basis polynomials and established relations between the generalized q-Bernstein-type basis polynomials the Bernoulli polynomials of higher-order and the generalized Stirling numbers of the second kind. Goldman et. al. derived the generating function of q-Bernstein basis fuctions by using q- calculus and constructed some new and interesting results of this function. Han et.al. a new generalization of Bzier curves obtained with related to based on the Lupa q-analogue of Bernstein operator.

Now, we present miscellaneous definitions and notations with related to q-calculus, Bernstein-Schurer polynomials and q-Bernstein-Schurer polynomials.

Let $0 < q \leq 1$ and $n \in \mathbb{Z}$. Then the q-analogue of n is defined by

$$[n]_q = \frac{q^x - 1}{q - 1}.$$

The q-Binomial coefficients are defined as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! \, [k]_q!}$$

where

$$[n]_q! = \begin{cases} 1 & , n = 1 \\ [n]_q \, . \, [n-1]_q \, . \cdots . \, [1]_q & n \neq 1 \end{cases} .$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1 & , n = 0 \\ (x-a) \, . \, (x-qa) \, . \cdots . \, (x - q^{n-1}a) & , n \geq 1 \end{cases}$$

The q-derivative operator is determined at the following equality

$$\mathbf{D}_q\left[f\left(x\right)\right]=\frac{f\left(qx\right)-f\left(x\right)}{\left(q-1\right)x}$$

where $f:\mathbb{R}\rightarrow\mathbb{R}$ and $x\in\mathbb{R}$.

The q-integral operator is defined as below:

$$\int f\left(x\right)d_qx=\left(1-q\right)x\sum_{j=0}^{\infty}q^jf\left(q^jx\right).$$

We have two types of q-exponential functions as in the q-calculus. These functions are defined as follows:

$$\begin{aligned} e_q^x &= \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} \\ E_q^x &= \sum_{n=0}^{\infty}q^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_q!}. \end{aligned}$$

In 1962, Schurer defined Bernstein-Schurer operators as follows:

$$\widetilde{B}_{m,p}\left(f,x\right)=\sum_{k=0}^{m+p}f\left(\frac{k}{m}\right)\left(\begin{array}{c} m+p \\ k \end{array}\right)x^k\left(1-x\right)^{m+p-k}$$

where $p,m\in\mathbb{N}$ and $\widetilde{B}_{m,p}\left(f,x\right):C\left(\left[0,p+1\right]\right)\rightarrow C\left(\left[0,1\right]\right)$ .By using above equation, we give Bernstein-Schurer polynomials at the following equation:

$$\widetilde{B}_{m,p}\left(x,k\right)=\left(\begin{array}{c} m+p \\ k \end{array}\right)x^k\left(1-x\right)^{m+p-k},k\leq m+p.$$

In 2011, Muraru defined q-Bernstein-Schurer operators and q-Bernstein-Schurer polynomials, respectively, as follows:

$$\widetilde{B}_{m,p}\left(f;q;x\right)=\sum_{k=0}^{m+p}f\left(\frac{k}{m}\right)\left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k\left(1-x\right)_q^{m+p-k}$$

and

$$\widetilde{B}_{m,p}\left(x;q,k\right)=\left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k\left(1-x\right)_q^{m+p-k}.$$

Main Results

In this part, we investigate results of q-Bernstein-Schurer polynomials under the q-calculus. Furthermore, we give the generating function of these polynomials.

Now, we show that $\widetilde{B}_{m,p}\left(x;q,k\right)$ is described under the derivative operator depand on x as below:

Corollary 1. *For $x\in[0,1]$, $k\leq m+p$ and $0<q\leq 1$, we get*

$$\frac{d}{dx}\left(\widetilde{B}_{m,p}\left(x;q,k\right)\right)=\frac{k\left(1-q^{m+p-k-1}x\right)}{[k]_q\left[m+p-k\right]_q}\widetilde{B}_{m,p}\left(x;q,k-1\right)+\frac{[k+1]_q\left(m+p-k\right)}{x\left[m+p-k\right]_q}\widetilde{B}_{m,p}\left(x;q,k+1\right).$$

Proof. By using the definition of q-Bernstein-Schurer polynomials, we have

$$\begin{aligned}
\frac{d}{dx}\left(\widetilde{B}_{m,p}(x;q,k)\right) &= \frac{d}{dx}\left(\left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k (1-x)_q^{m+p-k}\right) \\
&= k\left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^{k-1}(1-x)_q^{m+p-k} + \\
&\quad \left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k (m+p-k)(1-x)_q^{m+p-k-1} \\
&= \left[\begin{array}{c} \frac{k}{[k]_q [m+p-k]_q}\left(1-q^{m+p-k-1}x\right). \\ \widetilde{B}_{m,p}(x;q,k-1)) \end{array}\right] \\
&\quad + \frac{[k+1]_q (m+p-k)}{x[m+p-k]_q}\widetilde{B}_{m,p}(x;q,k+1).
\end{aligned}$$

Therefore, we arrive at the desired result. □

Corollary 2. *For $x \in [0,1]$, $k \leq m+p$ and $0 < q \leq 1$, we get*

$$\widetilde{B}_{m,p}(x;q,k) = \frac{[m+p-k]_q}{[k]_q (1-q^{m+p-k+1}x)}\widetilde{B}_{m,p}(x;q,k-1).$$

Proof. By using definition of $\widetilde{B}_{m,p}(x;q,k-1)$, we get

$$\begin{aligned}
\widetilde{B}_{m,p}(x;q,k-1) &= \left[\begin{array}{c} m+p \\ k-1 \end{array}\right]_q x^k (1-x)_q^{m+p-k+1} \\
&= \frac{[k]_q}{[m+p-k+1]_q (1-q^{m+p-k+1}x)}\left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k (1-x)_q^{m+p-k}.
\end{aligned}$$

After some basic mathematical operations, we have at the desired result. □

Corollary 3. *For $x \in [0,1]$, $k \leq m+p$ and $0 < q \leq 1$, we obtain*

$$\int_0^1 \widetilde{B}_{m,p}(x;q,k)\,d_qx = \left[\begin{array}{c} m+p \\ k \end{array}\right]_q \sum_{l=0}^{m+p-k}\left[\begin{array}{c} m+p-k \\ l \end{array}\right]_q (-1)^l q^{\binom{l}{2}}\frac{1}{[l+k+1]_q}.$$

Proof. By using definition of $\widetilde{B}_{m,p}(x;q,k)$,we have

$$\begin{aligned}
\int_0^1 \widetilde{B}_{m,p}(x;q,k)\,d_qx &= \int_0^1 \left[\begin{array}{c} m+p \\ k \end{array}\right]_q x^k (1-x)_q^{m+p-k}\,d_qx \\
&= \left[\begin{array}{c} m+p \\ k \end{array}\right]_q \int_0^1 x^k (1-x)_q^{m+p-k}\,d_qx.
\end{aligned}$$

If we use q-binomial expansion at the last equality, we get

$$\begin{aligned}
\int_0^1 \widetilde{B}_{m,p}(x;q,k)\,d_qx &= \left[\begin{array}{c} m+p \\ k \end{array}\right]_q \int_0^1 \sum_{l=0}^{m+p-k}\left[\begin{array}{c} m+p-k \\ l \end{array}\right]_q q^{\binom{l}{2}}(-1)^l x^{l+k} d_qx \\
&= \left[\begin{array}{c} m+p \\ k \end{array}\right]_q \sum_{l=0}^{m+p-k}\left[\begin{array}{c} m+p-k \\ l \end{array}\right]_q q^{\binom{l}{2}}(-1)^l \int_0^1 x^{l+k} d_qx \\
&= \left[\begin{array}{c} m+p \\ k \end{array}\right]_q \sum_{l=0}^{m+p-k}\left[\begin{array}{c} m+p-k \\ l \end{array}\right]_q (-1)^l q^{\binom{l}{2}}\frac{1}{[l+k+1]_q}.
\end{aligned}$$

Therefore, the proof is completed. □

Conclusion

In this paper, we obtained some results for q-Bernstein-Schurer Polynomials by using ordinary and q-calculus. This results might useful likely for further studies such as (p, q)-Bernstein-Schurer polynomials.

Acknowledgements

The present investigation was supported by the Gaziantep College Foundation, Gaziantep.

References

[1] H. Oru, G. M. Phillips, and P. J. Davis, A generalization of the Bernstein polynomials, Proceedings of the Edinburgh Mathematical society 42.2, 403-413, 1999.

[2] T. Kim, A note on q-Bernstein polynomials, Russian Journal of Mathematical Physics 18.1, 2011.

[3] Y. Simsek and M. Acikgoz, A New Generating Function of q- Bernstein-Type Polynomials and Their Interpolation Function, Abstract and Applied Analysis. Vol. 2010. Hindawi, 2010.

[4] Y. Simsek and M. Acikgoz, A New Generating Function of q- Bernstein-Type Polynomials and Their Interpolation Function, Abstract and Applied Analysis. Vol. 2010. Hindawi, 2010.

[5] R. Goldman, P. Simeonov and Y. Simsek, Generating functions for the q-Bernstein bases, SIAM Journal on Discrete Mathematics 28.3 (2014): 1009-1025, 2014.

[6] L. W. Han, Y. Chu and Z. Y. Qiu, Generalized Bzier curves and surfaces based on Lupa q-analogue of Bernstein operator, Journal of Computational and Applied Mathematics 261 (2014): 352-363, 2014.

[7] F. Schurer, Linear positive operators in approximation theory, Math. Inst. Techn. Univ. Delft Report, 1962.

[8] V. Kac, P. Cheung, Quantum Calculus, Universitext, Springer, 2002.

[9] C. V. Muraru, Note on q-Bernstein-Schurer operators, Stud. Univ. Babes-Bolyai Math 56.2 (2011): 489-495, 2011.

[1] Gaziantep College Foundation, Gaziantep TR-27090, Turkey.
E-mail : eagyuz86@gmail.com

Determining a Global Optimum of a Nonconvex Function in R^n Box

Fadila Leslous[1] , *Mohand Ouanes*[2]

Abstract

This paper presents a generalization of the proposed method for nonconvex functions, using the difference of convex functions algorithm and the minimum of the average of approximations of the function from the vertices of the box. This strategy has the advantage of giving in general a minimum to be situated in the attraction zone of the global minimum searched. After applying the difference of convex functions algorithm from this minimum we arrive, certainly, at the global minimum searched.

2010 Mathematics Subject Classifications : 90C26, 90C34, 90C25
Keywords: Optimization DC and DCA, Global optimization, Nonconvex optimization.

Introduction

In recent years, global optimization [1-4] has been the subject of several studies due to new theoretical results, strong demand in several fields including industrial applications, and the development of computing resources [5-7].

Global optimization did not inherit the easiness of the numerical techniques of local optimization. Indeed, the latter use for the most part descent directions, which makes it possible to converge naturally to a local minimum point[8-14].

Global optimization avoids staying at such a point as they must be escaped. This is why many approaches have been used in the attempt to solve problems. They mostly consist of finding a state of minimum and to stop only if it is the best (the global optimum). One of the most widely used methods is the Difference of Convex functions Algorithm , which is a descent method without a linear search (greatly favored by large dimensions). This method is applied with great success to numerous nonconvex optimization problems that rise in diverse fields of applied sciences such as: transportation logistics, telecommunications, finance, data mining, robotics,...
In this work we propose a generalization of method which is based on the decomposition of the function to be minimized into a difference of convex functions and the application of the difference of convex functions algorithm.
From a good initial point, the difference of convex functions algorithm furnishes a global minimum, we propose an approach to find a good initial point. For that we minimize the average of approximations of the function from the vertices of the box. This strategy has the advantage of giving generally a minimum to be located in the attraction zone of the global minimum searched. We apply the difference of convex functions algorithm from the minimum found and we arrive certainly to the global minimum searched.

Problem formulation

We consider the optimization difference of convex functions problem (DC) as follows:
$(P) \Longleftrightarrow \min\{f(x) = g(x) - h(x), x \in B \subset \mathbb{R}^n\}$
$B = \{x = (x_1, x_2...x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, ..., n\}$ with:
a_i and b_i Constants in $\mathbb{R}$.
f :$R^n \longrightarrow R$ nonconvex of class $\mathcal{C}^2$.
g :$R^n \longrightarrow R$ convex
h :$R^n \longrightarrow R$ convex
We want to solve the problem (P) by applying difference of convex functions algorithm (DCA) to the minimum of the average of approximations of the function from the vertices of the box B.

The principle of difference of convex functions algorithm (DCA)

Note that DCA works only with DC components g and h [5,6,14].
At the k-th iteration of DCA, h is replaced by its affine minorant
$h_k(x) = h(x^k) + \langle x - x^k, y^k \rangle$ in the neighborhood of x^k.
Knowing that h is a convex function and B is a box subset of $\mathbb{R}^n$, we have therefore $h(x) \geq h_k(x)$,$\forall x \in B$. As a result, $g(x) - [h(x^k) + \langle x - x^k, y^k \rangle] \geq g(x) - h(x), \forall x \in B$. That is to say, $g(x) - [h(x^k) + \langle x - x^k, y^k \rangle]$ is a majorant function of function f (x). Indeed, the surface of f^k can be imagined as a bowl being placed directly above the surface of f; Moreover, the two surfaces ar touching at point $(x^k, f(x^k))$.

The principle of the proposed method (MDCA)

From a good initial point the DCA furnishes a global minimizer[1,2,4].
In the case of a minimizing a nonconvex funtion of classe $\mathcal{C}^2$ defined in $\mathbb{R}^n$, The minimum found starting from a vertices of the box B will generally be different from that found starting from another vertices of the box B.
We propose to find a good initial point.Instead we want to minimize the average of approximations to f from the vertices $s_1, s_2, s_3, ..., s_{2^n}$ of the box B as follows:
$min\frac{1}{2}(\sum_{i=1}^{2^n} f(x; s_i))$ with:
$f(x; s_i) = g(x) - \nabla h(s_i)(x - s_i) - h(s_i)$
This strategy has the advantage of providing in general a minimum to be located in the attraction zone of the minimum global searched.

Properties of the proposed method MDCA

1- The MDCA constructs an initial point x^0 which is the solution of a convex problem, and a sequence $\{x^k\}$ generated by DCA, then it converges to an optimal solution of our problem.
2- It can be seen that MDCA has the option to skip certain neighborhoods of local minima, then arrives at a neighborhood of the global solution. we can understand that the performance of MDCA is the position of the initial point.

The algorithm of the proposed method (MDCA)

Step 0: k=0; n given; $s_1, s_2, s_3, ..., s_{2^n}$ given.
Step 1: $x^0 = min\frac{1}{2}(\sum_{i=1}^{2^n} f(x; s_i)); f(x; s_i) = g(x) - \nabla h(s_i)(x - s_i) - h(s_i)$.
Step 2: We search $y^k \in \partial h(x^k)$.
Step 3: We determine $x^{k+1} \in \partial g^*(y^k)$.

The application of the proposed method

Example 1.

$$(P) \Leftrightarrow \begin{cases} f(x,y) = y^2 + x - x^2 + y \longrightarrow Min \\ x \in [-3,+3], y \in [-3,+3] \end{cases}$$

$h(x, y) = x^2 - y$
$g(x, y) = y^2 + x$
$\nabla h(x, y) = (2x, -1)$
We aim to minimize the average of the approximations of f from s_1, s_2, s_3 *and* s_4, *with:*
$s_1 = (-3, -3), s_2 = (3, -3), s_3 = (3, 3), s_4 = (-3, 3)$.
$f(x; s_i) = g(x) - \nabla h(s_i)(X - s_i) - h(s_i), i = 1, 2, 3, 4$.
$f(x; s_1) = y^2+7x+y+9, f(x; s_2) = y^2-5x+y+9, f(x; s_3) = y^2-5x+y+9, f(x; s_4) = y^2 + 7x + y + 9$.
Step 0:
Solve the problem convex P′ following:
$(P\prime) \Longleftrightarrow \min \frac{1}{4}(f(x, s_1)+f(x, s_2)+f(x, s_3)+f(x, s_4))$. $(P\prime) \Longleftrightarrow \min(y^2+y+x+9)$.
$x = (-3, -\frac{1}{2})$. *is the solution of* $(P\prime)$
Step 1:
Application of DCA from $x = (-3, -\frac{1}{2})$.
$x = (-3, -\frac{1}{2})$ *is the optimal solution (global minimum) of Problem (P).*

Conclusion

This paper proposes the generalization of the MDCA method on an $\mathbb{R}$ box, using the DCA (difference of convex Algorithm) which concernes a particular class of optimization problems, namely: nonconvex problems.
The strategy of minimizing the average followed by the standard application of DCA has led to the production of the global minimum of function f, while the standard function fminbnd of MATLAB found a non global, local minimum. It now remains to test other examples to better evaluate the pertinence of this strategy, reinforcing the importance of DCA in solving nonconvex problems.

Acknowledgements

We would like to thank the anonymous reviewers for their future valuable suggestions and remarks .

References

[1] Le Thi, H.A., N.V. Vinh, N.V., Phan Din, T.: A Combined DCA and New Cutting Plane Techniques for Globally Solving Mixed Linear Programming, SIAM Conference on Optimization, Stockholm (2005)

[2] Horst, R., Hoang T.: Global Optimization : Deterministic Approches, 2nd revised edition, Springer, Berlin (1993)

[3] Nguyen, C.N., Le Thi, H.A., Phan Din. T.: A Branch and Bound Algorithm Based on DC Programming and DCA for Strategic Capacity Planning in supply Chain Design for a New Market Opportnity. Operations rechearch proceedings (2006)

[4] Chih-Sheng Chuang, Chi-Ming Chen.: Split proximal linearized algorithm and convergence theorems for the split DC program. Journal of Inequalities and Applications (2019)

[5] Messine, F., Lagouanelle, J.L.: Enclosure Methods for Multivariate Differentiable Functions and Application to Global Optimization. J. Universal Computer Science. 4, 589-603 (1998)

[6] Ratschek, H., Rokne, J.G.: New Computers Methods for Global Optimization, Wiley, New-York (1988)

[7] Vinko, T., Lagouanelle, J.L., Csendes, T.: A New Inclusion Function for Optimization: K ite- The One Dimensional Case. J. Global Optimization (2004)

[8] Horst, R., Pardalos, P.M., Romeijn, H.E.: Handbook of global optimization, Springer Publisher, Kluwer (1995)

[9] Huang, Z., Miao, X., Wang, P.:A revised cut-peak function method for box constrained continuous global optimization. J. Applied Mathematics and Computation. 194, 224-233 (2007)

[10] Liberti, L., Maculan, N.: Global optimization: from theory to implementation, Springer Publisher, (2006)

[11] Liu, C.X., Chi, G.: structural optimization using matlab. Engineering Systems and Design Lab, KAIST (2006)

[12] Nocedal, J., Wright, S.J.: Numerical optimization. Springer Series in Operations Research, Springer, (1999)

[13] Chih-Sheng Chuang, Chi-Ming Chen.: Hybrid proximal linearized algorithm for the split DC program in infinite dimensional real Hilbert spaces. Journal of Inequalities and Applications (2018)

[14] Ratschek, H., Rokne, J.: New computer methods for global optimization. Ellis Horwood Limited Market Cross House, Cooper Street, Chichester, West Sussex, PO19 1EB, England (1988)

[1]Department of Mathematics, Faculty of Science University of Mouloud Mammeri, Tizi Ouzou, Algeria., University of First Author

[2]Department of Mathematics, Faculty of Science University of Mouloud Mammeri, Tizi Ouzou, Algeria.

E-mail : fadila.leslous@yahoo.fr, ouanes_mohand@yahoo.fr

Semi-direct product of universal α-central extensions of Hom-Leibniz algebras

N. Pacheco Rego[1] ,

Abstract

We construct the endofunctor $\mathfrak{uce}_\alpha(-)$ between the α-perfect Hom-Leibniz algebras category, which assigns to an α-perfect Hom-Leibniz algebra its universal α-central extension. Moreover, we obtain some results concerning the lifting of automorphisms in an α-cover. Finally we analyse the compatibility of the semi-direct product of two α-perfect Hom-Leibniz algebras with the universal α-central extension .

2010 Mathematics Subject Classifications : 17A32, 16E40, 17A30
Keywords: universal (α)-central extension 1, Hom-action 2, Keyword semi-direct product

Introduction

Hom-Lie algebras were introduced in [12] as a generalization of Lie algebras, where the usual Jacobi identity is twisted by a linear self-map, that leads us to the so-called Hom-Jacobi identity. One of the main reasons of the introduction of Hom-Lie algebras was the construction of deformations of the Witt algebra, which is the Lie algebra of derivations on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$.

The investigation of several kinds of Hom-structures is in progress (for instance, see [2, 3, 4] and references given therein). In this context, during the last years many papers appeared dealing with the investigations of Hom-Lie structures. Naturally, the non-skew-symmetric version of Hom-Lie algebras, the so called Hom-Leibniz algebras, was considered as well (see [3, 8, 11, 13, 14, 15]). A Hom-Leibniz algebra is a triple $(L, [-,-], \alpha_L)$ consisting of a K-vector space L, a bilinear map $[-,-] : L \times L \to L$ and a homomorphism of K-vector spaces $\alpha_L : L \to L$ satisfying the Hom-Leibniz identity:

$$[\alpha_L(x), [y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)]$$

for all $x, y, z \in L$. When $\alpha_L = Id$, the definition of Leibniz algebra is recovered. If the bracket is skew-symmetric, then we recover the definition of Hom-Lie algebra [12].

Lie and Leibniz algebras have found important applications in Mathematics and Physics, in particular degenerations, contractions and deformations. The analysis of these properties in the Hom-Lie setting [10] have led to deal with universal central extensions.

In this paper, we have chosen to work with Hom-Leibniz algebras. This work is a continuation with the investigations on universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras initiated in [8], and a generalization of the work already begun in [7] to the framework of Hom-Leibniz algebras.

First at all, we construct the covariant right exact functor $\mathfrak{uce}_\alpha(-)$ of $\mathsf{Hom-Leib}^{\alpha-\mathrm{perf}}$ which assigns to an α-perfect Hom-Leibniz algebra its universal α-central extension (see Definition 3.2 in [8]). Then, we provide that under some conditions an automorphism can be lifted in an α-cover (a central extension $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ where $(L', \alpha_{L'})$ is α-perfect $(L' = [\alpha_{L'}(L'), \alpha_{L'}(L')])$. Finally we analyze the relationships between the universal α-central extension of the semi-direct product of two α-perfect Hom-Leibniz algebras, such that one of them Hom-acts over the other one, and the semi-direct product of the universal α-central extensions of both of them.

Main Results

Concerning the functorial properties of the universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras, we prove the following. For a homorphism of $\alpha-$perfect Hom-Leibniz algebras $f : (L', \alpha_{L'}) \to (L, \alpha_L)$, there exists a homomorphism of Hom-Leibniz algebras $\mathfrak{uce}_\alpha(f) : (\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}) \to (\mathfrak{uce}_\alpha(L), \overline{\alpha})$ such that the following diagram is commutative,

$$\begin{array}{ccc}
Ker\,(U_{\alpha'}) & & Ker\,(U_{\alpha}) \\
\downarrow & & \downarrow \\
(\mathfrak{uce}_\alpha(L'), \overline{\alpha'}) & \xrightarrow{\mathfrak{uce}_\alpha(f)} & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \\
{\scriptstyle U_{\alpha'}}\downarrow & & \downarrow{\scriptstyle U_{\alpha}} \\
(L', \alpha_{L'}) & \xrightarrow{f} & (L, \alpha_L)
\end{array}$$

From which one derives the existence of a covariant right exact functor $\mathfrak{uce}_\alpha$: $\mathsf{Hom-Leib}^{\alpha-\mathrm{perf}} \to \mathsf{Hom-Leib}^{\alpha-\mathrm{perf}}$ between the α-perfect Hom-Leibniz algebras category.
Here $\mathfrak{uce}_\alpha(L)$ denotes the quotient vector space $\frac{\alpha_L(L)\otimes\alpha_L(L)}{I_L}$, I_L denotes the vector subspace spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3]$, $\forall x_1, x_2, x_3 \in L$.

Then, we establish under what conditions an automorphism or a derivation can be lifted in an α-cover .
Namely, let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an $\alpha-$cover. For any $h \in Aut\,(L, \alpha_L)$, there exists a unique $\theta_h \in Aut\,(L', \alpha_{L'})$ such that the following diagram is commutative:

$$\begin{array}{ccc}
(L', \alpha_{L'}) & \xrightarrow{f} & (L, \alpha_L) \\
{\scriptstyle \theta_h}\downarrow & & \downarrow{\scriptstyle h} \\
(L', \alpha_{L'}) & \xrightarrow{f} & (L, \alpha_L)
\end{array}$$

if and only if the automorphism $\mathfrak{uce}_\alpha(h)$ of $(\mathfrak{uce}_\alpha\,(L)\,, \overline{\alpha})$ satisfies $\mathfrak{uce}_\alpha(h)\,(C) = C$. Moreover, the following group isomorphism holds,

$$\begin{array}{rcl}
\Theta : \{h \in Aut\,(L, \alpha_L) : \mathfrak{uce}_\alpha(h)\,(C) = C\} & \to & \{g \in Aut\,(L', \alpha_{L'}) : g\,(Ker(f)) = Ker(f)\} \\
h & \mapsto & \theta_h
\end{array}$$

Finally we analyze the relationships between the universal α-central extension of the semi-direct product of two α-perfect Hom-Leibniz algebras, such that one of them Hom-acts over the other one, and the semi-direct product of the universal α-central

extensions of both of them.
Namely for a split extension of α-perfect Hom-Leibniz algebras

$$0 \longrightarrow (M,\alpha_M) \xrightarrow{\ t\ } (G,\alpha_G) \underset{s}{\overset{p}{\rightleftarrows}} (Q, Id_Q) \longrightarrow 0$$

where the induced Hom-action of (Q, Id_Q) on (M,α_M) is given by $q \centerdot m = [s(q), t(m)]$ and $m \centerdot q = [t(m), s(q)], q \in Q, m \in M$ is symmetric. From the functorial properties of $\mathfrak{uce}_\alpha(-)$, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & Ker(U_\alpha^M) & & Ker(U_\alpha^G) & & HL_2(Q) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) & \xrightarrow{\tau} & (\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & (\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}) & & \\
 & & \downarrow{\scriptstyle U_\alpha^M} & & \downarrow{\scriptstyle U_\alpha^G} & & \downarrow{\scriptstyle u_Q} & & \\
0 & \longrightarrow & (M,\alpha_M) & \xrightarrow{t} & (G,\alpha_G) & \underset{s}{\overset{p}{\rightleftarrows}} & (Q, Id_Q) & \longrightarrow & 0
\end{array}$$

Here $(uce(Q); Id_{uce(Q)})$ denotes the universal central extension of $(Q; Id_Q)$, $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G})$ denotes the universal α-central extension of (G, α_G), $\tau = \mathfrak{uce}_\alpha(t)$,
$\pi = \mathfrak{uce}_\alpha(p)$ and $\sigma = \mathfrak{uce}_\alpha(s)$.
The main results establish that the following statements hold:

1. $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = \tau\left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right) \rtimes \sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$.

2. $\sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right) \cong \left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$.

3. $\left(Ker(U_\alpha^G), \overline{\alpha_G}_{|}\right) \cong \tau\left(Ker(U_\alpha^M), \overline{\alpha_M}_{|}\right) \oplus \sigma\left(HL_2(Q), Id_{\mathfrak{uce}(Q)_{|}}\right)$.

4. The homomorphism of Hom-Leibniz algebras

$$\Phi : \left(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \to (G, \alpha_G)$$

given by $\Phi\left(\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\}\right) = \left(t\left[\alpha_M(m_1), \alpha_M(m_2)\right], s\left[q_1, q_2\right]\right)$ is an epimorphism that makes commutative the following diagram, and its kernel is $Ker(U_\alpha^M) \oplus HL_2(Q)$.

$$\begin{array}{ccc}
\left(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) & \xrightarrow{\tau \rtimes \sigma} & \left(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}\right) \\
 & \searrow^{\Phi} \quad \swarrow_{U_\alpha^G} & \\
 & (G, \alpha_G) &
\end{array}$$

5. $Ker(\tau \rtimes \sigma) \cong \mathfrak{uce}(Q) \centerdot Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \centerdot \mathfrak{uce}(Q)$

and that the following statements are equivalent:

a) $\Phi = (t \circ U_\alpha^M) \rtimes (s \circ u_Q) : \left(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}_\alpha(Q)}\right) \to (G, \alpha_G)$ is a central extension, hence is an α-cover.

b) The Hom-action of $(\mathfrak{uce}(Q), Id_Q)$ on $(Ker(U_\alpha^M), \overline{\alpha_M}_{|})$ is trivial.

c) $\tau \rtimes \sigma$ is an isomorphism. Consequently $\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q)$ is the universal α-central extension of (G, α_G).

d) τ is injective.

In particular, for the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, Id_Q)$ the following isomorphism holds:

$$\left(\mathfrak{uce}_\alpha(M \times Q), \overline{\alpha_M \times Id_Q}\right) \cong \left(\mathfrak{uce}_\alpha(M) \times \mathfrak{uce}(Q), \overline{\alpha_M} \times Id_{\mathfrak{uce}(Q)}\right)$$

References

[1] J. T. Hartwing, D. Larson and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314–361.

[2] F. Ammar and A. Makhlouf, *Hom-Lie superalgebras and Hom-Lie admisible superalgebras*, J. Algebra **34** (7) (2010), 1513–1528.

[3] F. Ammar, S. Mabrouk and A. Makhlouf, *Representations and cohomology of n-ary multiplicative Hom-Nambu-Lie algebras*, J. Geom. Phys. **61** (10) (2011). 1898–1913.

[4] J. Arnlind, A. Makhlouf and S. Silvestrov, *Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras*, J. Math. Phys. **51** (4) (2010), 11 pp.

[5] H. Ataguema, A. Makhlouf and S. Silvestrov, *Generalization of n-ary Nambu algebras and beyond*, J. Math. Phys. **50** (8) (2009), 15 pp.

[6] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra **39** (6) (2011), 2216–2240.

[7] J. M. Casas and N. Corral, *On universal central extensions of Leibniz algebras*, Comm. Algebra **37** (6) (2009), 2104–2120.

[8] J. M. Casas, M. A. Insua and N. Pacheco Rego, *On universal central extensions of Hom-Leibniz algebras*, J. Algebra Appl. **13** (8) (2014), 22 pp.

[9] J. M. Casas, A. Kh. Khudoyberdiyev, M. Ladra and B. A. Omirov, *On the degenerations of solvable Leibniz algebras*, Linear Algebra Appl. **439** (2) (2013), 472–487.

[10] Y. S. Cheng and Y. C. Su, *Quantum deformations of the Heisenberg-Virasoro algebra*, Algebra Colloq. **20** (2) (2013), 299–308.

[11] Y. S. Cheng and Y. C. Su, *(Co)homology and universal central extensions of Hom-Leibniz algebras*, Acta Math. Sin. Vol. **27** (5) (2011), 813–830.

[12] J. T. Hartwing, D. Larson and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295** (2006), 314–361.

[13] A. Makhlouf and S. Silvestrov, *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. **22** (4) (2010), 715–739.

[14] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2) (2008), 51–64.

[15] D. Yau, *Hom-algebras as deformations and homology*, arXiv: 0712.3515 (2007).

[1](3)IPCA, Science Department, Campus do IPCA, Lugar do Aldo 4750-810 Vila Frescainha, S. Martinho, Barcelos, Portugal

E-mail : nrego@ipca.pt

Connectedness of Graphs and Omega Invariant

Ismail Naci Cangul[1], *Aysun Yurttas Gunes*[2], *Muge Togan*[3], *Sadik Delen*[4]

Abstract

Two of the authors recently defined a new graph invariant denoted by omega for a given degree sequence and determined several properties of the realizations of this degree sequence in a series of papers. In this paper, the authors study the connectedness and cyclicness properties of graphs by means of omega.

2010 Mathematics Subject Classifications: 05C07, 05C10, 05C30
Keywords: omega invariant, degree sequence, connectedness, cyclic graph, acyclic graph

Introduction

Let $G = (V, E)$ be a graph of order n and size m. The degree of a vertex v is denoted by d_v. The biggest vertex degree is denoted by Δ. A graph is called connected when there is a path between every pair of vertices. The degree sequence $DS(G)$ of a graph G is a non-decreasing sequence of non-negative integers which are the degrees of the vertices of G. Written with multiplicities, a degree sequence in general is written as $DS(G) = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$, where some of a_i's could be zero. Let $D = \{d_1, d_2, d_3, \cdots, \Delta\}$ be a set of non-decreasing non-negative integers. We say that a graph G is a realization of the set D if the degree sequence of G is equal to D. In such a case, we also say that D is realizable. In [1], a new topologic graph invariant denoted by omega was defined and some fundamental properties are studied:

Definition 1. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a set which also is the degree sequence of a graph G. The $\Omega(G)$ of the graph G is defined only in terms of the degree sequence as*

$$\Omega(G) = a_3 + 2a_4 + 3a_5 + \cdots + (\Delta - 2)a_\Delta - a_1 = \sum_{i=1}^{\Delta}(i-2)a_i.$$

After its definition, there is a series of papers studying several properties of this new invariant including [2] where some extremal problems are solved for the number of components and loops amongst all realizations of a given degree sequence; [3] where the effect of vertex and edge deletion on omega are studied; [4] where omega is used to study cyclicness of these realizations; [5] where some existing relations on the matching number, rank and nullity are restated in a more compact form by means of omega and also some new results are given and [6] where the connectedness of the realizations is determined according to omega. For the properties of omega used in this paper, see above references.

Main Results

The following is one of the main results on Ω:

Theorem 2. *[6] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a degree sequence and let G be a realization of D. If $\Omega(D) \leq -4$, then G is a disconnected graph. Equivalently, if $\Omega(D) \leq -4$, then G cannot be drawn as a connected graph.*

Here, note that when $\Omega(G) \leq -4$, G could be both cyclic or acyclic. We can use this result to decide whether a given degree sequence with Ω at most -4 is realizable as a disconnected graph only. Equivalently, if G is a connected graph, Theorem 2 guarantees that $\Omega(G) \geq -2$. We now prove the following useful result about the numbers of cyclic and acyclic components:

Theorem 3. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a degree sequence and let G be a realization of D. Let $r(G)$ denote the number of closed regions in G and let c_a and c_c denote the numbers of acyclic and cyclic components of G, respectively. If $\Omega(D) = -2k \leq -4$ where k is a positive integer, then the following properties hold:*

i) *$c_a \geq k$;*

ii) *if $c_c = 0$, then $c_a = k$;*

iii) *if $c_c = 1$, then $c_a = r(C_1) - 1 + k$ where C_1 is the unique cyclic component;*

iv) *if $c_c \geq 2$, then $c_a = \sum r(C_i) - c_c + k$ where C_i are the cyclic components of the realization.*

Proof.

i) Recall that every acyclic component has $\Omega = -2$ and every cyclic component has $\Omega \geq 0$. So if $\Omega(D) = -2k \leq -4$, then the number of -2's, which is the number of acyclic components is exactly k if all components are acyclic, or is $\geq k$ if there is at least one cyclic component. So the claim is proven.

ii) Let $c_c = 0$. That is, there is no cyclic component and hence all components are acyclic. As each acyclic component c_i has $\Omega(C_i) = -2$, the number c_a of such components would be $\Omega(G)/-2 = -2k/-2 = k$, giving the result.

iii) Let $c_c = 1$. Let the unique cyclic component be C_1. As we have $c_a = \frac{\Omega(C_1)-\Omega(G)}{2}$. Then by Theorem 3.1 in [1], we have $c_a = \frac{2r(C_1)-2+2k}{2}$ giving the result.

iv) Let $c_c \geq 2$. Let these cyclic components be denoted by C_i's. Then

$$c_a = \frac{\sum \Omega(C_i) - \Omega(G)}{2} = \frac{\sum (2r(C_i) - 2) + 2k}{2} = \sum r(C_i) - c_c + k.$$

□

When $\Omega(D) \geq -2$, we cannot decide about the connectedness of a realization G of D that easily. G could be connected or disconnected. We can collect all our findings on the connectedness and the structure of the given graph G according to $\Omega(G)$ in the following result which is obvious by Theorem 2 and the results in [1, 6]:

Theorem 4. *Let D be a realizable degree sequence, let c and t denote the number of components and the number of acyclic components of a realization G, respectively, and let C_i and A_i denote the cyclic and acyclic components of a realization of D, respectively. Recall that $\Omega(A_i) = -2$. Let $\Omega(C_i) = k_i$. Then*

i) *If $\Omega(D) \leq -4$, then all realizations of D are disconnected and*

$$G = \left(\cup_{i=1}^{c-t} C_i\right) \cup \left(\cup_{i=1}^{t} A_i\right).$$

Also $\Omega(G) = 2(k_1 + k_2 + \cdots + k_{c-t}) - 2t \leq -4$.

ii) *If $\Omega(D) = -2$, then a realization G of D could be connected or disconnected. If G is connected, then it is acyclic and $G = A_1$. If G is disconnected, then*

$$G = \left(\cup_{i=1}^{c-t} C_i\right) \cup \left(\cup_{i=1}^{t} A_i\right).$$

Also $\Omega(G) = 2(k_1 + k_2 + \cdots + k_{c-t}) - 2t = -2$.

iii) *If $\Omega(D) \geq 0$, then all realizations are cyclic and a realization G of D could be connected or disconnected. If G is connected, then it is cyclic and $G = C_1$. If G is disconnected, then again G is cyclic and*

$$G = \left(\cup_{i=1}^{c-t} C_i\right) \cup \left(\cup_{i=1}^{t} A_i\right).$$

Also $\Omega(G) = 2(k_1 + k_2 + \cdots + k_{c-t}) - 2t \geq 0$.

In particular, when $\Omega(D) = 0$, if G is connected, then it is unicyclic, and if G is disconnected, then the number of acyclic components of G is equal to half of the sum of k_i's.

By Theorems 2 and 4, we can give the following result:

Corollary 5.

i) *If all components of a graph G are cyclic, then $\Omega(G) \geq 0$.*

ii) *If all components of a graph G are acyclic, then $\Omega(G) \leq -2$.*

iii) *If the components of a graph G are both cyclic and acyclic, then $\Omega(G)$ could be any even integer.*

Proof.

i) Let all components G_i of G be cyclic. As $\Omega(G_i) \geq 0$, $\Omega(G) \geq 0$ by the additivity of Ω.

ii) Let all components G_i of G be acyclic. As $\Omega(G_i) = -2$, $\Omega(G) = -2c$ giving the result.

iii) Let c_c of the c components of G be cyclic and c_a of the c components be cyclic. By the reasoning in the proofs of **i)** and **ii)**, we have $\Omega(G) = (-2) \cdot c_a + (g) \cdot c_c$ where g is an even non-negative number. Hence the result follows. □

In case of $\Omega(D) = -2$, we have

Theorem 6. *Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \cdots, \Delta^{(a_\Delta)}\}$ be a degree sequence and let G be a realization of D. Let $r(G)$ denote the number of closed regions in G. If $\Omega(D) = -2$, then the following properties hold:*

i) *$c_a \geq 1$;*

ii) *if $c_c = 0$, then $c_a = 1$;*

iii) *if $c_c = 1$, then $c_a = 1 + \Omega(C_1)/2 = r(C_1)$ where C_1 is the unique cyclic component;*

iv) *if $c_c \geq 2$, then*

$$c_a = \frac{\sum_{i=1}^{c_c} \Omega(C_i)}{2} + 1 = \sum_{i=1}^{c_c} (r(C_i) - 1) + 1 = \sum_{i=1}^{c_c} r(C_i) - c_c + 1.$$

References

[1] S. Delen, I. N. Cangul, A New Graph Invariant, Turkish Journal of Analysis and Number Theory, 6 (1) (2018), 30-33.

[2] S. Delen, I. N. Cangul, Extremal Problems on Components and Loops in Graphs, Acta Mathematica Sinica, English Series, 35 (2) (2019), 161-171.

[3] S. Delen, M. Togan, A. Yurttas Gunes, U. Ana, I. N. Cangul, The Effect of Edge and Vertex Deletion on Omega Invariant, Applicable Analysis and Discrete Mathematics, Special Issue, vol. II.

[4] S. Delen, A. Yurttas Gunes, M. Togan, I. N. Cangul, Omega Invariant of Graphs and Cyclicness, Applied Sciences, 21 (2019), 91-95.

[5] M. S. Oz, I. N. Cangul, Matching Number in Relation with Maximal-Minimal Nullity Conditions and Cyclomatic Number by Coefficient Relations, Advanced Studies in Contemporary Mathematics, 30 (2020).

[6] U. Sanli, F. Celik, S. Delen, I. N. Cangul, Connectedness Criteria for Graphs by Means of Omega Invariant, FILOMAT 34 (2020).

[1,2,3,4]Mathematics Department, Bursa Uludag University, Gorukle 16059 Bursa-Turkey

E-mail : cangul@uludag.edu.tr, ayurttas@uludag.edu.tr, capkinm@uludag.edu.tr, sd.mr.math@gmail.com

φ(Ric)-vector fields on almost pseudo Ricci symmetric manifolds

Hülya Bağdatli Yilmaz[1] , *S.Aynur Uysal*[2] , *Bahar Kirik*[3]

Abstract

In this paper, we consider an $n-$dimensional $(n > 3)$ almost pseudo Ricci symmetric manifolds denoted by $A(PRS)_n$ and examine φ(Ric)-vector fields on these manifolds.

2010 Mathematics Subject Classifications : 53B05, 53B20
Keywords: Almost pseudo Ricci Symmetric, R-harmonic, φ(Ric)-vector field

Introduction

In the literature, there are two different notions of pseudosymmetric manifolds that were introduced by M. C. Chaki [1] and R. Deszcz [2] and such manifolds have been studied by many authors.

Firstly, we mention about some basic concepts related to this investigation by giving some definitions.

Definition 1 ([3]). *A non-flat n-dimensional Riemannian manifold (M, g), $(n > 3)$ is called a pseudo Ricci symmetric manifold if the Ricci tensor S of type (0,2) is non-zero and satisfies the condition*

$$(\nabla_X S)(Y, Z) = 2\pi(X)S(Y, Z) + \pi(Y)S(X, Z) + \pi(Z)S(Y, X) \tag{1}$$

where ∇ denotes the Levi - Civita connection and A is a non-zero 1-forms such that $g(X, \rho) = \pi(X)$ for all vector fields X; ρ being the vector field corresponding to the associated 1-forms π. If $\pi = 0$ then the manifold is called Ricci symmetric.

Pseudo Ricci symmetric manifolds are generalized by Chaki and Kawaguchi in 2007, [4]. The authors defined the notion of almost pseudo Ricci symmetric manifolds as follows:

Definition 2. *A non-flat n-dimensional Riemannian (or Semi - Riemannian) manifold (M, g) $(n > 3)$ is called an almost pseudo Ricci symmetric manifold [4] if the Ricci tensor S of type (0,2) is not identically zero and satisfies the condition*

$$(\nabla_X S)(Y, Z) = [\pi(X) + \omega(X)]S(Y, Z) + \pi(Y)S(X, Z) + \pi(Z)S(Y, X) \tag{2}$$

where ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g and π, ω are non-vanishing 1-forms such that $g(X, \rho) = \pi(X)$ and $g(X, \nu) = \omega(X)$ for all X and ρ, ν are called the basic vector fields of the manifold. The 1-forms π and ω are called the associated 1-forms of the manifold.

If $\omega = \pi$, then an almost pseudo Ricci symmetric manifold reduces to a pseudo Ricci symmetric manifold. Such a manifold will be denoted by $A(PRS)_n$.

$\varphi(Ric)$-vector fields on $A(PRS)_n$

A $\varphi(Ric)$-vector field is a vector field on an n-dimensional Riemannian manifold (M, g) and Levi-Civita connection ∇, which satisfies the condition

$$\nabla\varphi = \mu Ric \tag{3}$$

where μ is a constant and Ric is the Ricci tensor, [5]. When (M, g) is an Einstein manifold, the vector field φ is concircular. If $\mu \neq 0$, then we call that the vector field φ is proper $\varphi(Ric)$-vector field. Moreover, when $\mu = 0$, the vector field φ is covariantly constant.

It was shown that [5], Riemannian spaces (M, g) with a $\varphi(Ric)$-vector field of constant length have constant scalar curvature. In [6], it is proved that the converse is also true.

This section provides an investigation of $\varphi(Ric)$-vector fields on $A(PRS)_n$.

Theorem 3. *Let each of the basic vector fields π_i and ω_i of $A(PRS)_n$ be $\varphi(Ric)$-vector fields of constant length. Then the following condition is satisfied:*

$$\nabla_{[k}\nabla_{i]}S_{lp} + \nabla_{[i}\nabla_{l]}S_{pk} + \nabla_{[l}\nabla_{p]}S_{ik} + \nabla_{[p}\nabla_{k]}S_{li} = 0$$

Proof. Let us write the equation (2) in the local coordinate. Then we have

$$\nabla_i S_{kl} = (\pi_i + \omega_i)S_{kl} + \pi_k S_{il} + \pi_l S_{ki}, \tag{4}$$

taking the covariant derivative of the equation (4) and again using the equation (4), we get

$$\begin{aligned} \nabla_p\nabla_i S_{kl} = (\nabla_p\pi_i + \nabla_p\omega_i)S_{kl} + (\pi_i + \omega_i)\left[(\pi_p + \omega_p)S_{kl} + \pi_k S_{pl} + \pi_l S_{kp}\right] \\ + (\nabla_p\pi_k)\, S_{il} + \pi_k\left[(\pi_p + \omega_p)S_{il} + \pi_i S_{pl} + \pi_l S_{ip}\right] \\ + (\nabla_p\pi_l)\, S_{ki} + \pi_l\left[(\pi_p + \omega_p)S_{ki} + \pi_k S_{pi} + \pi_i S_{kp}\right]. \end{aligned} \tag{5}$$

Let us now assume that each of the basic vector fields π_i and ω_i of $A(PRS)_n$ is $\varphi(Ric)$-vector fields of constant length. From the equation (3), there are the following relations

$$\nabla_p\pi_i = \lambda S_{pi} \quad \text{and} \quad \nabla_p\omega_i = \nu S_{pi} \tag{6}$$

where λ and ν are constants, and we also know that since the basic vector fields π_i and ω_i of $A(PRS)_n$ is $\varphi(Ric)-$vector fields of constant length, the following relations are satisfied:

$$S_{pi}\pi^i = 0 \quad \text{and } S_{pi}\omega^i = 0. \tag{7}$$

Let us rewrite the equation (5) by replacing indices i with p and subtract it from the equation (5). So we get

$$\nabla_p\nabla_i S_{kl} - \nabla_i\nabla_p S_{kl} = \pi_k\pi_i S_{pl} + \pi_i\pi_l S_{kp} - \pi_k\pi_p S_{il} - \pi_l\pi_p S_{ki}. \tag{8}$$

By writing the equation (8) for all cyclic permutations of indices p, i, k, l and summing them up, we reach

$$\nabla_{[k}\nabla_{i]}S_{lp} + \nabla_{[i}\nabla_{l]}S_{pk} + \nabla_{[l}\nabla_{p]}S_{ik} + \nabla_{[p}\nabla_{k]}S_{li} = 0. \tag{9}$$

□

Theorem 4. *Let only the basic vector field π_i of $A(PRS)_n$ be a $\varphi(Ric)$-vector field of constant length. Then, the basic vector field ω_i is closed if and only if the equation (9) holds.*

Proof. Suppose that only the basic vector field π_i of $A(PRS)_n$ is a $\varphi(Ric)$-vector field of constant length. Then, from the equation (5), we get

$$\nabla_p\nabla_i S_{kl} - \nabla_i\nabla_p S_{kl} = (\nabla_p\omega_i - \nabla_i\omega_p)\, S_{kl} + \pi_k\pi_i S_{pl} + \pi_i\pi_l S_{kp} - \pi_k\pi_p S_{il} - \pi_l\pi_p S_{ki}. \tag{10}$$

By using the equation (10), let us again contruct the equation (9). Then the right-hand side of this equation becomes as follows:

$$= (\nabla_k\omega_i - \nabla_i\omega_k)\, S_{lp} + (\nabla_i\omega_l - \nabla_l\omega_i)\, S_{pk} + (\nabla_l\omega_p - \nabla_p\omega_l)\, S_{ik} + (\nabla_p\omega_k - \nabla_k\omega_p)\, S_{li}. \tag{11}$$

Therefore, it is clear that when the vector field ω_i is closed, then we have the equation (9).

The converse is apparent, by the same equation. □

Thus, from the above theorems, the following is true:

Theorem 5. *Let the basic vector field π_i of $A(PRS)_n$ be a $\varphi(Ric)$-vector field of constant length. Then, the equation (9) holds if either the basic vector field ω_i is closed or it is a $\varphi(Ric)$-vector field.*

Theorem 6. *Let each of the basic vector fields π_i and ω_i of $A(PRS)_n$ be $\varphi(Ric)$-vector fields of constant length. Then, the scalar curvature of $A(PRS)_n$ is zero.*

Proof. Contracting the equation (4) with g^{il}. So we get

$$\nabla_i S^i_l = (\pi^l + \omega^l) S_{kl} + \pi_k r + \pi^i S_{ki}. \tag{12}$$

From the equation (7), the first and last terms of the right-hand side of the above equation vanish. Let us now remember $\nabla_i S^i_l = \frac{1}{2}\nabla_l r$, and hence since the basic vector fields are $\varphi(Ric)$-vector fields of constant length, the scalar curvature is constant. Thus, the left-hand side of the above equation is equal to zero. Therefore we arrive $\pi_k r = 0$.

From $\pi_k \neq 0$, it follows that $r = 0$. □

$\varphi(Ric)$-vector fields on conformally flat $A(PRS)_n$

In this section, we consider the case when $A(PRS)_n$ is conformally flat. In a conformally flat space, it is known that the following equation holds:

$$\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)}[(\nabla_j r) g_{kl} - (\nabla_k r) g_{jl}]. \tag{13}$$

Definition 7. *A symmetric tensor field T of type (0,2) on a Riemannian manifold (M, g) is said to be a Codazzi tensor if it satisfies the condition*

$$(\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z); \quad X, Y, Z \in \chi(M). \tag{14}$$

Definition 8. *A Riemannian manifold M is called $R-$harmonic if its curvature tensor field R is harmonic [7].*

Theorem 9. *Let $A(PRS)_n$ be conformally flat and let the basic vector field π_i of $A(PRS)_n$ be a $\varphi(Ric)$-vector field of constant length. Then the following conditions hold:*

i. $A(PRS)_n$ is $R-$harmonic.

ii. the Ricci tensor S of $A(PRS)_n$ is of Codazzi type.

Proof. We know that if the basic vector field π_i is a $\varphi(Ric)$-vector field of constant length, then the scalar curvature is constant. So, using the equation (13), we have

$$\nabla_j S_{kl} - \nabla_k S_{jl} = 0. \tag{15}$$

By considering

$$\mathrm{div}(R^i_{jkl}) = \nabla_j S_{kl} - \nabla_k S_{jl}, \tag{16}$$

and Definition 3.1, the equation (13) implies that both $\mathrm{div}(R^i_{jkl}) = 0$ and the Ricci tensor is of Codazzi type. Further, by using the information which a curvature tensor field R is harmonic if $\mathrm{div}(R^i_{jkl}) = 0$, the proof is completed. □

Theorem 10. *Let $A(PRS)_n$ be conformally flat and let the basic vector field π_i of $A(PRS)_n$ be a $\varphi(Ric)$-vector field of constant length. Then the other basic vector field ω is the null vector.*

Proof. Again considering the equation (4) and replacing indices j with k, we reach $\nabla_j S_{kl} - \nabla_k S_{jl} = \omega_j S_{kl} - \omega_k S_{jl}$. From (4), it follows that
$\omega_j S_{kl} - \omega_k S_{jl} = 0$.
This equation implies that ω is the null vector. □

References

[1] M.C.Chaki, On pseudo symmetric manifolds, Analele Sti. Ale Univ. 'AL. I. CUZA'din Iasi. 33 (1987), 53–58.

[2] R.Deszcz, On pseudosymmetric spaces, Bull. Belg. Math. Soc., Ser. A 44 (1992), 1–34.

[3] M.C.Chaki, On pseudo Ricci symmetric manifolds, Bulgar. J. Phys. 15 (1988), 526–531.

[4] M.C.Chaki, T.Kawaguchi, On almost pseudo Ricci symmetric manifolds, Tensor N.S. 68 (2007), 10–14.

[5] I.Hinterleitner, V.A.Kiosak, φ(Ric)-vector fields in Riemannian spaces, Archivum Mathematicum 44(5) (2008), 385–390.

[6] B.Kırık, F.Ö.Zengin, Generalized quasi-Einstein manifolds admitting special vector fields, Acta Math. Academia Paedagogicae Nyiregyhaziensis 31(1) (2015), 61–69.

[7] K.Arslan, C.Murathan, C.Özgür and A.Yıldız, On contact metric R-harmonic manifolds, Balkan.J. Geo. and Its App. 5 (2000), 1–6.

[1,3]Marmara University, Faculty of Arts and Sciences, Department of Mathematics, İstanbul, Turkey
[2]Doğuş University, Faculty of Arts and Sciences, Department of Mathematics, İstanbul, Turkey

E-mail : hbagdatli@marmara@edu.tr, auysal@dogus.edu.tr, bahar.kirik@marmara.edu.tr

Optimal Curve Fitting Model Using Nature Inspired Optimization Algorithm

Mitat Uysal [1] , *M.Ozan Uysal* [2]

Abstract

In this study, we propose a new model for Bezier curve fitting problem based on a new nature inspired heuristic optimization algorithm called as Migration Birds Optimization (MBO). The control points of the Bezier curve are generated randomly in the initial phase of the MBO. The randomly generated control points are used for calculating the Bezier curve point. The required fitness function of the MBO algorithm is the sum of error values in each point. The goal of the MBO is to find the minimum value of the error function. The new found MBO algorithm (E.Duman, M.Uysal, A.F.Alkaya) obtains better results than the other heuristic algorithms like Simulated Annealing and Tabu Search for finding the optimal values of the fitness function.

Keywords: curve fitting, Bezier curves, optimization, natural inspired algorithms, migrating birds optimization

Introduction

Computer graphics and computational geometry are two very important and popular research area. New developing topics like artificial intelligence and machine learning are applied to above mentioned areas. Especially bio-inspired meta-heuristic optimization algorithms or more generally nature inspired optimization algorithms play a very important role in curve fitting problems. In this study, we propose a new model for Bezier curve fitting problem based on a new nature inspired heuristic optimization algorithm called as Migration Birds Optimization (MBO).

Related Works

Z.Li and C.Li [1] developed a hybrid model to forecast windspeed based on Particle Swarm Optimization (PSO) and Support Vector Machine (SVM). Results indicated that the Morlet+RBF combined kernel function is considerably effective in enchancing the forecasting accuracy. W.V.Loock et al.[2] developed a convex optimization model to curve fitting with B-Splines. They presented a framework for approximating data with smooth splines.

Y.Jixin et al.[3] applied the evolutionary algorithm in B-Spline Curve Fitting problem. H.Ni et al.[4] studied on moving least square curve and surface fitting with interpolation conditions. They presented a method for moving least square curve and surface fitting with interpolation conditions. The main advantage of this model is the low degree of fitting function. A.Galvez and A.Iglesias [5] studied on the Efficient Particle Swarm Optimization Approach for Data Fitting with free knot B-Splines. The method in this paper applies Particle Swarm Optimization (PSO) to compute an appropriate location knots automatically.

M.J.P.Corpa et al.[6] tried to fit the experimental chemical data using bio-inspired optimization algorithm. They applied various optimization heuristics to the experimental data. As the results of the study, the Cooperative Convolution algorithm (CC) and Evolutionary Programming algorithm (EP) outperform the results of PSO (Particle Swarm Optimization) and VNS (Variable Neighborhood Search) heuristic approaches. Z.Wu et al.[7] studied on fitting scattered data points with ball B-spline curves using Particle Swarm Optimization.

J.Ding et al.[8] studied on "A Hybrid Particle Swarm Optimization-Cuckoo Search Algorithm and Its Engineering Applications". M.A.Ismail et al.[9] developed a model on "A Hybrid of Optimization Method for Multi-objective Constraint Optimization of Biochemical System Production". The effectiveness of the proposed method was evaluated using two benchmark biochemical systems and the experimental results showed that the proposed method was able to generate the highest results compare to other existing works.[9]

Parametric Bezier Curve

The parametric Bezier curve at parameter t can be defined as below:

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t) \quad 0 \leq t \leq 1 \tag{1}$$

[10] where the n is degree of the curve, $J_{n,i}(t)$ is the Bernstein polynomial basis function and t is the parameter.

Bernstein Basis Function

Bernstein Basis Function is defined as below :

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i} \tag{2}$$

$J_{n,i}(t)$ is the $nth - order$ Bernstein basis function. t is defined as below:[10]

$$t_i = t_{i-1} + \frac{|x_i - x_{i-1}|}{\sum_{i=1}^{n} |x_i - x_{i-1}|} \quad i = 1, 2, ...n \tag{3}$$

Bezier Curve Fitting and Fitness Function of The Optimization Problem

Curve fitting problem focuses on two different case:
I. To find a set of control points that interpolates a sequence of data points
II. To find a set of control points that approximates a sequence of data points
In this study our problem is the second case. To determine a set of control points that approximates a sequence of data points with minimum error. This last situation is an optimization problem.

Fitness Function

Approximation of function can be done by minimizing the difference between the determined curve and the given sequence of points :

$$F(p_0, p_1, .., p_n) = \sum_{k=1}^{m-1} |d_k - P(u_k)|^2 \tag{4}$$

where 0< k < m , n+1 is the number of control points and d is the sequence of points with m+1 elements. $|d_k - P(u_k)|$ is distance between one point d_k and Bezier curve. Points d_0 and d_m are not added in the sum because the first and last control points are always interpolated.[11]

Migrating Birds Optimization (MBO) Algorithm

MBO is a new nature inspired meta-heuristic approach based on the V flight formation of the migrating birds which is proven to be an effective formation in energy saving. Its performance is tested on quadratic assignment problem instances arising from a real life problem and very good results are obtained. The quality of the solutions are better than simulated annealing, tabu search, genetic algorithm, scatter search, particle swarm optimization, differential evolution and guided evolutionary simulated annealing approaches.[12] More details can be obtained from [12].

Main Results

Proposed Model

Figure 1 shows the applying the MBO algorithm to Bezier Curve Fitting problem. Until the required convergency is obtaines in each point, MBO algorithm continues. After completing the algorithm, optimal control points are produced.

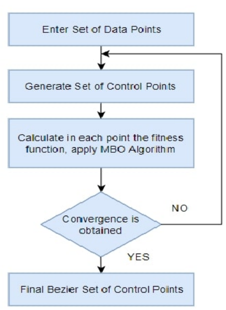


Figure 1: Bezier Curve Fitting using MBO algorithm

Experimental Results

In our study, 2D data points are used. A Java code is developed for the proposed model. Figure 2 shows the results of the proposed model in 2 dimensional spaces.

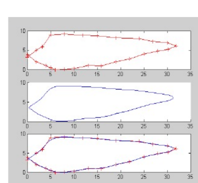


Figure 2: The result of Bezier Curve Fitting using MBO algorithm

Conclusion and Future Work

In this study, MBO Algorithm has been used for Bezier Curve fitting problem. MBO Algorithm finds the optimal set of control points that yield curve that best fit to the given data points. For future work, we will try the other curve fitting and surface fitting methods with MBO and the other famous meta-heuristic optimization algorithms.

References

[1] Z.Li, C.Li, Selection of Kernel Function for Least Squares Support Vector Machines in Downburst Wind Speed Forecasting, IEEE XPlore (2018), 337–341.

[2] W.V.Loock, G.Pipeleers, J.D.Schutter, J.Swevers, A Convex Optimization Approach to Curve Fitting with B-Splines, IFAC (2011), 2290–2295.

[3] Y.Jixin, L.Fang, T.Xuecheng, W.Xuejun, The Application of Evolutionary Algorithm in B-Spline Curve Fitting, IEEE Xplore (2012), 2809–2812.

[4] H.Ni, Z.Li, H.Song, Moving Least Square Curve and Surface Fitting with Interpolation Conditions, IEEE XPlore (2010), V13-300–V13-304.

[5] A.Galvez, A.Iglesias, Efficient particle swarm optimization approach for data fitting with free knot B-Splines, Computer Aided Design 43.12 (2011), 1683–1692.

[6] M.J.P.Corpa, S.S.Sanz, A.M.P.Bellido, P.L.Espi, R.Benavente, E.Perez, Curve fitting using heuristics and bio-inspired optimization algorithms for experimental data processing in chemistry, Chemometrics and Intelligent Laboratory Systems 96 (2009), 34–42.

[7] Z.Wu, X.Wang, Y.Fu, J.Shen, Q.Jiang, Y.Zhu, Fitting scattered data points with ball B-Spline curves using particle swarm optimization, Computers&Graphics 72 (2018), 1–11.

[8] J.Ding, Q.Wang, Q.Zhang, Q.Ye, Y.Ma, A Hybrid Particle Swarm Optimization - Cuckoo Search Algorithm and Its Engineering Applications, Hindawi Mathematical Problems in Engineering 2019 (2019), Article ID 5213759, 12 pages.

[9] M.A.Ismail, S.Deris, M.S.Mohamad, M.A.Isa, A.F. Abdullah, M.A. Remli, S.M.M.Aldeen, A Hybrid of Optimization method for multiobjective constraint optimization of biochemical system production, Journal of Theoretical&Applied Information Technology, 81.3 (2015), 502–513.

[10] P.Pandunata, S.M.H.Shamsuddin, Differential Evolution Optimization for Bezier Curve Fitting, 2010 Seventh International Conf. on Computer Graphics, Imaging and Visualization, IEEE XPlore (2010), 68–72.

[11] E.K.Ueda, M.S.G.Tsuzuki, R.Y.Takimoto, A.K.Sato, T.C.Martins, P.E.Miyagi, R.S.U.Rosso Jr, Piecewise Bezier Curve Fitting by Multiobjective Simulated Annealing, IFAC PapersOnLine 49.31 (2016), 49–54.

[12] E.Duman, M.Uysal, A.F.Alkaya, Migrating Birds Optimization: A new metaheuristic approach and its performance on quadratic assignment problem, Information Sciences 217 (2012), 65–77.

[1]Dogus University, Engineering Faculty, Acibadem 34722, Istanbul , Turkey
[2]Appcent Limited, Fikirtepe, Istanbul, Turkey
E-mail : muysal@dogus.edu.tr, ozanuysal@appcent.mobi

Some remarkable combinatorial relations based on methods of hypercomplex function theory

Isabel Cação[1], *M. Irene Falcão*[2], *Helmuth R. Malonek*[1], *Graça Tomaz*[1,3]

Abstract

Recently methods of hypercomplex function theory have successfully been used for studying number sequences arising in harmonic analysis as well as combinatorial identities and new number theoretic properties of central binomial coefficients. Following this approach, the aim of this work is to prove a theorem about a family of number sequences, some of them connected with Catalan numbers, and a new recurrence formula for generalized Vietoris numbers.

2010 Mathematics Subject Classifications : 30G35, 05A10, 11B83

Keywords: hypercomplex Appell polynomials, recurrence relation, Catalan's numbers, Vietoris' numbers.

Introduction

In this contribution we will focus on two sequences of rational numbers arising as coefficients of hypercomplex polynomials of arbitrary $(n+1)$ real variables. These polynomials were introduced in [9] and since then they have been studied by several authors due to their importance in theory and applications (see for instance [1, 2]). Recently, the relation between coefficients of different representations of the same hypercomplex polynomial caught attention, and their arithmetical and combinatorial properties have been studied ([8, 10]) as well as their relation to other well known sequences ([6, 7]).

In order to frame the subject which has its roots in Clifford Analysis, we will start by introducing essential notation and definitions. The reader interested in more details can find them in [4].

We denote by $\mathcal{C}\ell_{0,n}$ the real Clifford algebra generated by the standard basis $\{e_1, \ldots, e_n\}$ of the Euclidean vector space $\mathbb{R}^n$. The multiplication in $\mathcal{C}\ell_{0,n}$ is determined by the rules

$$e_i e_j + e_j e_i = -2\delta_{i,j},\ i,j = 1, \ldots, n,$$

where $\delta_{i,j}$ is the Kronecker symbol. An element $z \in \mathcal{C}\ell_{0,n}$ is written as $z = \sum_A z_A e_A$, $z_A \in \mathbb{R}$, with the basis $\{e_A : A \subseteq \{1, \ldots, n\}\}$ formed by $e_A = e_{j_1} \cdots e_{j_k}, 1 \leq j_1 < \cdots < j_k \leq n$. For $A = \emptyset$, we set $e_\emptyset = 1$, which is the unit of the algebra. In particular, $x = x_0 + \underline{x} := x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathcal{A}_n := \text{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$ is called paravector; the identification of $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with x shows that the vector space $\mathbb{R}^{n+1}$ is embedded in the Clifford algebra $\mathcal{C}\ell_{0,n}$. The conjugate and the norm of x are given by $\bar{x} = x_0 - \underline{x}$ and $|x| = (x\bar{x})^{\frac{1}{2}} = (x_0^2 + x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$, respectively.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by $\bar{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$, with $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$.

We consider functions $f : \Omega \subset \mathbb{R}^{n+1} \cong \mathcal{A}_n \longmapsto \mathcal{C}\ell_{0,n}$ defined and real differentiable in an open subset Ω of $\mathbb{R}^{n+1}$, $n \geq 1$. Functions that are solutions of the differential equation $\bar{\partial} f = 0$ (resp.$f\bar{\partial} = 0$) are generalized holomorphic functions, often called *left monogenic* (resp. *right monogenic*).

A monogenic function f defined in Ω is hypercomplex differentiable if and only if it has a uniquely defined areolar derivative f' in each point of Ω (for details, see [12]). The hypercomplex derivative f' of a monogenic function f is given by $f' = \partial f$ where $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$ is the conjugate generalized Cauchy-Riemann operator (see [11]).

Finally, we recall the definition of a generalized Appell sequence of monogenic polynomials associated to ∂ (cf. [13]):

A sequence of $\mathcal{A}_n$-valued monogenic polynomials $(\mathcal{Q}_k(x))_{k\geq 0}$ is called a generalized Appell sequence, if $\mathcal{Q}_k(x)$ is of exact degree k, for each $k = 0, 1, \ldots$, and $\partial \mathcal{Q}_k(x) = k\mathcal{Q}_{k-1}(x),\ k = 1, 2, \ldots$.

Some relations of generalized Appell polynomials

Let $(\mathcal{P}_k^n(x))_{k\geq 0}$, for each $n \geq 1$, be the sequence of generalized Appell polynomials whose elements can be represented by

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n) x^{k-s} \bar{x}^s \quad \text{or} \quad \mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s.$$

If $(a)_m$ denotes the Pochhammer symbol, given by $(a)_m := a(a+1)\cdots(a+m-1)$, for any integer $m \geq 1$, $(a)_0 := 1$, $(0)_0 := 1$), and $\lfloor \cdot \rfloor$ stands for the floor function, then the corresponding coefficients $T_s^k(n)$ and $c_s(n)$ have the form

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \tag{1}$$

and

$$c_s(n) = \frac{\left(\frac{1}{2}\right)_{\left\lfloor \frac{s+1}{2} \right\rfloor}}{\left(\frac{n}{2}\right)_{\left\lfloor \frac{s+1}{2} \right\rfloor}}, \tag{2}$$

respectively (cf. [5, 13]). Our concern is now on the underlying sequences of rational positive numbers $T(n) := (T_s^k(n))_{k\geq 0}$ and $S(n) := (c_s(n))_{s\geq 0}$. For each value of n, the numbers $T_s^k(n)$ may be arranged in a triangle with lines of height $k = 0, 1, \ldots$, and ordered from $s = 0$ up to $s = k$ (see Table 1 for the case $n = 2$, and $k = 0, \ldots, 5$). In the Table 2 the first elements of $(c_{2k}(n))$, $n = 2, \ldots, 5$, are shown. We remark that $c_{2k-1}(n) = c_{2k}(n)$.

Two remarkable properties of the coefficients $T_s^k(n)$ can be proved straightforward. First of all, the elements of each line of the mentioned triangle form a partition of 1, secondly their alternating sum is just equal to the element $c_k(n)$ of $S(n)$ (cf. [10]). But as an example of a relation less directly to obtain about the elements of the non-symmetric triangle of the $T_s^k(n)$ we can prove the following theorem.

Theorem 1. *For* $k = 1, 2, \ldots$,

$$\sum_{s=1}^{k} s\, T_s^k(n) = \frac{k(n-1)}{2n}. \tag{3}$$

Proof. Recalling that

$$s\binom{k}{s} = k\binom{k-1}{s-1} \quad \text{and} \quad \left(\frac{n-1}{2}\right)_s = \frac{n-1}{2}\left(\frac{n-1}{2}+1\right)_{s-1},$$

from (1) we can write

$$\sum_{s=1}^{k} sT_s^k(n) = \sum_{s=1}^{k} k\binom{k-1}{s-1}\frac{\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)_{s-1}\left(\frac{n+1}{2}\right)_{k-s}}{(n)_k}$$
$$= \frac{k(n-1)}{2(n)_k}\sum_{s=1}^{k}\binom{k-1}{s-1}\left(\frac{n+1}{2}\right)_{s-1}\left(\frac{n+1}{2}\right)_{k-s}.$$

Thus, by using the Chu-Vandermonde convolution identity $(x+y)_m = \sum_{k=0}^{m}\binom{m}{k}(x)_k(y)_{m-k}$, with $x = y = \frac{n+1}{2}$, and since $(n)_k = n(n+1)_{k-1}$, we get the result. □

We observe that the binomial sum $\sum_{s=1}^{k} s\binom{k}{s} = 2^{k-1}k$ (see [3], p.14) is a particular case of the last result. In fact that sum can be obtained from (3) with $n \to \infty$, and taking into account that $T_s^k(\infty) = 2^{-k}\binom{k}{s}$ (cf. [10]).

Other properties and relations on $T(n)$, as well as the close relation to the Pascal triangle, can be found in [10].

Table 1: First rows of $T(2)$

$k \backslash s$	0	1	2	3	4	5
0	1					
1	$\frac{3}{4}$	$\frac{1}{4}$				
2	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$			
3	$\frac{35}{64}$	$\frac{15}{64}$	$\frac{9}{64}$	$\frac{5}{64}$		
4	$\frac{63}{128}$	$\frac{7}{32}$	$\frac{9}{64}$	$\frac{3}{32}$	$\frac{7}{128}$	
5	$\frac{231}{512}$	$\frac{105}{512}$	$\frac{35}{256}$	$\frac{25}{256}$	$\frac{35}{512}$	$\frac{21}{512}$

Table 2: First elements of $(c_{2k}(n))$, $(n = 2, \ldots, 5)$

n	
2	$1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256}, \frac{231}{1024}, \ldots$
3	$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \ldots$
4	$1, \frac{1}{4}, \frac{1}{8}, \frac{5}{64}, \frac{7}{128}, \frac{21}{512}, \frac{33}{1024}, \ldots$
5	$\frac{3}{3}, \frac{3}{15}, \frac{3}{35}, \frac{3}{63}, \frac{3}{99}, \frac{3}{143}, \frac{3}{195}, \ldots$

Furthermore, there is a particular connection between $T(n)$ and $S(n)$ that allowed the study of interesting combinatorial identities [8]. A more detailed analysis for particular values of n further revealed links to other well-known numerical sequences. It is the case of the relation of $S(2)$, and also $T(2)$, with the Vietoris sequence ([6, 7]). In this context it was proved that, for $k = 0, 1, \ldots$, and $m \leq \log_2(k+1)$,

$$m_k = k + \sum_{l=1}^{m}\left\lfloor\frac{k+1}{2^l}\right\rfloor$$

is the least non-negative integer such that $2^{m_k}T_0^k(2) \in \mathbb{N}$. We note that m_k plays the same role for the diagonal entries of the Table 1, $T_k^k(2)$, i.e., $2^{m_k}T_k^k(2) \in \mathbb{N}$. The integer sequences $(m_k)_{k\geq 0}$ and $(2^{m_k}T_k^k(2))_{k\geq 0}$ are listed in *The On-Line Encyclopedia of Integer Sequences* [14] (A283208, A098597).

The diagonal entries $T_k^k(2)$ are related to the celebrated Catalan numbers, $C(k)$, $k = 0, 1, \ldots$. In fact, according to (1),

$$T_k^k(2) = \frac{(2k-1)!!}{(k+1)!2^k} = \frac{(2k)!!(2k-1)!!}{2^{2k}k!(k+1)!} = \frac{1}{2^{2k}}\frac{1}{k+1}\binom{2k}{k} = \frac{1}{2^{2k}}C(k). \tag{4}$$

Thus, $2^{m_k}T_k^k(2) = 2^{m_k-2k}C(k)$, and we can write

$$A098597(k) = A00108(k)2^{A283208(k)-A005843(k)}.$$

It is worth noting that all the elements of the triangle in the Table 1 are related to Catalan numbers. It is enough to use (4) together with the relation between adjacent elements in the row k and an element in the row $k-1$ [10],

$$(k-s)T_s^k(2) + (s+1)T_{s+1}^k = kT_s^{k-1}(2),\ k = 1, 2, \ldots, \text{and } s = 0, \ldots, k-1,$$

in order to obtain any element of the triangle as a linear combination of Catalan numbers.

Remark 3. *The elements of $S(4)$ are also related to the Catalan numbers. Indeed, from (1) and (2), as well as (4) we get*

$$c_{2k+1}(4) = T_{k+1}^{k+1}(2) = \frac{1}{4^{k+1}}C(k+1).$$

Often generating functions of sequences arise from recurrence formulas. However, the opposite is also possible and may allow new insights into the nature of the involved sequence. Based on a generating function of the generalized Vietoris sequence $S(n)$, a new recurrence formula for the elements of $S(4)$ can be obtained (see [7] for the case $S(2)$).

Theorem 2. *The elements of the generalized Vietoris' number sequence $S(4)$ satisfy the recurrence relation*

$$c_{k+3} = -c_{k+2} + \tfrac{1}{2}c_{k+1} + \tfrac{1}{4}\sum_{s=0}^{k} c_s c_{k-s},\ k \geq 0,$$

where $c_k := c_k(4)$, and $c_0 = 1, c_1 = c_2 = \frac{1}{4}$.

Proof. Setting $n = 4$ in the generating function of the sequence $S(n)$ (cf. [7] Theorem 3),

$$G(t;n) = \begin{cases} \frac{1}{t}\left[(1+t)\,{}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; t^2) - 1\right], & \text{if } t \in]-1,1[\setminus\{0\} \\ 1, & \text{if } t = 0 \end{cases},$$

where ${}_2F_1(a,b;c;z)$ is the Gauss' hypergeometric function, we get $F := G(t;4) = \frac{2t+1-\sqrt{1-t^2}}{t(1+\sqrt{1-t^2})}$, which leads to

$$(Ft+1)\sqrt{1-t^2} = 2t + 1 - Ft. \tag{5}$$

Squaring both sides and multiplying by t, (5) becomes

$$(Ft^2)^2 = (4 + 4t - 2t^2)Ft - 5t^2 - 4t. \tag{6}$$

Since $F = \sum_{k=0}^{+\infty} c_k t^k$,

$$Ft = \sum_{k=0}^{+\infty} c_k t^{k+1},\ Ft^2 = \sum_{k=0}^{+\infty} c_k t^{k+2}, \text{and } (Ft^2)^2 = \sum_{k=0}^{+\infty}\left(\sum_{s=0}^{k} c_s c_{k-s}\right)t^{k+4}.$$

Replacing in (6) $(Ft^2)^2$ and Ft by the corresponding series, the left hand side and the right hand side can be written, respectively, as

$$\sum_{k=0}^{+\infty}\left(\sum_{s=0}^{k} c_s c_{k-s}\right)t^{k+4},$$

and

$$(4c_0-4)t+(4c_1+4c_0-5)t^2+(4c_2+4c_1-2c_0)t^3+4\sum_{k=0}^{+\infty}(c_{k+3}+c_{k+2}-\tfrac{1}{2}c_{k+1})t^{k+4}.$$

The comparison of the two sides leads to

$$c_0=1,\ c_1=c_2=\tfrac{1}{4},\ \sum_{s=0}^{k}c_sc_{k-s}=4(c_{k+3}+c_{k+2}-\tfrac{1}{2}c_{k+1}),$$

and the result follows immediately. □

Conclusion

A theorem on numbers connected with Catalan numbers, and a new recurrence formula for generalized Vietoris numbers have been proved. Both are examples of combinatorial relations suggested by the study of generalized Appell polynomials in the hypercomplex context. It seems to be remarkable that the structures of hypercomplex polynomials allow to obtain combinatorial relations going beyond the use of polynomials over the complex field.

Acknowledgements

The work of the second author was supported by Portuguese funds through the CMAT - Centre of Mathematics and FCT within the Project UID/MAT/00013/2013. The work of the other authors was supported by Portuguese funds through CIDMA - Center for Research and Development in Mathematics and Applications, and FCT–Fundação para a Ciência e a Tecnologia, within project UID/MAT/04106/2019.

References

[1] L. Aceto, H. R. Malonek, and G. Tomaz, Matrix approach to hypercomplex Appell polynomials, Appl. Numer. Math. 116 (2017), 2–9.

[2] S. Bock and K. Gürlebeck, On a generalized Appell system and monogenic power series, Math. Methods Appl. Sci., 33(4) (2010), 394–411.

[3] G. Boros and V. Moll, Irresistible Integrals: Symbolics, Analysis and Experiments in the Evaluation of Integrals, Cambridge, England: Cambridge University Press; 2004.

[4] F. Brackx, R. Delanghe, and F. Sommen, Clifford Analysis, Boston-London-Melbourn: Pitman; 1982.

[5] I. Cação, M. I. Falcão, and H. R. Malonek, Combinatorial identities in the context of hypercomplex function theory, In: AIP Conference Proceedings 1978 (2018), 2800041–2800044.

[6] I. Cação, M. I. Falcão, and H. R. Malonek, Hypercomplex polynomials, Vietoris' rational numbers and a related integer numbers sequence, Complex Anal. Oper. Theory (2017), 1059–1076.

[7] I. Cação, M. I. Falcão, and H. R. Malonek, On generalized Vietoris' number sequences, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.10.017.

[8] I. Cação, H. R. Malonek, M. I. Falcão, and G. Tomaz, Combinatorial Identities Associated with a Multidimensional Polynomial Sequence, J. Integer Seq. 21 (2018), Article 18.7.4.

[9] M. I. Falcão, J. Cruz, and H. R. Malonek, Remarks on the generation on monogenic functions, In: K. Gürlebeck and C. Könke (eds.), Proc. 17th IKM Weimar, (2006).

[10] M. I. Falcão and H. R. Malonek, A note on a one-parameter family of non-symmetric number triangles, Opuscula Math. 32 (2012), 661–673.

[11] K. Gürlebeck and H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{m+1}$ and its applications, Complex Variables 39 (1999), 199–228.

[12] H. R. Malonek, Selected topics in hypercomplex function theory, In: S.-L. Eriksson (ed.), Clifford Algebras and Potential Theory 7 (2004), 111–150.

[13] H. R. Malonek and M. I. Falcão, Special monogenic polynomials - properties and applications, In: AIP Conference Proceedings 936 (2007), 764–767.

[14] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[1]CIDMA-Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal

[2]CMAT and Department of Mathematics, University of Minho, Portugal

[3] UDI-Research Unit for Inland Development, Polytechnic of Guarda, Portugal

E-mail : isabel.cacao@ua.pt, mif@math.uminho.pt, hrmalon@ua.pt, gtomaz@ipg.pt

Minimal time synthesis for nonlinear Bloch model in quantum dots

R. EL Assoudi-Baikari[1], *N. Forcadel*[2], *I. E. Zibo*[3]

Abstract

In this paper, we analyze the time optimal control for the population transfer in quantum dots at two energy levels under the action of a laser field by taking into the presence of Coulomb interaction. This problem is modeled by a nonlinear Bloch equation with the nonlinearity arising from Coulomb interaction. We show the non trivial role of the Coulomb parameter on the population dynamics. Using tools of geometric optimal control theory, we determine different optimal synthesis for specific values of the system parameters.

2010 Mathematics Subject Classifications : 49J15, 81Q37, 93B50.
Keywords: Optimal control synthesis problems, Quantum dots.

Introduction

In these recent years, quantum systems have been the subject of several studies. Indeed, in [7] the minimum time population transfer problem for a spin particle driven by a magnetic field on Bloch sphere where population dynamics is influenced by a parameter which depends on the maximum amplitude of control field and the energy level, was studied. In [5] the time minimal of control problem taking into account the dissipative terms and the effects of damping which act on the dynamics of the populations was solved, the system is modeled by the nonlinear Bloch equation which one finds in Nuclear Magnetic Resonance. Here, we study the changes made by Coulombian effects on population dynamics taking into account inter-band transition frequencies of quantum dots. We consider a non-linear Bloch model in [1] whose nonlinear terms come from Coulomb interaction.

$$i\hbar\dot{\rho}(t) = [H, \rho] + [V(\rho), \rho] \qquad (1),$$

where $\hbar$ is the constant of Plancks (we will take $\hbar = 1$), $\rho(t)$ is the density matrix which is hermitian, positive and with trace equal to 1. It represents pure and mixed quantum states if and only if trace$(\rho(t)^2) \leq 1$. The total hamiltonian $H = H_0 + H_L$, with H_0 is the free hamiltonian of the system and H_L is the hamiltonian of interaction between quantum dot and laser field. The matrix $V(\rho)$ is *Coulomb interaction matrix*, which is hermitian.

In this paper, we consider the quantum dot as a quantum system with two energy levels. We use the known parametrization of the density matrix for two-level quantum system given in [2] by $\rho(x) = \frac{1}{2}(I_2 + \sum_{i=1}^{3} x_i\sigma_i)$, where σ_i, $i = 1, 2, 3$ are Pauli matrix, I_2 is identity matrix of two order and $x = (x_1, x_2, x_3)$ is the Bloch vector and it belongs to unit ball of $\mathbf{R}^3$. Also, we take $H_0 = diag(E_1^c, E_2^v)$, with E_1^c, E_2^v are free energies, we denote c for conduction band and v for valence band; and $H_L = E(t)M$, where $E(t)$ is an electric field defining the control function taken to be real and bounded,

M is a dipolar operator moment, and $V(\rho) = \begin{pmatrix} -R\rho_{11} & -R\rho_{12} \\ -R\rho_{12} & R\rho_{11} \end{pmatrix}$, where R is *the Coulomb parameter* which is a real constant and $\rho = (\rho_{ij})$.

After replacing $\rho(x)$ in (1) we get by a simple calculation the following affine control system of the form $\dot{x} = F(x) + uG(x)$:
(Σ) $\dot{x}_1 = ((E_2^v - E_1^c) + R)x_2$, $\dot{x}_2 = ((E_1^c - E_2^v) - R)x_1 - ux_3$ and $\dot{x}_3 = ux_2$.

Clearly, $\forall u$, $x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 = 0$, and if $\| x(0) \| = 1$ then $x(t) \in S^2$.

We are interested in time-optimal problem, we will find pair trajectory-control $(x(t), u(t))$ defined on $[0, T]$ joining an initial point $x(0) = x_0$ to a final point $x(T) = x_f$ in minimal time T.

We are led to use the techniques developed in the literature to solve time-optimal control problem on two-dimensional manifolds [4, 6]. The main tool is the Pontryagin Maximum Principle (PMP) [3]. Here, we give a version adapted to the affine control system [4]. It states that the optimal trajectories are solutions of the following equations:
$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t), u(t))$ and $\dot{p}(t) = -\frac{\partial H}{\partial x(t)}(x(t), p(t), u(t))$,
$H(x(t), p(t), u(t)) = max_{v(t)\in[-1,1]} \{H(x(t), p(t), v(t))\}$, and
$H(x(t), p(t), u(t)) = 0$, for almost any $t \in [0, T]$,
where $H(x(t), p(t), v(t)) = p(t)(F(x(t)) + v(t)G(x(t)) + p_0$, with $p_0 \leq 0$ a constant and $p(t)$ is the nonzero line adjoint vector.

If $p_0 \neq 0$, the trajectory is called a *normal extremal*. If $p_0 = 0$, the trajectory is called an *abnormal extremal*. The maximization condition of the PMP gives us, $max_{v(t)\in[-1,1]} \{H(x(t), p(t), v(t))\} = max_{v(t)\in[-1,1]} \{v(t)\Phi(t)\}$, where $\Phi(t) = p(t)G(x(t))$ called the *switching function*. The analyze of $\Phi(t)$ gives a strategy of switching [4].

Denote by $-E_{21} = E_1^c - E_2^v$ the *inter-band transition frequencies*, and set $a = E_{21} + R$. We will assume $a \neq 0$. The associated hamiltonian to (Σ) is $H(x, p, u) = ax_2p_1 - ax_1p_2 + u(t)(p_3x_2 - x_3p_2) + p_0$. The adjoint equations are $\dot{p}_1(t) = ap_2$, $\dot{p}_2(t) = -ap_1 - up_3$ and $\dot{p}_3(t) = up_2$, and $\Phi(t) = p_3x_2 - p_2x_3$. The sets and functions calculated below are details in [6]. We find $\Delta_1(x) = F(x) \wedge G(x) = -ax_2x$ and $\Delta_2(x) = G(x) \wedge [F, G](x) = ax_3x$, with $x \in S^2$ ($\wedge$ is vector product). Clearly, $\Delta_1^{-1}(0) = \{x_2 = 0\} \cap S^2$ and $\Delta_2^{-1}(0) = \{x_3 = 0\} \cap S^2$. A normal trajectory solution of control system (Σ) can bifurcate at a possible switching time $\bar{t}$ if $\dot{\Phi}(\bar{t}) = \frac{x_3(\bar{t})}{x_2(\bar{t})}$, $x_2(\bar{t}) \neq 0$.

Main Results

We present in the following proposition a time optimal synthesis which involves Coulomb parameter for a quantum dot with $-E_{21} = 0.287$ (see [8]).

Proposition. 1) If $-E_{21} = 0.287$ and $R = 10(-E_{21})$ then :
i) normal optimal trajectory starting from north pole and reaches a point over plan $x_3 = 0$ is respectively B_-; B_-B_+; $B_-B_+B_-$; $B_-B_+B_-B_+$ and $B_-B_+B_-B_+B_-$.

ii) every optimal abnormal trajectory switches at most five times before loses its optimality and has the form $B_-B_+B_-B_+B_-B_+$.

2) If $-E_{21} = 0.287$ and $R = \frac{1}{10}(-E_{21})$ then :
i) normal optimal trajectory starting from north pole is respectively B_-; B_-S; B_-SB_+; B_-SB_- and $B_-B_+B_-$.
ii) every optimal abnormal trajectory switches at most thirteen times before loses its optimality and has the form $B_-B_+B_-B_+...B_-B_+$.

Proof: we use the necessary condition of switching time given by $\dot{\Phi}(\bar{t}) = \frac{x_3(\bar{t})}{x_2(\bar{t})}$, $x_2(\bar{t}) \neq$

0 to analyze the behavior of optimal trajectories in four areas depending on the sign of $\frac{x_3}{x_2}$. Set $d_1 = \{x \in S^2 : x_3 > 0; x_2 > 0\}$ and $d_2 = \{x \in S^2 : x_3 > 0; x_2 < 0\}$. From d_1 (resp. d_2) we find d_3 (resp. d_4) by taking $x_3 < 0$. For 1) along a normal optimal extremal with control $u = -1$ denoted by x^-, starting from north pole, switching can happen when it enters in the area d_1 at a certain time $\bar{t}$ from $u = -1$ to $u+1$ because $\dot{\Phi}(\bar{t}) > 0$. A normal optimal extremal x^+ corresponding to the control $u = +1$ can not bifurcate from x^- in d_2, because in this part we have $\dot{\Phi}(\bar{t}) < 0$. When the trajectory x^- it self-intersects it loses its optimality.

• The trajectory x^+ bifurcating from the point $x^-(\bar{t})$ in d_1 can switch from $u = +1$ to $u = -1$ in the area d_2 at time $\overline{t_1} > \bar{t}$ due to $\dot{\Phi}(\overline{t_1}) < 0$. When the trajectory x^- is intersected in the area d_2 by trajectory x^+ coming from the point $x^-(\bar{t})$, it loses its optimality at intersection point (locus point).

• Trajectory x^- coming from point $x^+(\overline{t_1})$ in the area d_2 can switch from $u = -1$ to $u = +1$ in d_1 at a time $\overline{t_2} > \overline{t_1}$ due to $\dot{\Phi}(\overline{t_2}) > 0$ in this area.

• The trajectory x^+ bifurcating from the point $x^-(\overline{t_2})$ in d_1, is no longer optimal when it intersects a trajectory x^- located at the top of the Bloch sphere, as it is shown in left figure in Figure 2. And this point corresponds to a cut locus point.

• Therefore, when trajectory x^+ doesn't intersect a trajectory x^- located at the top of the Bloch sphere, it can switch at certain time $\overline{t_3} > \overline{t_2}$ when it enters in the area d_2 due to the fact that $\Phi(\overline{t_3}) < 0$, and the trajectory x^- bifurcating from the point $x^+(\overline{t_3})$ in d_2 is no longer optimal when it intersects a trajectory x^+ in the same area and this intersection point is a cut locus point.

• If a optimal extremal x^- starting from north pole with control $u = -1$ is abnormal then switching can happen on $\{x_2 = 0\} \cap S^2$. Therefore we see that trajectory makes half turn when it reaches the set $\{x_2 = 0\} \cap S^2$, so switching happen at the time $t = \pi$. After five switching, trajectory loses its optimality at a cut locus point represented with magenta color in Figure 1 at right side.

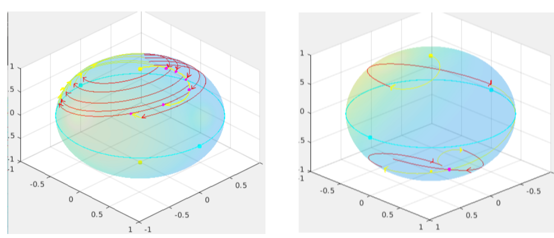


Figure 1: Red color curves are for control $u = +1$, yellow color curves are for control $u = -1$, the cyan curve is singular locus. In left figure: Normal, *bang; bang-bang; bang-bang-bang; bang-bang-bang-bang; bang-bang-bang-bang-bang* optimal trajectories. In the right side figure : Abnormal, *bang-bang-bang-bang-bang-bang* optimal trajectory.

For 2) we use the same technique as before to describe the optimal trajectories. But the difference is that here the trajectory x^- coming from north pole cross the *turnpike* singular locus. Indeed, let $\bar{t}$ the first time at which it reaches this singular locus at point $x^-(\bar{t}) = \overline{x}$, switching can happen from $u = -1$ to $u = +1$ at the point $\overline{x}$ in a small neighborhood of $\overline{x}$ in d_1. Then singular locus can contain optimal singular trajectory [4]. Trajectory x^+ coming from the point $x^-(\bar{t})$ passes by the south pole, so the south pole is reached by optimal $bang-bang$ trajectory. If a optimal extremal x^- starting from north pole with control $u = -1$ is abnormal switching happen at time $t = \pi$ on $\{x_2 = 0\} \cap S^2$. By simulation on Figure 2 at right side, this trajectory loses its optimality after thirteen switching.

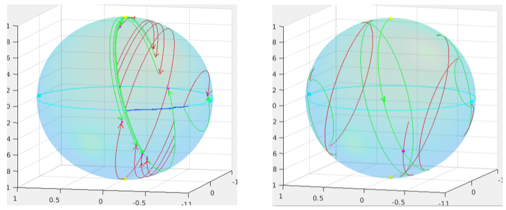


Figure 2 Red color curves are for control $u = +1$, green color curves are for control $u = -1$, the cyan curve is singular locus, blue curve is singular optimal trajectory, and magenta color points are cut locus points. Left figure : Normal, *bang; bang-bang; bang-singular-bang; bang-bang-bang* optimal trajectories. Right side figure: Abnormal *bang-bang...bang-bang* optimal trajectory, magenta color point is cut locus point.

Conclusion

In this paper, using the non linear Bloch model, we have bring out the optimal synthesis for the cases when the Coulomb parameter is great than and weak than transition frequency. There is no optimal trajectories that reach south pole when Coulomb parameter is greatly added to the transition frequencies, whereas in other case there are optimal trajectories that reach south pole. For futur investigations we are interesting in the study of the above system with two controls, and to consider (1) with 3 energy levels.

Acknowledgements

The authors thank the Region Haute-Normandie for supporting this work, via the project M2Sinum: 18P03390/18E01750/18P02733.

References

[1] B. Bidégaray-Fesquet and K. Keita, (2014), A nonlinear Bloch model for Coulomb interaction in quantum dots. Journal of Mathematical Physics, 55, 021501.

[2] E. Brüning, H. Mäkelä and A. Messina, (2011), Parameterizations of density matrices, Journal of Modern.

[3] A.A. Agrachev and Y.L. Sachkov, (2004), Control theory from the geometric viewpoint, Vol. 87. Encyclopedia of mathematical sciences. Control theory and optimization, II. Berlin, Springer.

[4] U. Boscain and B. Piccol, (2004), Optimal Synthesis for Control Systems on 2-D Manifolds, Springer, SMAI, Vol. 43.

[5] M. Lapert, E. Assemat, Y. Zhang, S.J. Glaser and D. Sugny, (2013), Time-optimal control of spin 1/2 particles with dissipative and generalized radiation-damping effects, Physical Review A, APS.

[6] B. Piccol, (1996), Regular time-optimal syntheses for smooth planar systems. Rend. Sem. Mat. Univ. Padova - numdam.org.

[7] U. Boscain and P. Mason, (2006), Time minimal trajectories for a spin $\frac{1}{2}$ particle in a magnetic field, Journal of Mathematical Physics, - aip.scitation.org.

[8] C. Tonin, (2012), Manipulation cohérente de l'émission résonnante d'une boîte quantique unique tel.archives-ouvertes.fr.

INSA Rouen Normandie, LMI. France.
E-mail : rachida.el-assoudi@insa-rouen.fr, nicolas.forcadel@insa-rouen.fr, edouard.zibo@insa-rouen.fr

Applications of generating functions derived from Möbius-Bernoulli numbers

Daeyeoul Kim[1], *Abdelmejid Bayad*[2], *Hyungyu Ahn*[3*]

Abstract

In this paper, we define generating functions derived from Möbius-Bernoulli numbers and double Möbius-Bernoulli numbers. And Using Melin transformation, we consider L-function connected to these generating functions. Finally, we find relationship between these functions and L-functions.

2010 Mathematics Subject Classifications : 11A05, 33E99
Keywords: Möbius-Bernoulli numbers

Introduction

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $k \geq 0$ the Möbius-Bernoulli numbers $M_k(n)$ is defined as follows:

$$\sum_{k=0}^{\infty} M_k(n)\frac{t^k}{k!} = \sum_{d|n} \mu(d)\frac{t}{e^{dt}-1}, \quad |t| < \frac{2\pi}{n}.$$

If $n = 1$, the Möbius-Bernoulli numbers $M_k(n)$ are eqaul to Bernoulli numbers B_k, where $\mu(n)$ is the Möbius function.
Let n' be a positive integer, the double Möbius-Bernoulli numbers $M_k(n, n')$ given by

$$M_k(n, n') = \sum_{j=0}^{k} \binom{k}{j} M_j(n) M_{k-j}(n'). \tag{1}$$

Similar results are in [3].

Generating functions derived from Möbius-Bernoulli numbers

We consider these generating function. For $n \in \mathbb{N}$, $f(t, x)$ and $g(t)$ are defined as follows:

$$f(t, x) := \sum_{k=0}^{\infty} M_k(x, n)\frac{t^k}{k!} = \sum_{d|n} \mu(d)\frac{te^{xt}}{e^{dt}-1}$$

$$g(t) := \sum_{k=0}^{\infty} \widetilde{M}_k(n)\frac{t^k}{k!} = \sum_{d|n} \mu(d)\frac{te^{dt}}{e^{nt}-1}.$$

*Corresponding author

where $f(t,x)$ is called Möbius-Bernoulli polynomial, and $g(t)$ is called Modified Möbius-Bernoulli numbers.
Now we consider L-functions connected to these numbers. Let $x > 0$. We consider L-finction of $f(t,x)$ by using Mellin transform.

$$\begin{aligned} L(s,f;x) &= \frac{1}{\Gamma(s)}\int_0^\infty (\sum_{d|n}\mu(d)\frac{-t}{e^{-dt}-1})e^{-xt}t^{s-2}dt \\ &= \sum_{d|n}\mu(d)\frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-xt}}{1-e^{-dt}}t^s\frac{dt}{t} \quad (\text{ for Re}s > 1), \end{aligned} \tag{2}$$

By changing variable t to t/d in (2), then we get

$$L(s,f;x) = \sum_{d|n}\frac{\mu(d)}{d^s}\frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-(x/d)t}}{1-e^{-t}}t^s\frac{dt}{t} \quad (\text{ Re}s > 1).$$

Using Proposition 10.2.2 of [2] and formula (2.2) of [4], we can get the following theorem.

Proposition 1.

$$L(s,f;x) = \sum_{d|n}\frac{\mu(d)}{d^s}\zeta(s;\frac{x}{d})$$

and

$$L(-k,f;x) = -\frac{M_{k+1}(x,n)}{k+1}.$$

Now, we consider Modified Möbius-Bernoulli numbers $\widetilde{M}_k(n)$.
For $n=1$,

$$\sum_{k=0}^\infty \widetilde{M}_k(n)\frac{t^k}{k!} = t + \frac{t}{e^t-1} = t + \sum_{k=0}^\infty B_k\frac{t^k}{k!}.$$

So $\widetilde{M}_k(1)$ is equal to Bernoulli number B_k. Now let $n \geq 2$. Then we have

$$\sum_{k=0}^\infty \widetilde{M}_k(n)\frac{t^k}{k!} = \sum_{d|n}\mu(d)\frac{te^{dt}}{e^{nt}-1}.$$

Changing variable t to t/n. we get

$$\sum_{k=0}^\infty \widetilde{M}_k(n)\frac{1}{n^k}\frac{t^k}{k!} = \frac{1}{n}\sum_{d|n}\mu(d)\frac{te^{\frac{d}{n}t}}{e^t-1} = \frac{1}{n}\sum_{d|n}\mu(d)\sum_{k=0}^\infty B_k(\frac{d}{n}).$$

Thus we have the following theorem.

Proposition 2. *Let $n \geq 2$. Then*

$$\widetilde{M}_k(n) = n^{k-1}\sum_{d|n}\mu(d)B_k(\frac{d}{n}).$$

Now, we consider L-finction of $g(t)$ by using Mellin transform.

$$L(s,g)=\frac{1}{\Gamma(s)}\int_0^\infty(\sum_{d|n}\mu(d)\frac{-te^{-dt}}{e^{-nt}-1})t^{s-2}dt. \tag{3}$$

Changing variable t to t/n, we get

$$L(s,g)=\sum_{d|n}\frac{\mu(d)}{n^s}\frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{e^{-\frac{d}{n}t}}{1-e^{-t}}\frac{dt}{t}.$$

Similarly as Proposition 1, we have the following theorem.

Proposition 3. *Let $n\geq 1$. Then we have*

$$L(s,g)=n^{-s}\sum_{d|n}\mu(d)\zeta(s;d/n)$$

and

$$L(-k,g)=-\frac{\widetilde{M}_{k+1}(n)}{k+1}\quad (\textit{for } k\geq 0).$$

The above results are in [1]

Main Result

We consider another generating functions derived from Möbius-Bernoulli numbers

$$h_n(t):=\sum_{k=0}^\infty M_k^+(n)\frac{t^k}{k!}=\sum_{d|n}\mu(d)\frac{te^{nt}}{e^{dt}-1}.$$
$$U_n(t):=\sum_{k=0}^\infty M_k^{++}(n)\frac{t^k}{k!}=\sum_{d|n}\mu(\frac{n}{d})\frac{te^{dt}}{e^{nt}-1}$$
$$V_n(t):=\sum_{k=0}^\infty M_k^{*}(n)\frac{t^k}{k!}=\sum_{d|n}\mu(\frac{n}{d})\frac{te^{nt}}{e^{dt}-1}$$

For $n=1$, $M_k^+(n)$, $M_k^{++}(n)$, and $M_k^*(n)$ are equal to Bernoulli number B_k. In a similar way to $\widetilde{M}_k(n)$, we have the following theorem for each cases.

Theorem 4. *Let $n\geq 2$. Then*

$$M_k^+(n)=\sum_{d|n}\mu(d)B_k(\frac{n}{d})d^{k-1}.$$

$$M_k^{++}(n)=\sum_{d|n}\mu(\frac{n}{d})B_k(\frac{d}{n})n^{k-1}$$

$$M_k^{*}(n)=\sum_{d|n}\mu(\frac{n}{d})B_k(\frac{n}{d})d^{k-1}$$

Also, we consider L-function using Melin transfrom for each case, we obtain this theorem.

Theorem 5. *Let $n \geq 1$. Then we have*

$$L(s,h) = \sum_{d|n} \mu(d)\zeta(s; \frac{n}{d})d^{-s}$$
$$L(s,u) = \sum_{d|n} \mu(n/d)\zeta(s; \frac{d}{n})n^{-s}$$
$$L(s,v) = \sum_{d|n} \mu(n/d)\zeta(s; \frac{n}{d})d^{-s}$$

and

$$L(-k,h) = -\frac{M_k^+(n)}{k+1} \quad (\textit{for } k \geq 0).$$
$$L(-k,u) = -\frac{M_k^{++}(n)}{k+1} \quad (\textit{for } k \geq 0).$$
$$L(-k,v) = -\frac{M_k^*(n)}{k+1} \quad (\textit{for } k \geq 0).$$

Acknowledgements

This research is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(NRF-2018R1D1A1B07041132).

References

[1] A. Bayad, D. Kim and Y. Li, *Arithmetical properties of double Mobius-Bernoulli numbers*, Open Math. **17** (2019), no. 1, 32–42.

[2] H. Cohen, *Number Theory Volume II: Analytic and Modern Tools*, Springer, GTM240, 2007.

[3] D. Kim, A. Bayad , H. Ahn *A study of Möbius-Bernoulli numbers*, to appear in Appl. Anal. Discrete Math.

[4] M. O. Rubinstein, *Identities for the Hurwitz zeta function, Gamma function, and L-functions*, Ramanujan J. **32**, 421-464, 2013.

[1]Department of Mathematics and Institute of Pure and Applied Mathematics
Chonbuk National University
567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896
South Korea
[2]Université d'Evry Val d'Essonne, Université Paris-Saclay, CNRS (UMR 8071)
Laboratoire de Mathématiques et Modélisation d'Évry
Bâtiment I.B.G.B.I., 3ème étage, 23 Boulevard de France
91037 Evry cedex
[3]Department of Mathematics and Institute of Pure and Applied Mathematics
Chonbuk National University
567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896

South Korea

E-mail : kdaeyeoul@jbnu.ac.kr, abdelmejid.bayad@univ-evry.fr, hgahn2413@naver.com

Solution for a functional equation $B(x) = H(xB(x)^r)$

Dmitry Kruchinin[1] , *Shablya Y.V.*[1] , *Vladimir Kruchinin*[2]

Abstract

Using the notion of the composita and Lagrange inversion theorem, we present techniques for solving the following functional equation $B(x) = H(xB(x)^r)$, where $H(x)$;$B(x)$ are generating functions, and r is any rational number. That is a generalization of Lagrange inverse formula for a rational power r in terms of compositae. Also we give some examples.

2010 Mathematics Subject Classifications : 05A15, 30D05, 39B12
Keywords: Composita; Generating function; Lagrange inversion theorem; Functional equation

Introduction

The Lagrange inversion theorem is one of the important results in combinatorics and analysis. The Lagrange inversion formula (LIF) is presented as follows:

Suppose $H(x) = \sum_{n\geq 0} h(n)x^n$ with $h(0) \neq 0$, and let $A(x)$ be defined by

$$A(x) = xH(A(x)). \tag{1}$$

Then

$$n[x^n]A(x)^k = k[x^{n-k}]H(x)^n, \tag{2}$$

where $[x^n]A(x)^k$ is the coefficient of x^n in $A(x)^k$ and $[x^{n-k}]H(x)^n$ is the coefficient of x^{n-k} in $H(x)^n$.

So the formula gives a solution for the equation $A(x) = xH(A(x))$ in terms of the coefficients of the generating function $A(x)$.

The LIF and this type functional equations often are useful as application in many areas of mathematics. Readers can start study the Lagrange inverse theorem with the book "Enumerative Combinatorics, Volume 2" of Stanley [1].

In the literature there exist different cases of generalization and forms of the Lagrange inversion theorem, that can be found in reviews of Lagrange inversion made by Gessel [2] or Merlini, Sprugnoli, and Verri [3].

In [4] we generalized the equation (1) for the case of natural power m:

$$B(x) = H(xB(x)^m), \tag{3}$$

where a generating function $xB(x)$ was substituted for $A(x)$ in the equation (1) and m is a natural number. We found that coefficients of powers of generating function $xB(x)$ (or composita of $xB(x)$) are defined by

$$B_x^{\Delta}(n,k,m) = \frac{k}{i_{m-1}} H_x^{\Delta}(i_m, i_{m-1}), \tag{4}$$

where $i_m = (m+1)n - mk$ and $H_x^{\Delta}(n,k)$ and $B_x^{\Delta}(n,k)$ are the compositae of the generating functions $xH(x)$ and $xB(x)$, respectively.

By the composita introduced in the papers [6, 5, 7] we mean the following notion: The composita is the function of two variables defined by

$$F^{\Delta}(n,k) = \sum_{\pi_k \in C_n} f(\lambda_1) f(\lambda_2) \dots f(\lambda_k), \tag{5}$$

where C_n is a set of all compositions of an integer n, π_k is the composition $\sum_{i=1}^{k} \lambda_i = n$ into k parts exactly.

Suppose $F(x) = \sum_{n>0} f(n)x^n$ is the generating function, in which there is no free term $f(0) = 0$. From this generating function we can write the following equation

$$[F(x)]^k = \sum_{n>0} F(n,k)x^n. \tag{6}$$

The expression $F(n,k)$ is the composita, and it is denoted by $F^{\Delta}(n,k)$. In this case composita is a expression for the coefficients of powers of generating function with free term equal to 0.

In this paper we study application of the composita for solving the following generalized equation

$$B(x) = H(xB(x)^r), \tag{7}$$

where $H(x)$ and $B(x)$ are generating functions with free term not equal to 0, and r is any rational number.

Main Results

Firstly we give the formula for the composita of $x[H(x)]^r$.

Theorem 1. The composita of $x[H(x)]^r$ is defined by

$$H_x^{\Delta}(n,m,r) = \begin{cases} H^{\Delta}(1,1)^{m\,r}, & n = m \\ \sum\limits_{k=1}^{n-m} H_m^{\Delta}(n-m,k)\binom{mr}{k} H^{\Delta}(1,1)^{mr-k} & n > m, \end{cases} \tag{8}$$

where $H_m^{\Delta}(n,k)$ is the coefficients of generating function $[H(x)-h(0)]^k$ that is defined by the composita $H^{\Delta}(n,k)$ of the generating function $xH(x)$.

By the formula (8) we introduce an operator $R(n,k,r,H^{\Delta})$ that transforms formula of coefficients $[H(x)-h(0)]^k$ to the composita of $x[H(x)]^r$.

Let us consider the equation (7) and represent it in the following way:

$$A(x) = xH^r(A(x)) \tag{9}$$

The equation (9) has type of Lagrange inversion formula with rational power r.

Theorem 2.

For the functional equation $B(x) = H(xB(x)^r)$, the solution in term of composita is defined by

$$B_x^{\Delta}(n,k,r) = R\left(n,k,\frac{1}{r},A^{\Delta}\right), \tag{10}$$

where $A^{\Delta}(n,k,r) = \frac{k}{n} H_r^{\Delta}(2n-k,n)$.

Example 1. Applying obtained results for the equation

$$B(x) = \frac{1}{1 - x\,B(x)^r} \tag{11}$$

we get that the composita of $xB(x)$ is equal to

$$B^{\Delta}(n,m,r) = \sum_{k=0}^{n-m} \binom{\frac{m}{r}}{k} \sum_{j=0}^{n-m} \frac{j\ (-1)^{j-k}}{n-m+j} \binom{k}{j} \binom{r\ (n-m+j)+n-m-1}{n-m} \tag{12}$$

and expression for the coefficients for $B(x)$ is equal to

$$b(n,r) = \sum_{k=0}^{n} \binom{\frac{1}{r}}{k} \sum_{j=0}^{n} \frac{j\ (-1)^{j-k}}{n+j} \binom{k}{j} \binom{r\ (n+j)+n-1}{n} \tag{13}$$

Conclusion

Using properties of composita and the Lagrange inversion theorem we have obtained the explicit formula for the solution of the following generalized equation $B(x) = H(xB(x)^r)$, where $H(x)$ and $B(x)$ are generating functions with free term not equal to 0, and r is any rational number. Those results can contribute to the development of methods for solving functional and iterative equations related to the Lagrange inversion theorem.

Acknowledgements

The reported research was funded by Russian Foundation for Basic Research and the government of the Tomsk region of the Russian Federation, grant number 18-41-703006.

References

[1] R. P. Stanley, Enumerative Combinatorics, Volume 2, Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press; 1999.

[2] Ira M. Gessel, Lagrange inversion, J. COMB. THEORY A. 144 (2016), 212–249.

[3] D. Merlini, R. Sprugnoli, and M. C. Verri, Lagrange inversion: when and how, Acta Appl. Math. 94 (2006), no. 3, 233–249.

[4] D. V. Kruchinin, On solving some functional equations, Adv. Differ. Equ. 1 (2016), 1–7.

[5] D. V. Kruchinin and V. V. Kruchinin, A method for obtaining generating functions for central coefficients of triangles, J. Integer Seq. 15 (2012), no. 9, 1–10.

[6] D. V. Kruchinin and V. V. Kruchinin, Application of a composition of generating functions for obtaining explicit formulas of polynomials, J. Math. Anal. Appl. 404 (2013), no. 1, 161–171.

[7] V. V. Kruchinin and D. V. Kruchinin, Composita and Its Properties, Journal of Analysis and Number Theory 2 (2014), no. 2, 1–8.

[1] Department of Complex Information Security of Computer Systems, Tomsk State University of Control Systems and Radioelectronics (TUSUR)

[2] Department of Applied Mathematics and Informatics, Tomsk State University of Control Systems and Radioelectronics(TUSUR)

E-mail : KruchininDm@gmail.com, shablya-yv@mail.ru, kru@ie.tusur.ru

Application of the method of compositae in combinatorial generation

Yuriy Shablya[1], *Dmitry Kruchinin*[1], *Vladimir Kruchinin*[2]

Abstract

In this paper we study an application of the method for obtaining explicit expressions for the coefficients of generating functions in combinatorial generation. This method is based on the notion of compositae and applied to find an expression of the cardinality function of a combinatorial set by using the known expression of the generating function for the sequence of values of this cardinality function.

2010 Mathematics Subject Classifications: 05A15, 68R05
Keywords: Combinatorial generation, Combinatorial set, Cardinality, Generating function, Composita

Introduction

A combinatorial set is a finite set whose elements have some structure and there is an algorithm for constructing the elements of this set. Elements of combinatorial sets (combinatorial objects) like combinations, permutations, partitions, graphs, trees, etc. play an important role in mathematics and computer science and also have many applications in practice.

D. Knuth [1] gives an overview of formation and development of the direction related to designing combinatorial algorithms, and special attention is paid to the procedure for traversing all possible elements of the given combinatorial set. F. Ruskey [2] introduces the concept of combinatorial generation and distinguishes the following four problems in this area:

1. listing – generating elements of the given combinatorial set sequentially;
2. ranking – ranking (numbering) elements of the given combinatorial set;
3. unranking – generating elements of the given combinatorial set in accordance with their ranks (numbers);
4. random selection – generating elements of the given combinatorial set in a random order.

Methods of combinatorial generation

There are several basic general methods for developing algorithms for combinatorial generation:

- backtracking [3];
- ECO-method [4, 5];

- P. Flajolet's method [6, 7];
- B.Y. Ryabko's method [8];
- V.V. Kruchinin's method [9].

Also, there are many algorithms for combinatorial generation that are based on features of the applied combinatorial set. Therefore, the methods for developing such algorithms cannot be universal.

Table 1: Comparing methods of combinatorial generation

Method	Characteristic				
	Listing	Ranking/ Unranking	More than one parameter	Bijection	Additional requirements
Backtracking	+	−	+	−	
ECO-method	+	−	−	+	Cardinality, ECO-rule
Flajolet's method	+	+	−	+	Cardinality, generating function, specification
Ryabko's method	+	+	+	+	Cardinality, additional functions
Kruchinin's method	+	+	+	+	Cardinality in $\{\mathbb{N}, +, \times, R\}$

Table 1 presents the results of comparing the main characteristics of the general methods for developing algorithms for combinatorial generation:

- "Listing": there is an opportunity to develop algorithms for listing combinatorial objects;
- "Ranking/Unranking": there is an opportunity to develop algorithms for ranking and unranking combinatorial objects;
- "More than one parameter": there is an opportunity to apply the method for combinatorial sets described by more than one parameter;
- "Bijection": there is a requirement to represent combinatorial objects in a special form;
- "Additional requirements": there are requirements for additional information describing a combinatorial set.

Thus, it can be noted that the main characteristic, which is necessary in developing algorithms for combinatorial generation, is the cardinality function of a combinatorial set. If the expression for the cardinality function is unknown, then the development of algorithms for combinatorial generation becomes a difficult task. For solving this problem, we propose to apply the theory of generating functions, since it is one of the basic approaches in modern combinatorics and generating functions are already known for many combinatorial sets.

Method of compositae

For the coefficients of the powers of generating functions, the notion of the composita of a generating function was introduced in [10]. The compositae formed the basis of the mathematical apparatus of the powers of generating functions.

The composita of an ordinary generating function $G(t) = \sum_{n>0} g_n t^n$ is the function with two variables $G^{\Delta}(n,k)$, which is a coefficients function of the k-th power of the generating function $G(t)$:

$$(G(t))^k = \sum_{n \geq k} G^{\Delta}(n,k) t^n.$$

This mathematical apparatus provides such operations on compositae as shift, addition, multiplication, composition, reciprocation, and compositional inversion of generating functions. Such operations on compositae allows us to obtain explicit expressions for the coefficients of generating functions.

For example, if we get a generating function $G(t) = \sum_{n>0} g_n t^n$ with its composita $G^{\Delta}(n,k)$, then the values of the coefficients g_n can be obtained by using

$$g_n = G^{\Delta}(n,1).$$

Also if we can represent the given generating function in the form of a composition of two generating functions

$$G(t) = R(F(t)) = \sum_{n \geq 0} g_n t^n,$$

where $R(t) = \sum_{n \geq 0} r_n t^n$ and $F(t) = \sum_{n>0} f_n t^n$, then the values of the coefficients g_n can be obtained by using

$$g_n = \begin{cases} r_0, & n = 0; \\ \sum\limits_{k=1}^{n} F^{\Delta}(n,k) r_k, & n > 0. \end{cases}$$

To obtain an explicit expression for the coefficients of a generating function using the method of compositae, it is necessary to decompose the given generating function into functions for that the compositae are known, and then apply the corresponding operations to them.

If for the given combinatorial set A we consider its subset $A_n \subset A$, which contains only the combinatorial objects of size n, then the cardinality function $f(n) = |A_n|$ of this combinatorial set A_n can be described by a generating function

$$F(t) = \sum_{n \geq 0} f_n t^n = \sum_{n \geq 0} f(n) t^n = \sum_{n \geq 0} |A_n| t^n.$$

Hence, to obtain an expression for the cardinality function $f(n)$ of the combinatorial set for that the generating function is known, we can apply the method of composita for obtaining explicit expressions of the coefficients of generating functions.

For example, in [11] we have studied the combinatorial set, for that the cardinality function is defined by the elements of Euler-Catalan's number triangle. For this combinatorial set using the method of compositae, we have obtained a bivariate generating function and an explicit expression for the cardinality function.

Conclusion

Using the method of compositae, it is possible to obtain explicit expressions for the coefficients of generating functions. This method can be applied in combinatorial generation when we need to get an expression of the cardinality function of a combinatorial set. For that we must know an expression of the generating function for the sequence of values of this cardinality function.

Acknowledgements

The reported study was funded by Russian Science Foundation (project no. 18-71-00059)

References

[1] D. E. Knuth, The Art of Computer Programming (Volume 4A: Combinatorial Algorithms, Part 1), Addison-Wesley; 2011.

[2] F. Ruskey, Combinatorial generation; 2003. URL: http://page.math.tu-berlin.de/ felsner/SemWS17-18/Ruskey-Comb-Gen.pdf

[3] D. L. Kreher, D. R. Stinson, Combinatorial Algorithms: Generation, Enumeration, and Search, CRC Press; 1999.

[4] E. Barcucci, A. Lungo, E. Pergola, R. Pinzani, ECO: A methodology for the enumeration of combinatorial objects, Journal of Difference Equations and Applications 5 (1999), 435–490.

[5] S. Bacchelli, E. Barcucci, E. Grazzini, E. Pergola, Exhaustive generation of combinatorial objects by ECO, Acta Informatica 40 (2004), 585–602.

[6] P. Flajolet, P. Zimmerman, B. Cutsem, A calculus for the random generation of combinatorial structures, Theoretical Computer Science 132 (1994), 1–35.

[7] C. Martinez, X. Molinero, A generic approach for the unranking of labeled combinatorial classes, Random Structures and Algorithms 19 (2001), 472–497.

[8] B. Y. Ryabko, Fast enumeration of combinatorial objects, Discrete Mathematics and Applications 8 (1998), 163–182.

[9] V. V. Kruchinin, Methods for developing algorithms for ranking and unranking combinatorial objects based on and-or trees, Tomsk: V-Spektr; 2007. (in rus)

[10] V. V. Kruchinin, D. V. Kruchinin, Composita and its properties, Journal of Analysis and Number Theory 2 (2014), 37–44.

[11] Y. V. Shablya, D. V. Kruchinin, Euler-Catalan's number triangle and its application, Proceedings book of the Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (2018), 212–216.

[1] Department of Complex Information Security of Computer Systems, Tomsk State University of Control Systems and Radioelectronics (TUSUR)

[2] Department of Applied Mathematics and Informatics, Tomsk State University of Control Systems and Radioelectronics (TUSUR)

E-mail : KruchininDm@gmail.com, shablya-yv@mail.ru, kru@ie.tusur.ru

Remarks and Observations On Combinatorial and Changhee Numbers and Polynomials

Deayeoul Kim[1] , *Yilmaz Simsek*[2] , *Ji Suk So*[3]

Abstract

In this presentation, we study and survey some properties of combinatorial numbers and polynomials with their generating functions. We also investigate some properties of Changhee numbers and polynomials. We give some relations and identities including Changhee numbers and polynomial, combinatorial numbers and polynomials, Apostol-type numbers and polynomials and others special numbers and polynomials. Finally, we give some notes and observations on these numbers and polynomials with their generating functions.

2010 Mathematics Subject Classifications : 05A15, 05A10, 11B83, 26C05
Keywords: Generating function, Euler numbers and polynomials, Stirling numbers, Changhee numbers, Combinatorial Numbers and Polynomials

Introduction

Generating functions for special numbers and polynomials have various applications in mathematics and in other applied sciences. Recently, different methods and definitions, many new kinds of special numbers and polynomials have been presented (*cf.* [1]-[20]).

In this presentation, we use the following generating functions for special polynomials and numbers which were studied and investigated in [1]-[20]:

The Euler polynomials of order k are defined by

$$F_E(t,x;k) := \left(\frac{2}{e^t+1}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}.$$

The Stirling numbers of the first kind are defined by

$$F_{S_1}(t,k) := \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!}.$$

The λ-Stirling numbers are defined by

$$F_{S_2}(t,k;\lambda) = \frac{(\lambda e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}. \tag{1}$$

Setting $\lambda = 1$ in (1), we have the Stirling numbers of the second kind:

$$S_2(n,k) = S_2(n,k;1).$$

Let m be any integer. The Changhee polynomials of order m are given by the following generating function (*cf.* [3]-[19]):

$$F(t,x,m) = \frac{2^m(1+t)^x}{(2+t)^m} = \sum_{n=0}^{\infty} Ch_n^{(m)}(x)\frac{t^n}{n!}. \tag{2}$$

Setting $x = 0$ into equation (2), we have the Changhee polynomials of order d, which defined by means of the following generating function:

$$\frac{2^d}{(2+t)^d} = \sum_{n=0}^{\infty} Ch_n^{(d)}\frac{t^n}{n!} \tag{3}$$

(*cf.* [4]) which, for $d = 1$, yields $Ch_n(x) = Ch_n^{(1)}(x)$ denote the Changhee polynomials and $Ch_n = Ch_n(0)$ denotes the Changhee numbers which is given by

$$Ch_n = \left(-\frac{1}{2}\right)^n n!$$

(*cf.* [3], [4]).

Main Results

In [2], the authors have studied and investigated various properties of the Changhee numbers and polynomials of negative order with the light of combinatorial numbers and polynomials which were defined by Simsek [8]-[17]. In this section, we give some identities and relations for the Changhee numbers and polynomials of negative order and also combinatorial numbers and polynomials.

The authors [2] defined the following generating functions for the Changhee polynomials of order $-k$:

$$H(t,x,-k) = \frac{(1+t)^x\,(2+t)^k}{2^k} = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x)\frac{t^n}{n!}, \tag{4}$$

where k is a positive integers.

The authors gave the following computation formula for the Changhee polynomials of order $-k$:

Theorem 1. *([2]) Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)}(x) = \frac{1}{2^k}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(j)_{n-l}(x)_l. \tag{5}$$

The authors [2] defined the following generating functions for the Changhee numbers of order $-k$:

$$K(t,k) = \frac{(2+t)^k}{2^k} = \sum_{n=0}^{\infty} Ch_n^{(-k)}\frac{t^n}{n!}. \tag{6}$$

The authors gave the following computation formula for the Changhee numbers of order $-k$:

Theorem 2. *([2]) Let $n \in \mathbb{N}_0$. Then we have*

$$Ch_n^{(-k)} = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(j)_n. \tag{7}$$

Conclusions

It is well-known that special polynomials and numbers with their generating functions have very important applications in many branches of mathematics, probability, statistics, mathematical physics, engineering, and also applied sciences. Thanks to the relationship between generating functions, (partial) differential equations and functional equations, very useful and novel identities and relations for special numbers and polynomials are investigated and found. Special numbers and polynomials of this presentation are related to many well-known numbers and polynomials such as the Apostol-type numbers and polynomials, the Stirling numbers, the Boole numbers and polynomials, Peters type polynomials and numbers, the Changhee numbers and polynomials, and also the other combinatorial numbers and polynomials. The results of this paper including especially the Changhee numbers and polynomials of negative order and combinatorial numbers and polynomials defined by Simsek [8]-[17]. Thus, we think that the aforemationed results may be used in mathematics, in mathematics, probability, statistics, mathematical physics, engineering, and also applied sciences.

Acknowledgements

This research is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(NRF-2018R1D1A1B07041132)

The present investigation was supported by Scientific Research Project Administration of Akdeniz University

References

[1] N. P. Cakic, B. S. El-Desouky, G. V. Milovanovic, *Explicit formulas and combinatorial identities for generalized Stirling numbers*, Mediterr. J. Math. **10** (2013), 57–72.

[2] D. Kim, Y. Simsek and J. S. So, *Computation formulae for Combinatorial numbers and Changhee polynomials*, preprint

[3] D. S. Kim, T. Kim, J. Seo, *A note on Changhee numbers and polynomials*, Adv. Stud. Theor. Phys. **7** (2013), 993–1003.

[4] D. S. Kim, T. Kim, J. J. Seo, *Higher-order Changhee Numbers and Polynomials*, Adv. Studies Theor. Phys. **8**(8) (2014), 365–373.

[5] T. Kim, D. V. Dolgy, D. S. Kim, J. J. Seo, *Differential equations for Changhee polynomials and their applications*, J. Nonlinear Sci. Appl. **9** (2016), 2857–2864.

[6] D. V. Kruchinin, V. V. Kruchinin, *Application of a composition of generating functions for obtaining explicit formulas of polynomials*, J. Math. Anal. Appl. **404**(1) (2013), 161–171.

[7] S. Roman, *The Umbral Calculus. Academic Press*, New York. (1984).

[8] Y. Simsek, *Generating functions for generalized Stirling type numbers, array type polynomials*, Eulerian type polynomials and their alications. Fixed Point Theory Appl. **87** (2013), 343–355.

[9] Y. Simsek, *Analysis of the p-adic q-Volkenborn integrals: An approach to generalized Apostol-type special numbers and polynomials and their applications*, Cogent Math. **3**(1269393) (2016), 1–17.

[10] Y. Simsek, *Apostol type Daehee numbers and polynomials*, Adv. Stud. Contemp. Math. **26**(3) (2016), 1–12.

[11] Y. Simsek, *Identities on the Changhee numbers and Apostol-Daehee polynomials*, Adv. Stud. Contemp. Math. **27**(2) (2017), 199-212.

[12] Y. Simsek: *Construction of some new families of Apostol-type numbers and polynomials via Dirichlet character and p-adic q-integrals*, Turk. J. Math. **42** (2018), 557–577.

[13] Y. Simsek, *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials*, Appl. Anal. Discrete Math., **12** (2018), 1–35. .

[14] Y. Simsek, *Computation methods for combinatorial sums and Euler type numbers related to new families of numbers*, Math. Meth. Appl. Sci., **40** (2017), 2347–2361.

[15] Y. Simsek, *Construction method for generating functions of special numbers and polynomials arising from analysisof new operators*, Math. Meth. Appl. Sci., **41** (2018), 6934–6954.

[16] Y. Simsek, *Generation functions for finite sums involving higher powers of binomial coeffcients: Analysis of hypergeometric functions including new families of polynomials and numbers*, J. Math. Anal. Appl. **477** (2019), 1328–1352.

[17] Y. Simsek, *Peters type polynomials and numbers and their generating functions: Approach with p-adic integral method*, Math Meth Appl Sci. 2019;1–17; DOI: 10.1002/mma.5807.

[18] Y. Simsek, J. S. So, *Identities, inequalities for Boole type polynomials: approach to generating functions and infinite series*, J. Inequal. Appl. 2019:62 (2019), 1-11, https://doi.org/10.1186/s13660-019-2006-x.

[19] H. M. Srivastava, I. Kucukoglu, Y. Simsek, *Partial differential equations for a new family of numbers and polynomials unifying the Apostol-type numbers and the Apostol-type polynomials*, J. Number. Theory. **181** (2017), 117–146.

[20] H. M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York. (2012).

[1]Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University

[2]Department of Mathematics, Faculty of Science, Akdeniz University

[3]Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University

E-mail : kdaeyeoul@jbnu.ac.kr, ysimsek@akdeniz.edu.tr, goleta961@jbnu.ac.kr

Thue equations and Indices in number fields

Abdelmejid Bayad[1], *Mohammed Seddik*[2]

Abstract

Let $f \in \mathbb{Z}[X, Y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and k be an integer. In 1909 Thue proved that the diophantine equation

$$f(x, y) = k$$

has only finitely many solutions in integers x and y.
His method does not give explicitly the solutions. In this talk, first we review old and new results on this direction.
We investigate the theory of *indices in cubic number fields*. We then obtain new method to study Thue Equations. We obtain precise results when f is irreducible binary cubic form.
As a consequence of our study we will be able to obtain information on the number of integers and also of the number of rational points of the *Mordell's elliptic curves*:

$$E_d : dy^2 = x^3 + Ax + B.$$

These families are extremely studied in the literature and are sources of intensive research at present in the field of elliptic curves.
In particular, there is connection between the number of integers points on the curves E_d and the rank of E_d.

2010 Mathematics Subject Classifications : 11R04; 11DXX; 11R16; 11R33; 11R09.

Keywords: Cubic Thue equations, cubic fields, common index divisors of cubic fields.

Introduction and motivation

Let $f \in \mathbb{Z}[X, Y]$ be a homogeneous irreducible polynomial of degree $n \geq 3$ and k be an integer. In 1909 Thue proved that the diophantine equation

$$f(x, y) = k$$

has only finitely many solutions in integers x and y.
His method does not give explicitly the solutions. In this talk, first we review old and new results on this direction.

We investigate the theory of *indices in cubic number fields* . We then obtain new method to study Thue Equations. We obtain precise results when f is irreducible binary cubic form.
As a consequence of our study we will be able to obtain information on the number of integers and also of the number of rational points of the *Mordell's elliptic curves* :

$$E_d : dy^2 = x^3 + Ax + B.$$

These families are extremely studied in the literature and are sources of intensive research at present in the field of elliptic curves.
In particular, there is connection between the number of integers points on the curves E_d and the rank of E_d.

Main results

Let N be any integer. We denote by $v_2(N)$ the greatest exponent s such that 2^s divides N. The discriminant of the form

$$F(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

is the invariant

$$D = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2. \tag{1}$$

The binary form F has the quadratic and cubic covariants

$$H(x,y) = A_0x^2 + B_0xy + C_0y^2, \tag{2}$$
$$G(x,y) = A_1x^3 + B_1x^2y + C_1xy^2 + D_1y^3. \tag{3}$$

where

$$\begin{aligned} A_0 &:= b^2 - 3ac, B_0 := bc - 9ad, C_0 := c^2 - 3bd; \\ A_1 &:= -2b^3 - 27a^2d + 9abc, B_1 = 3(b^2c - 9abd + 6b^2d), \\ C_1 &= 3(bc^2 + 9acd - 6b^2d), D_1 = (2c^3 + 27ad^2 - 9bcd). \end{aligned} \tag{4}$$

the quadratic form H is the Hessian and G is the gradient of F, see [29, pp.213].

Throught this paper, we assume that $D \neq 0$ and $\gcd(a,b,c,d) = 1$. Now we state our main result.

Theorem 1. *Let a, b, c, d and k be integers such that*

$$2v_2(A_1) = 3v_2(A_0), \ v_2(k) \equiv 1, 2 \pmod 3.$$

Then the cubic Thue Diophantine equations

$$ax^3 + bx^2y + cxy^2 + dy^3 = k$$

has no integer solution (x, y).

Corollary 2. *Let $(n, k) \in \mathbb{Z}^2$ where $3 \nmid v_2(k)$. Then the following two families of cubic Thue equations*

1. $x^3 - (n^3 - 2n^2 + 3n - 3)x^2y - n^2xy^2 - y^3 = k, \ n \not\equiv 1 \pmod 4,$
2. $x^3 - n(n^2 + n + 3)(n^2 + 2)x^2y - (n^3 + 2n^2 + 3n + 3)xy^2 - y^3 = k.$

have no integer solution.

Corollary 3. *Let a, b, c, d, k as in Theorem 1. Then the homogeneous form*

$$ax^3 + bx^2y + cxy^2 + dy^3 = kz^3$$

has only the integer solution $(x, y, z) = (0, 0, 0)$.

Corollary 4. *Let a, b, c, d as in Theorem 1, and e be integers such that $v_2(e) \geq v_2(A_1)$ and $v_2(e) \equiv 1 \pmod 3$. Then the familly of twisted ellitpic curves*

$$E: \ ax^3 + bx^2 + cx + d = ey^2 \tag{5}$$

have no integer points (x, y).

References

[1] E. T. Avanesov: *A bound for the number of representations by a special class of binary cubic forms of positive discriminant.* Acta Arith, **20** (1972), 17-31.

[2] M. Ayad: *Automorphismes d'une forme binaire cubique et représentation d'entiers.* C. R. Acad. Sci. Paris Sér. I Math, **299** (1984), no. 19, 959-962.

[3] A. Baker: *Contribution to the theory of Diophantine equations.* Phil. Trans. Roy. Soc. London, Ser. A, **167** (1967/68), 717-208.

[4] M. A. Bennett: *On the representation of unity by binary cubic forms.* Trans. American Math. soc. Vol, **353** (2000), 1507-1534.

[5] M. A. Bennett, A. Ghadermarzi: *Extremal families of cubic Thue equations.* Journal de Théorie des nombres de Bordeaux, **27** (2015), 389-403.

[6] Yu. Bilu, G. Hanrot: *Solving Thue equations of high degree.* J. Number Theory, **442** (1993), 19-62.

[7] Y. Bugeaud, K. Györy: *Bounds for the solutions of Thue-Mahler equations and norm form equations.* Acta Arith, **74** (1996), 273-292.

[8] G. V. Chudnovsky: *On the Thue-Siegel.* Ann. of Math. **117** (1983), 325-382.

[9] T. W. Cusick: *Lower bound for regulators.* Number Theory Noordwijkerhout Springer Verlag (1983), 63-73, Ed : H. Jager, LNM Vol. **1068**, Springer Verlag (1984).

[10] B. Delauney: *Über die Darstellung der Zahlen durch die binären kubischen Formen von negative Discriminante.* Math. Z. **31** (1930), 1-26.

[11] H. Davenport, K. F. Roth: *Rational approximations to algebraic numbers.* Mathematika **2** (1955), 160-167.

[12] P. Erdös, K. Mahler: *On the number of integers which can be represented by a binary form.* J. London Math. Soc, **13** (1938), 134-139.

[13] J. H. Evertse: *On the Representation of Integers by Binary Cubic Forms of Positive Discriminant.* Invent. Math. **73** (1983), 117-138.

[14] C. Heuberger, A. Pëtho, R. F. Tichy: *Complete solution of parametrized Thue equations.* Acta Math. Inform. Univ. Ostraviensis. **6** (1998), 91-113.

[15] A. Hoshi: *On correspondence between solutions of a family of cubic Thue equations and isomorphism classes of the simplest cubic fields.* J. Number Theory. **131** (2011), 2135-2150.

[16] J. G. Huard, B. K. Spearman, K. S. Williams: *A short proof of the formula for the conductor of an abelian cubic field.* Norske Vid. Selsk. Skr. **2** (1994), 3-8.

[17] Y. Kishi: *A family of cyclic polynomials whose roots are systems of fundamental units.* J. Number Theory **102** (2003), 90-106.

[18] O. Lecacheux: *Units in number fields and elliptic curves.* Advances in Number Theory, Oxford Science Publications, Oxford University Press, New York (1993), 293-301.

[19] E. Lee: *Studies on diophantine equation.* PhD thesis, Cambridge University, 1992.

[20] F. Lemmermeyer, A. Pëtho: *Simplest cubic fields.* Manuscripta Math. **88** (1995), 285-321.

[21] G. Lettl, A. Pëtho, P. Voutier: *Simple families of Thue inequalities.* Trans. Amer. Math. Soc. **351** (1999), 1871-1894.

[22] P. Llorente, E. Nart: *Effective determination of the decomposition of the rational primes in a cubic field.* Proc. Amer. Math. Soc. **87** (1983), 579-585.

[23] D. Lewis, K. Mahler: *Representations of integers by binary forms.* Acta. Arith. **6** (1961), 333-363.

[24] K. Mahler: *Zur Approximation algebraischer Zahlen.* II Über die Anzahl der Darstellungen ganzer Zahlen durch Binärformen, Math. Ann. **108** (1933), 37-55.

[25] M. Mignotte: *Verification of a conjecture of E. Thomas.* J. Number Theory **44** (1993), 172-177.

[26] M. Mignotte, A. Pëtho, F. Lemmermeyer: *On the family of Thue equations* $x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = k$. Acta Arith. **76** (1996), no. 3, 245-269.

[27] M. Mignotte, N. Tzanakis: *On family of cubics.* J. Number Theory. **39** (1991), 41-49.

[28] M. Mignotte, N. Tzanakis: *Pethö cubics.* Preprint.

[29] L. J. Mordell: *Diophantine Equations.* Academic Press, London, (1969).

[30] T. Nagell: *Remarques sur une classe d'équations indéterminées.* Arkiv för Math. **8** (1969), 199-214.

[31] Blair K. Spearman, Kenneth S. Williams: *Indices of Integers in Cyclic Cubic Fields.* International Mathematical Forum, **3**, 2008, no. 32, 1595 - 1606.

[32] A. Pethö, R. Schulenberg: *Effektives Lösen von Thue Gleichungen.* Publ. Math. Debrecen. **34** (1987), 189-196.

[33] D. Shanks: *The simplest cubic fields.* Math. Comp. **28** (1974), 1137-1152.

[34] C. L. Siegel: *Über einige Anwendungen diophantischer Approximationen.* Abh. preuss. Akad. Wiss. Phys. Math. Kl. (1929) No. 1.

[35] E. Thomas: *Complete solutions to a family of cubic diophantine equations.* J. Number Theory. **34** (1990), 235-250.

[36] E. Thomas: *Solutions to certain families of Thue equations.* J. Number Theory. **43** (1993), 319-369.

[37] A. Thue: *Über Annäherungswerte algebraischer Zahlen.* J. reine angew. Math. **135** (1909), 284-305.

[38] A. Togbé: *A parametric family of cubic Thue equations.* J. Number Theory. **107** (2004), 63-79.

[39] A. Togbé: *Complete solutions of a family of cubic Thue equations.* Journal de Théorie des Nombres de Bordeaux. **18** (2006), 285-298.

[40] A. Togbé: *On the solutions of a parametric family of cubic Thue equations.* Bull. Braz. Math. Soc. **39** (4) (2008), 537-554.

[41] N. Tzanakis, B. M. M. de Weger: *On the practical solution of the Thue equation.* J. Number Theory. **32** (1989), 31-99.

[42] I. Wakabayashi: *On family of cubic Thue equations with 5 solutions.* Acta Arithmetica. **109** (2003), 285-298.

[43] I. Wakabayashi: *Number of solutions for cubic Thue equations with automorphisms.* Ramanujan J. **14** (2007), 131-154.

[44] I. Wakabayashi: *Simple families of Thue inequalities.* Ann. Sci. Math. Comp. **31** (2007), 211-232.

[45] I. Washington: *A family of cubic fields and zeros of 3-adic L-functions.* J. Number Theory. **63** (1997), 408-417.

[1]Université d'Évry Val d'Essonne
[2]University of Paris Saclay

E-mail : abdelmejid.bayad@univ-evry.fr, seddik.mohamed2011@gmail.com

Polygon shape numbers derived from restricted divisor functions

Daeyeoul Kim[1] , *Abdelmejid Bayad*[2] , *SungHo Yoo*[3]

Abstract

In this paper, we'll create the polygons using restricted divisor functions that by specific number and explain it. We shall define the m-gonal shape numbers and non-polygon(or sandglass) shape numbers. Also we will define high flat or low flat as similar to the method in [1].

2010 Mathematics Subject Classifications : 11M36, 11F11, 11F30
Keywords: Restricted divisor function, divisor functions, Mersenne prime.

Introduction

A study of the divisor function is closely related to Mersenne primes, Euler φ-function, perfect number, basic hyper-geometric series, partition theory, coding theory, modular form theory, Riemann zeta functions, etc. Moreover, the divisor function has many unsolved problems in [7, Chapter B].

Divisor function was studied by Euler, Lambert, Waring, Jacobi, Dirichlet, Stern, Bouniakowsky, Liouville, Hermite, Glaisher, Ramanujan, Williams, etc [5, Chapter X].

Recently, several restricted divisor functions have been studied.

Iterating the sum of divisor function and their applications have been investigated by Cohen and Riele in [4]. A few more related references are [6], [13], [14]. It should be mentioned that iteration theory associated with odd divisor functions have been studied by Bayad and Kim in [1]. We want to consider the properties of the restricted divisor functions as in [1].

In this paper we use the following notations:
$\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ will be denoted by the set of natural numbers, the set of non-negative integers and the ring of integers, respectively.

Let $a, d, m \in \mathbb{N}$ and $t \in \mathbb{N}_0$.

- $D_1^{(t)}(n) := \sum_{d|n} \mu(d)^2 d^t$,
- $S(n) := \sum_{\substack{d|n \\ d \text{ odd}}} d$,
- $D_m^{(t)}(n) := D_1^{(t)}(D_{m-1}^{(t)}(n))$,
- $D_m(n) := D_1^{(1)}(D_{m-1}^{(1)}(n))$,
- $D(n) := D_1^{(1)}(n) = D_1(n)$,
- $D_0(n) := n$,
- $\mathbb{M}_p$: Mersenne primes,
- $\mu(n)$: Möbius μ-function of n.

*Corresponding author

A natural number n has

$$\begin{cases} \text{Type } \mathbf{A} & \text{if } n > D(n) > D_2(n) \text{ and } n - 2D(n) + D_2(n) < 0, \\ \text{Type } \mathbf{B} & \text{if } D(n) > n > D_2(n), \\ \text{Type } \mathbf{C} & \text{if } n > D(n) > D_2(n) \text{ and } n - 2D(n) + D_2(n) > 0, \\ \text{Type } \mathbf{D} & \text{if } n < D(n) < D_2(n) \text{ and } n - 2D(n) + D_2(n) > 0, \\ \text{Type } \mathbf{E} & \text{if } n < D(n) < D_2(n) \text{ and } n - 2D(n) + D_2(n) < 0, \\ \text{Type } \mathbf{F} & \text{if } n > D_2(n) > D(n), \\ \text{Type } \mathbf{G} & \text{if } n < D_2(n) < D(n). \end{cases}$$

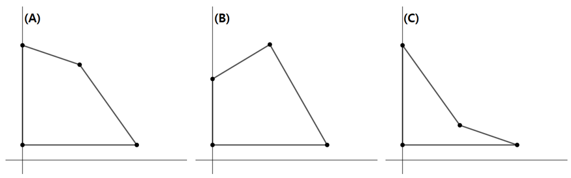


Figure 1: (A) Type **A**, (B) Type **B**, (C) Type **C**

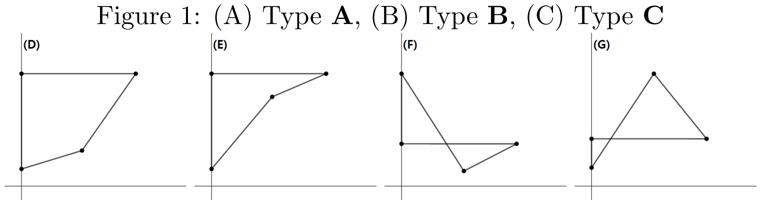


Figure 2: (D) Type **D**, (E) Type **E**, (F) Type **F**, (G) Type **G**

Main Results

Proposition 1. *Let n be a natural number with $ord(n) = 2$ and $w(n) \leq 2$. Then n has*

$$\begin{cases} \textit{Type } \mathbf{A} \textit{ if and only if } n = 2^2 \cdot \mathbb{M}_{p_1}(\neq 3),\ 2^2 \cdot \mathbb{P}_{i_1,j_1}, \\ \textit{Type } \mathbf{B} \textit{ if and only if } n = \mathbb{M}_{p_1}(\neq 3),\ \mathbb{P}_{i_1,j_1}(\neq 5, 11),\ 2 \cdot \mathbb{M}_{p_2}(\neq 3),\ 2 \cdot \mathbb{P}_{i_2,j_2}(\neq 5), \\ \qquad\qquad \mathbb{M}_{p_3} \cdot \mathbb{M}_{p_4},\ \mathbb{M}_{p_5} \cdot \mathbb{P}_{i_3,j_3},\ \mathbb{P}_{i_4,j_4} \cdot \mathbb{P}_{i_5,j_5}, \\ \textit{Type } \mathbf{C} \textit{ if and only if } n = \mathbb{M}_{p_1}^{m_1}(\neq 3),\ \mathbb{P}_{i_1,j_1}^{m_2}(\neq 5, 11),\ 2^{a_1} \cdot \mathbb{M}_{p_2}^{b_1}(\neq 3),\ 2^{a_2} \cdot \mathbb{P}_{i_2,j_2}^{b_2}, \\ \qquad\qquad \mathbb{M}_{p_3}^{c_1} \cdot \mathbb{M}_{p_4}^{d_1},\ \mathbb{M}_{p_5}^{c_2} \cdot \mathbb{P}_{i_3,j_3}^{d_2},\ \mathbb{P}_{i_4,j_4}^{c_3} \cdot \mathbb{P}_{i_5,j_5}^{d_3}, \\ \textit{Type } \mathbf{D} \textit{ if and only if } n = 5, \\ \textit{Type } \mathbf{F} \textit{ if and only if } n = 5^{m_1}, \\ \textit{Type } \mathbf{G} \textit{ if and only if } n = 2 \cdot 5. \end{cases}$$

Here, $m_\mu \geq 2$, $(a_\alpha, b_\alpha) \in \mathbb{N} \times \mathbb{N} - \{(1,1),(2,1)\}$ and $(c_\gamma, d_\gamma) \in \mathbb{N} \times \mathbb{N} - \{(1,1)\}$.

Remark 4. *Proposition* 1 *is in [9]*

Conclusion

A study of polygon shape numbers derived from restricted divisor function is an interesting topic introduced by Bayad and Kim [1], [2].

	n
Type **A**	$20, 28, 44, 68, 92, 124, 212, 284, \cdots$
Type **B**	$7, 14, 15, 17, 21, 22, 23, 31, 33, 34, 35, 46, 53, 55, 62, 71,$ $77, 85, 93, 106, 107, 115, 119, 127, 142, 155, 187, 214, 217, \cdots$
Type **C**	$40, 45, 49, 56, 63, 80, 88, 98, 100, 112, 135, 136, 147, 160,$ $175, 176, 184, 196, 200, 224, 225, 242, 245, 248, 272, 275, \cdots$
Type **D**	5
Type **E**	
Type **F**	$25, 125, \cdots$
Type **G**	10

Table 1: Type of n $(1 \leq n \leq 16)$

Polygon shape numbers with $ord_2(n) = 2$ given by [1] are 3-types but small differently, polygon shape numbers with $ord(n) = 2$ suggested by this paper are 6-types. That is, we can interpret that we can make several polygon.

Acknowledgements

This research is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education(NRF-2018R1D1A1B07041132).

References

[1] Bayad, A.; Kim, D. Polygon numbers associated to the sum of odd divisors function. *Exp. Math.* **2017**, 26, 287–297.

[2] Bayad, A.; Kim, D.; Park, S. The restricted divisor functions, Iterated Stable numbers and Iterated Amicable pairs. Submitted.

[3] Cheong, C. *Some Properties of Absolute Möbius Divisor Functions and Convolution Sums of Divisor Functions*, Chonbuk National University, 2016.

[4] Cohen, G.L.; Herman J. J. te Riele. Iterating the sum-of-divisors function. *Experiment. Math.* **1996**, 5, no. 2, 91–100.

[5] Dickson, L.E. *History of the theory of numbers. Vol. I: Divisibility and primality*, Cresse Chelsea Publishing Co: New York, 1966.

[6] Erdös, P. Some remarks on the iterates of the ϕ and σ functions. *Colloq. Math.* **1967**, 17, 195–202.

[7] Guy, R.K. *Unsolved Problems in Number Theory*, Springer: 2004.

[8] Jung, S.; Kim, D.; Kim, J. GeoMaTree : Geometric and Mathematical model based digital tree authoring system. *KSII T Internet Info.* **2018**, 12, no. 7, 3284–3306.

[9] Kim, D.; Bayad, A.; Yoo, S. Seven types of polygon shape numbers derived from restricted divisor functions. *submitted*

[10] Kim, D.; Kim, J. Procedural Modeling and Visualization of Multiple Leaves. *Multimed. Syst.* **2017**, 23, Issue 4, 435–449.

[11] K. I.; M. R. *A Classical Introduction to Modern Number Theory*, Springer-Verlag, Japan, 2014.

[12] Kim, J.; Kim, D.; Cho, H. Procedural Modeling of Trees based on Convolution Sums of Divisor Functions for Real-time Virtual Ecosystems. *Comp. Anim. Virtual Worlds.* **2013**, 24, 237—246.

[13] Lenstra, H.W. *Problem 6064*, Amer. Math. Monthly **1975** , 82, 1016; Solution by the proposer. **1977**, 84, 580.

[14] Lord, G. Even perfect and superperfect numbers. *Elem. Math.* **1975**, 30, 87–88.

[15] https://primes.utm.edu/mersenne/. applle

[1]Department of Mathematics and Institute of Pure and Applied Mathematics
Chonbuk National University
567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896
South Korea
[2]Université d'Evry Val d'Essonne, Université Paris-Saclay, CNRS (UMR 8071)
Laboratoire de Mathématiques et Modélisation d'Évry
Bâtiment I.B.G.B.I., 3ème étage, 23 Boulevard de France
91037 Evry cedex
France
[3]Department of Mathematics and Institute of Pure and Applied Mathematics
Chonbuk National University
567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do 54896
South Korea

E-mail : kdaeyeoul@jbnu.ac.kr, abayad@maths.univ-evry.fr, yoush333@jbnu.ac.kr

Recurrent Process on Appointing an Aid Site: A Case for Airports

Emre Demir[1]

Abstract

Mathematical models have been used for many decades in order to improve the infrastructure design of transportation facilities. In this study, locational best fit airport infrastructure is determined for providing an urgent assistance in case of an emergency requirement throughout the country. The data are supplied from civil airports in Turkey. In order to solve the problem, the mathematical models such as Weber Problem solution methodology is applied for assessing the results. Moreover, the limitations such as distance costs and also the weights of the regional airports such as the passenger counts and the approximate physical size of the airport field are taken into account. Optimizing the circumstance, the economic losses are aimed to be lower, and thus the financial improvement is targeted by the application of the mathematical model and the implementation of its results.

KEYWORDS: Airport, Iterative model, Location appointment, Recurrent algorithm, Weber problem

Introduction

Many research show that mathematical methodologies solve the problems of transportation activities such as discovering the directions with similarities and inferring the paths [1], [2], [3]. Accordingly, with the help of mathematical models, many policies are developed for better transportation activities at a specific region [4].

The concern in this study is to provide immediate air support in case of an emergency situation at any airport within the country. The aim is to determine the location of one large facility for storing the emergency necessities such as health-care equipment and professionals. Hence, the motivation of this aim is satisfied by decreasing the cost of immediate access to an unforeseen disaster. There are applications in the literature which consider the number of airplanes to determine the location of a support facility [5], [6]. Those research took the number of airplanes landed to the airport (i.e. the number of airplanes used the airport for landing and takeoff) as an indication of airport weights. However, since the number of lives are not taken into account for this kind of particular study (e.g. appointing an aid site), this research introduces the number of passengers used the airports rather than the number of airplanes occupied the airports. Therefore, this study aims at deciding on the most convenient transportation location in the case of an intervention, by considering the number of travelers at airports. To achieve the goal, a single point facility location is located by the techniques in the literature to solve the case such that a facility location is determined for a specific industry [7].

One of the best mathematical technique for finding a single facility location in two-dimensional space is called the Weber Problem (WP) [8], [9]. Because the weights of airports (i.e. the number of passengers used the specific airports and the estimated physical sizes of the airport terrains), an iterative technique has to be applied to get the optimum location. One of the most authoritative mathematical method for assessing the optimal facility location is the Weiszfeld method or algorithm (WM) for implementation [10], [11]. Hence, transportation information is analyzed by the help of WP and WM in this research.

Methodology

The operation research problem of WP is an optimization of a location. The main goal on this problem is to minimize the Euclidean distances which are weighted by some kind of costs, to find an optimal location meeting the requirements of all other demand locations [9], [11]. The explanation of WP is as follows.

$$\min \sum_{j=1}^{n} w_j |\bar{x} - \bar{x}_j| \tag{1}$$

based on $\bar{x} = (x, y)$ and $\bar{x}_j = (x_j, y_j)$. Moreover, the demand points $\bar{x}_j$ are members of the surface of E^2. Consequently, the optimization explanation (1) ascertains the most appropriate place of $\bar{x}$. To solve the mathematical problem of WP, a powerful method is introduced in the following.

A successful iterative methodology of Weiszfeld Method or Weiszfeld Algorithm (WM) can confidently solve WP with by utilizing the software of LINGO [10], [12], [13]. The mathematical location allocation problem is dealt with WM which optimizes the weights in terms of the demands. Hence, the following WM description can be implemented in this study.

$$\min \sum_{i=1}^{n} a_i D_i \tag{2}$$

so that $a_i D_i$ in the formulation (2) is the weight times the distance which means the Euclidean cost (e.g. the distance) which is weighted for the location of final site location (X_f, Y_f). n is the number of demand locations requesting service such that affecting the final decision of the site. Additionally the final position of the place must satisfy the mathematical inequality of $(X_f, Y_f) \geq 0$, as the final output must be on the fixed topography.

The computer program applies the methodology of the combination of WP and the recurrent method of WM to the weighted Euclidean costs (e.g. distances) of demand locations at a fixed topography surface. The program runs until the iterations give the same result accordingly. Then the optimization with mathematical iterations stops, and the final outcome can be stated. Because the site of the facility has a weighted importance such as the capacity, physical size of the area, etc., the inputs from WP can be accounted as the inputs for a solution of WM.

Application to the Case Study

First of all, the scope of this case includes not only the number of passengers that visit the airport in the last six months as *parameter-1*, but also the physical size of the airports lands as *parameter-2*. These parameters decide the weight of the airports included in the study. Totally 55 civil airports in Turkey are used in the analysis. *Parameter-1* having a weight of 0.8 is assigned, since the probability of having causalities increases as the number of lives around increases. In this manner, *parameter-2* having a weight of 0.2 is assigned, because the terrain area size is important in order to establish a facility. Required area should be selected as large as possible to store services for emergency situations. Thus, the total weight of the parameters from each demand source as demanding place is equal to 1.00. Additionally, the positional data of all demand sources (i.e. 55 airports) are considered to finalize the location of the aid site. Position information delivers very vital constraint for cost that is labeled as distance for this research.

Major Outcomes

The algorithm with the objective function of (2) that is applied to the data in this research is suitable to be launched in the software of LINGO 17.0 x64.lnk [13]. Therefore, this solver program was operated by a computer having Intel®Core^{TM} i7-2640M CPU@2.80GHz. The solver processed the data input and provided the result of (0.6956857, 0.5563164) as the optimum solution of (X_f, Y_f) coordinate. This spatial output of 0.6956857 and 0.5563164 serve as 39.85985448° N and 30.34723505° E respectively which belong to $39°51'35.5''$ N, $30°20'50.1''$ E correspondingly, according to the Mercator projection. The solver reported that this result is the local optimum solution with no infeasibilities. However, since the problem here was convex, the output of local optimum was literally the global optimum. Hence the verification of results was done.

Conclusions

Mathematical models are being used to solve several problems of infrastructures in many engineering areas and applications. In our case, the location of the aid site is appointed and proposed by the application of WP which is a mathematical problem, and WM which is a mathematical iterative technique. Spatial results are gained by the evaluation of the parameters of not only the weights but also the costs as distances of demand locations. As mentioned earlier, the weights are considered as the factors of the numbers of passengers visited the airport in the last six months, and the physical areas of the airports. In accordance with the mathematical appraisal in this research, the aid site for the airport throughout the country is proposed to be erected as soon as possible. By taking the outcomes of this study into consideration, making great contribution to the state planning, cutting the budgets of traveling distances and mileage use, and more importantly rescuing lives quicker than earlier times during the interventions are anticipated.

References

[1] T.M. Matisziw, E. Demir, Measuring Spatial Correspondence among Network Paths, Geographical Analysis 48(1) (2016), 3–17.

[2] T.M. Matisziw, E. Demir, Inferring Network Paths from Point Observations, International Journal of Geographical Information Science 26(10) (2012), 1979–1996.

[3] E. Demir, Assigning Convenient Paths by an Approach of Dynamic Programming, Proceedings Book of MICOPAM 2018, Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary, (2018), 196–199.

[4] D. Yordanova, M. Pencheva, Bulgarian Policy for Regional Development and its Implications on Air Transport System: The Case for Shtraklevo Airport, BulTrans-2018 Conference Proceedings (2018), 137–142.

[5] E. Demir, Havalimanlarında kalkış öncesi, acil durumlarda, yardım alınabilecek en uygun lokasyonun Weber problemine uyarlanarak belirlenmesi, Türk Coğrafya Dergisi 70, (2018), 81–85.

[6] E. Demir, Approach for siting a support facility for transporting supplies in emergency cases in the Republic of Bulgaria, MATEC Web of Conferences 234, EDP Sciences (2018), 06001.

[7] E. Demir, N.U. Kockal, Facility Location Determined by an Iterative Technique, Proceedings Book of MICOPAM 2018, Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70th Anniversary, (2018), 192–195.

[8] A. Weber, Alfred Webers Theory of the Location of Industries, Chicago: University of Chicago Press; 1929.

[9] L.N. Tellier, The Weber Problem: Solution and Interpolation, Geographical Analysis 4(3) (1972), 215–233.

[10] E. Weiszfeld, Sur le point pour lequal la somme des distances de n pointes donnes est minimum, The Tohoku Mathematical Journal 43 (1937), 335–386.

[11] H.M. Venceslau, M.B. Karam Venceslau, A.E. Xavier, N. Maculan, A Geometric Perspective of the Weiszfeld Algorithm for Solving the Fermat-Weber Problem, RAIRO - Operations Research 50(1) (2016), 157–173.

[12] H. Uster, R.F. Love, The Convergence of the Weiszfeld Algorithm, Computers and Mathematics with Applications 40(4) (2000), 443–451.

[13] LINDO Systems Inc. (2019), https://www.lindo.com/index.php/products/lingo-and-optimization-modeling. Last access April 24, 2019.

[1]Department of Civil Engineering, Antalya Bilim University
* For correspondence

E-mail : emre.demir@antalya.edu.tr

Relationship Between Traffic Density and Pavement Deflections

Emre Demir[1], *Niyazi Ugur Kockal*[2]

Abstract

Traffic density is one of the most important parameters to consider in road construction. The presence of different components in road construction makes it essential to know the individual layer and inter-layer behaviors. One of the physical factors that plays a major role in these layers is their displacement, which varies depending on the type and engineering properties of these layers. Road damaging is inevitable as different deformations also trigger mismatching. Due to incompatibilities between layers, the number and content of repairs are also high than they should be, which increase maintenance costs. Therefore, it is important to conduct experimental and theoretical studies on this subject. Although the effect of traffic density on a single layer has been studied, there are not many studies investigating the co-movement of different layers. Therefore, the aim of this paper is to provide a holistic approach to this subject by compiling different studies.

KEYWORDS: Displacements, Layers, Traffic density, Pavement

Introduction

According to the European Union Road Statistics, and European Commission, 50% of the employment in companies of all transport modes lies in activities related to roads [1]. Additionally, many research show that transportation and manufacturing activities are related to mathematical methodologies [2, 3, 4]. A key component in making design, construction, maintenance, and rehabilitation decisions for pavements, consists of evaluation like, assessing and measuring surface distresses namely, cracking and rutting, or structural properties say, deflection and strain and forecasting the effect of such conditions on future performances [5]. Condition forecasts are generated with performance models, which are mathematical expressions that relate condition data to a set of explanatory variables such as material properties, pavement design characteristics, traffic loading, environmental factors and the history of maintenance activities.

Some investigations related to the mentioned properties given below: Ibrahim et al. [6] studied the evaluation of flexible pavement deformation of roads over subgrade and concluded that increase in the thickness of flexible pavement increased the number of passes which reached the same value of rutting (value of failure) and led to decrease the displacement in subgrade and subbase layers. Muttashar et al. [7] stated that the subgrade layer characterizes as one of the important fundamentals in the pavement design. The subgrade layer responses impact the whole pavement behavior. Weak subgrade layers are difficulty task to the engineers always in case of the construction of such an embankment over weak foundations soil, especially for the evacuation of; slope stability, bearing capacity, lateral pressures, differential settlement and movements. Therefore, researchers attempted to improve weak subgrade soil by pile technique.

There are also some applications that are affected by soil quality and traffic loads. Thus, some investigators [8] studied the mechanism of the interaction of soil and an underground pipeline in order to determine the load arising on the pipeline from the pressure due to the backfill soil and automobile traffic.

In another investigation, general formulae for the stresses and displacements in layered elastic systems subjected to external loads are derived [9]. On the basis of these formulae the practical formulae for calculation of the stresses and displacements in layered elastic systems under the action of vertical, centripetal-horizontal, unidirectional-horizontal and rotational-horizontal loads distributed on circular areas and compressed by rigid circular plates are derived, and the corresponding computer programs are worked out. Some researchers also worked on effect of traffic load on behavior of flexible pavements [10]. Therefore, calculations of critical stresses and strains (tensile on the bottom of asphalt and cement stabilized layer, vertical compressive load on the top of the subgrade) were performed for selected pavements by the CIRCLY software.

Methodologies and Assessments

In the light of research in the literature recently, commonly used methodologies and assessments are described as follows. Traffic density is related to the individual spacing between successive vehicles (e.g. distance measured from front bumper of a vehicle to front bumper of the next vehicle) [11]. Therefore, traffic density is defined as

$$k = \frac{n}{l} \tag{1}$$

based on traffic density in vehicles per unit distance as k, number of vehicles occupying some length of roadway at some specified time as n, length of roadway as l. The roadway length, l, can be defined as

$$l = \sum_{i=1}^{n} s_i \tag{2}$$

where s_i is spacing of the i^{th} vehicle (e.g. the distance between vehicles i and $i-1$, measured from the front bumper to front bumper), and n is number of measured vehicle spacings [11]. By substituting into traffic density equation:

$$k = \frac{n}{\sum_{i=1}^{n} s_i} \tag{3}$$

or

$$k = \frac{1}{\bar{s}} \tag{4}$$

Accordingly, Rayleigh damping is one of the most popular phenomena about deflections and the traffic density proportion [12, 13]. By using this methodology, mass proportional damping can be combined with the stiffness proportional damping on the road. The description can be seen as

$$\left[C\right] = \alpha\left[M\right] + \beta\left[K\right] \tag{5}$$

where matrix $\left[\text{C}\right]$ is damping matrix, α is the Rayleigh coefficient for the mass proportional damping, and β is the Rayleigh coefficient for the stiffness proportional damping [13]. The relationship between α and β can be provided by the following equation considering the circular frequency of ω with one-degree of freedom:

$$\xi = \frac{1}{2}\left(\beta\omega + \frac{\alpha}{\omega}\right) \tag{6}$$

where ξ is the fraction of damping with the circular frequency of ω. A relationship exists between damping and the load frequency in a limited range of frequency. This frequency may be cased by the traffic density at a specific region containing specific construction materials in pavement.

Conclusions

Mathematical methods and models are used to overcome diverse phenomena of infrastructures in lots of engineering subjects and operations. Traffic load such as the amount of traffic density and other factors are important parameters in the design of roads. The presence of different components in the road structure causes changes in their behavior due to the effect of these parameters. Although individual component behaviors under different influences have been examined in the literature, there is no study on the entire road structure that has been found by considering all the active parameters. Such studies need to be increased.

References

[1] P. Pereira, J. Pais, Main flexible pavement and mix design methods in Europe and challenges for the development of an European method, Journal of Traffic and Transportation Engineering (English Edition) 4(4) (2017), 316–346.

[2] E. Demir, Havalimanlarında kalkış öncesi, acil durumlarda, yardım alınabilecek en uygun lokasyonun Weber problemine uyarlanarak belirlenmesi, Türk Coğrafya Dergisi 70 (2018), 81–85.

[3] E. Demir, N.U. Kockal, Facility Location Determined by an Iterative Technique, Proceedings Book of MICOPAM 2018, Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70^{th} Anniversary (2018), 192–195.

[4] E. Demir, Assigning Convenient Paths by an Approach of Dynamic Programming, Proceedings Book of MICOPAM 2018, Dedicated to Professor Gradimir V. Milovanović on the Occasion of his 70^{th} Anniversary (2018), 196–199.

[5] A. Gupta, P. Kumar, R. Rastogi, Critical review of flexible pavement performance models, KSCE Journal of Civil Engineering 18(1) (2014), 142–148.

[6] S.F. Ibrahim, G.G. Sofia, Z.T. Teama, An Approach in Evaluating of Flexible Pavement In Permanent Deformation of Paved and Unpaved Roads Over Sand Dunes Subgrade Under Repeated Loads, Journal of Environmental and Earth Science 4(14) (2014), 78–90.

[7] H.Y. Muttashar, Y.M. Al-Badran, N.G.A. Alkawaaz, Evaluation of Pile Technique to Improve Weak Subgrade Soil Under Traffic Loading, International Journal of Engineering & Technology 7 (4.20) (2018), 563–567.

[8] V.S. Shirokov, Soil and Traffic Loads on Underground Pipelines, Soil Mechanics and Foundation Engineering 55(2) (2018), 115–119.

[9] W. Kai, Analysis and calculation of stresses and displacements in layered elastic systems, Acta Mechanica Sinica 3(3) (1987), 251–260.

[10] J. Domitrović, T. Rukavina, M. Ožbolt, Effect of traffic load on behaviour of flexible pavements examples from Croatia, In CETRA 2010, January 2010.

[11] F.L. Mannering, S.S. Wasburn, Principles of Highway Engineering and Traffic Analysis, 5^{th} Ed., John Wiley & Sons Singapore Pte. Ltd., 2013.

[12] A. Lanzi, J.E. Luco, Elastic velocity damping model for inelastic structures, Journal of Structural Engineering 144(6) (2018), 04018065.

[13] J. Huang, M. Losa, P. Leandri, Determining the effect of damping layers in flexible pavements on traffic induced vibrations, Advances in Materials and Pavement Prediction: Papers from the International Conference on Advances in Materials and Pavement Performance Prediction (AM3P 2018), April 16-18, 2018, Doha, Qatar. CRC Press, 2018.

[1]Department of Civil Engineering, Antalya Bilim University
[2]Department of Civil Engineering, Akdeniz University
* For correspondence

E-mail : emre.demir@antalya.edu.tr, nukockal@akdeniz.edu.tr

Sequences on an integral domain

Mustafa Alkan[1]

Abstract

In this note, we study a sequence on an integral domain and give some general formulas for the Binet Formulas.

2010 MATHEMATICS SUBJECT CLASSIFICATIONS : 05A15,05A19, 11B39
KEYWORDS: Generalized Fibonacci Polynomials, Generalized Lucas Polynomials,Generalized Humbert Polynomials, Binet Formula, Combinatorics identities, Generating Function

Main Results

After Fibonacci number, many people have studied Fibonacci numbers and achieved many combinatorial results. They have been widely used in many research areas such as Physis, Engineering, Architecture Art and so on. Therefore, in literature, there are many sequences of integers and polynomials via the Fibonacci numbers such as Fibonacci polynomials, Lucas numbers and polynomials, Pell numbers and polynomials, Pell-Lucas numbers and polynomials, Jacobstal numbers and polynomials, Jacobstal-Lucas numbers and polynomials, Chebyshev numbers and polynomials and so on. Many people have studied these numbers and polynomials (([1],[2],[3],[4],,[6],[9],[12],[19], [20],[22],[23],[24]).

For integer numbers a, b, in [6], Horadam defined the sequence on the integer numbers $w_n(a, b, p, q)$ or briefly $\{w_n\}$ by the recurrence relation

$$w_n = pw_{n-1} - qw_{n-2} \ (n \geq 2)$$

with the initial conditions $w_0 = a$, $w_1 = b$.

Similarly Horadam polynomials $h_n(x, a, b, p, q)$ (briefly h_n) was defined in the following

$$h_1 = a, h_2 = bx, \text{ and } h_n = ph_{n-1} - qh_{n-2} \ (n \geq 3)$$

with the initial conditions $h_1 = a$, and $h_2 = bx$.

In this note, we generalize the sequences of both integers and polynomials with integers coefficient in the literature to a sequence on an integral domain via the Fibonacci numbers. By introducing the sequence on an integral domain R, we have a unified approach to deal with not only the existing sequences on both integers and polynomials but also some results in literature since the polynomial ring with finite indeterminates over an integral domain is an integral domain. By using some properties of matrix on an integral domain, we get some expression recurrences and combinatorial formulas on an integral domain.

In this paper, we use the notation R for an an integral domain, $R[X]$ for the extension ring containing R and $X = \{x_1, ..., x_n\}$ and α^{-1} or $\frac{1}{\alpha}$ for the inverse of α in a extension ring of R.

Let p, q, a and b be elements of R. Then for a positive integer n, we define the recurrence relations $w_{n+1} = pw_n + qw_{n-1}$ and we say that $w = \{w_{n+1} = pw_n + qw_{n-1} :$

$w_0 = a, w_1 = b,\ n \in \mathbb{Z}^+\}$ is the sequence of p, q with a, b on an integral domain R denoted $w_n(a, b, p, q)$ or briefly $\{w_n\}$.

If q^{-1} exits then $w_{-n} = q^{-1}(w_{-n+2} - pw_{-n+1})$ exists for integer n and so we may assume that q^{-1} is in R.

By the definitions, we may calculate that

$$\begin{aligned} w_{-1} &= q^{-1}(b - pa), \\ w_2 &= pb + qa, \\ w_7 &= \left(q^3 + p^6 + 6p^2q^2 + 5p^4q\right) b + \left(4p^2q + 3q^2 + p^4\right) pqa. \end{aligned}$$

For $a = 0$ and $b = 1$, we use the notation $k = \{k_{n+1} = pk_n + qk_{n-1} : k_0 = 0, k_1 = 1, n \in \mathbb{Z}\}$ and so

$$\begin{aligned} k_{-2} &= -\frac{p}{q}, \\ k_8 &= p\left(2q + p^2\right)\left(4p^2q + p^4 + 2q^2\right). \end{aligned}$$

By specifying the parameters,we get some well–known sequence such as Pell numbers, Jacobsthal numbers, Fermat numbers, Chebyshev numbers, Morgan-Voyce numbers, Delannoy numbers and so on.

Remark 5. *Let R be the ring of integers.*

1. *If $p = 2, q = 1$, $a = 0$ and $b = 1$ then w_n is the n-Pell number.*
2. *If $p = 1, q = 2$, $a = 0$ and $b = 1$ then w_n is the n-Jacobsthal number.*
3. *If $p = q = 1$, $a = 2$ and $b = 1$ then w_n is the n-Lucas number.*
4. *If $p = q = 1$, $a = 2$ and $b = 2$ then w_n is the n-Pell-Lucas number.*
5. *If $p = 1, q = 2$, $a = 2$ and $b = 1$ then w_n is the n-Jacobsthal-Lucas number.*

Remark 6. *Let $R = \mathbb{Z}[x]$.*

1. *If $p = x, q = 1$, $a = 0$ and $b = 1$ then w_n is the n-Fibonnaci polynomial.*
2. *If $p = x + 1, q = -2$, $a = 0$ and $b = 1$ then w_n is the n-Delannoy polynomial.*

Remark 7. *Let $R = \mathbb{Z}[x, y]$ and if $p = x, q = y$, $a = 0$ and $b = 1$ then w_n is the n-Fibonnaci polynomial with two variables for positive integer n.*

Let M be $n \times n$ matrix with entries from an integral domain R. Then we have that

$$M.Adj(M) = \det M.I$$

where $Adj(M)$ is the adjugate of M.

Therefore, $\det M$ is invertible in R if and only if M is an invertible matrix.

To use linear algebra method, we use the matrix notations for the sequence of p, q with a, b. Let us define the matrix

$$W(n) = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}.$$

Hence, we get that $W(0) = \begin{bmatrix} w_1 & w_0 \\ w_0 & q^{-1}(w_1 - pw_0) \end{bmatrix}$ and so for any integer n, we have

$$W(n) = K.W(n-1)$$

where $K = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$.

Now, we fix the notation $K = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$. If $w_1 = 1$ and $w_0 = 0$ then

$$W(n) = K(n) = \begin{bmatrix} k_{n+1} & k_n \\ k_n & k_{n-1} \end{bmatrix} \text{ and } W(0) = K(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{q} \end{bmatrix}.$$

If $w_0 = 0$ and $q = 1$ then

$$W(n) = L(n) = \begin{bmatrix} l_{n+1} & l_n \\ l_n & l_{n-1} \end{bmatrix} \text{ and } L(0) = w_1 I.$$

Theorem 1. *For any integer n, we have $K^n = k_n K + q k_{n-1}$.*

Corollary 2. *For an integer n, we have $K^n = \begin{bmatrix} k_{n+1} & qk_n \\ k_n & qk_{n-1} \end{bmatrix}$.*

Let R_1 be an extension of R containing both the inverse of q and the roots of $m(t)$ where $m(t) = t^2 - pt - q$ is the minimal polynomial for K on $R[t]$. Let α, β be elements of R_1. Then it follows that $m(t) = (t-\alpha)(t-\beta) \in R_1[t]$ and so $\alpha^{-1} = q^{-1}(\alpha - p), \beta^{-1} = q^{-1}(\beta - q) \in R_1$. Thus we may define the matrices P and D on R_1 such that

$$P = \begin{bmatrix} 1 & 1 \\ -q^{-1}\alpha & -q^{-1}\beta \end{bmatrix} \text{ and } D = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}.$$

Then it follows that $(detP)\, K = PDAdj(P)$. Hence for an integer n,

$$(detP)\, K^n = PD^n Adj(P).$$

Let R_2 be an extension integral domain of R_1 such that $(\alpha-\beta)^{-1} \in R_2$, *i.e.* $R_2 = R_1[(\alpha-\beta)^{-1}]$. Then for an integer n, we have that $K^n = PD^nP^{-1}$ since the inverse of $\Lambda = \frac{1}{q}(\alpha - \beta)$ is $\frac{q}{(\alpha-\beta)}$ in R_2.

If $R = \mathbb{Q}$ is the set of the rational numbers then $R_2 = \mathbb{C}$ is the set of complex numbers for integers p and q and $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$, $\beta = \frac{p-\sqrt{p^2+4q}}{2}$ and so $\Lambda = \frac{1}{q}\sqrt{p^2+4q}$. Hence we get Binet Theorem for an integral domain.

Theorem 3. *Let α β be above and q^{-1} exist. Then for any integer n, we have*

$$(\alpha-\beta)\, w_n = \alpha^{n-1}(qw_0 + \alpha w_1) - \beta^{n-1}(qw_0 + \beta w_1).$$

Now by specifying the parameters in Theorem 3, we get Binet formulas for the sequences of both integers and polynomials in the literature.

Remark 8. *1. If $p = q = 1$, $a = 2$ and $b = 2$ in Theorem 3, we have*

$$w_n = \frac{1}{5}\left(\alpha^{n-1}\left(3\sqrt{5}+5\right) + \beta^{n-1}\left(5 - 3\sqrt{5}\right)\right)$$

is Binet formula for Pell-Lucas numbers.

2. If $p = 1, q = 2$, $a = 2$ and $b = 1$ in Theorem 3, we have

$$w_n = 2\alpha^{n-1} - \beta^{n-1} = 2^n + (-1)^n$$

is Binet formula for Jacobsthal-Lucas numbers.

3. If $p = x+2, q = -1$, $a = 0$ and $b = 1$ in Theorem 3, then we have

$$w_n = \frac{(2+x)(\alpha^{n-1} - \beta^{n-1}-) + (\alpha^{n-1} + \beta^{n-1})\sqrt{(x+2)^2 - 4}}{2\sqrt{(x+2)^2-4}}$$

is Binet formula for Morgan-Voyce polynomials.

References

[1] Bergum, G. E., Hoggatt V. E.: Sums and products for recurring sequences, The Fibonaci Quaterly, **13(2)**, 115-120, (1975).

[2] Cheon G.S, Kim H, Shapiro L.W.: A generalization of Lucas polynomial sequence. Discrete Appl Math.;**157(5)**, 920–927 (2009).

[3] Dilcher K. A.: Generalization of Fibonacci polynomials and a representation of Gegenbauer polynomials of integer order. Fibonacci Quart. **25.4** 300-303 (1987)

[4] Falcón, S.: On the k-Lucas numbers. Int J Contemp Math Sci. **6(21–24),**1039–1050, (2011).

[5] Falcon, S. Plaza, A.: .On k-Fibonacci sequences and polynomials and their derivatives, Chaos, Solitons and Fractals 39 1005–1019, (2009)

[6] Horadam, A. F.: Basic properties of a certain generalized sequence of numbers, The Fibonaci Quaterly, **3(2)** , 161-176, (1965).

[7] Horadam,A.F.: Tschebyscheff and other functions associated with the sequence $W_n(a, b; p, q)$, The Fibonaci Quaterly, **7(1)**, 14-22, (1969).

[8] Horadam A.F.: A synthesis of certain polynomial sequences. In: Bergum GE, Philippou AN, Horadam AF, editors. Applications of Fibonacci numbers, Vol. 6. Dordrecht: Kluwer Academic Publishers; 215–229 (199

[9] Hsu L.C.: On stirling-type pairs and extended Gegenbauer-Humbert-Fibonacci polynomials. In: Bergum GE, Philippou AN, Horadam AF, editors. Applications of Fibonacci numbers, Vol. 5. Dordrecht: Kluwer Academic Publishers; 367–377, (1993).

[10] Hsu, L.C., Shiue P.J.S.: Cycle indicators and special functions. Ann Comb. **5(2)**, 179–196 (2001).

[11] Jacobson N. Basic algebra. I. 2nd ed. New York: W.H. Freeman and Company; 1985.

[12] Janjic M. Hessenberg matrices and integer sequences. J Integer Seq. **13(7)**,10 pp. Article 10.7.8. (2010).

[13] Janjic M. : Determinants and recurrence sequences. J Integer Seq. ;**15(3)**:21 pp. Article 12.3.5.(2012).

[14] Koshy, T.: Fibonacci and Lucas Numbers with Applications, Wiley-Interscience Publications, (2001).

[15] Lee, G, Asci M.: Some properties of the (p, q)-Fibonacci and (p, q)-Lucas polynomials. J Appl Math. 18 pp. Art. ID 264842, (2012).

[16] Mansour, T: A formula for the generating functionsof powers of Horadam's sequence, Australasian J. of Combinatorics **30**, 207-212,(2004).

[17] Nalli A, Haukkanen P. On generalized Fibonacci and Lucas polynomials. Chaos Solitons Fractals. **42(5)**, 3179–3186, (2009) .**5(4)**,

[18] Ozdemir G., Simsek Y.;Generating Functions For Two-Variable Polynomials Related To a Family of Fibonacci Type Polynomials and Numbers FILOMAT **30:4** 969–975, (2016),

[19] Ramírez J.L.: Some properties of convolved k-Fibonacci numbers. ISRN Combin. Art. ID 759641 (2013).

[20] Robbins, N. A new formula for Lucas numbers, The Fibonaci Quaterly, **29**, 362-363 (1991).

[21] Weiping, W., Hui W.:Some results on convolved (p, q)-Fibonacci polynomials Ing.Trans and Special Function, **26**-5, 340–356, 2015..

[22] Weiping, W., Hui W.:.Generalized Humbert polynomilas via generalizaed Fibonacci polynomials, 307, 204–2016, 2017..

[23] Yuan, Y., Zhang, W.: Some identities involving the Fibonacci polynomials. Fibonacci Quart. **40(4)**, 314–318, (2002).

[24] Zhang, W.: Some identities involving the Fibonacci numbers. Fibonacci Quart. **35(3)**, 225–229, (1997).

[1]Department of Mathematics, Faculty of Science, University of Akdeniz, Antalya TR-07058, Turkey.

E-mail : alkan@akdeniz.edu.tr

Dual Zariski Topology related to Ideals of a Ring

Ortaç Öneş[1] , *Mustafa Alkan*[2]

Abstract

We deal with the dual Zariski topology on comultiplication modules and also give some characterizations for the module by using dual Zariski topology related to any ideals of a ring.

2010 Mathematics Subject Classifications : 13C13, 13C99
Keywords: Comultiplication Modules, Second Submodules, Dual Zariski Topology

Introduction

In this study, all rings are commutative with identity, all modules are unital, R will denote a ring and M will denote an R-module.

A submodule N of M is said to be a second submodule provided N is non-zero and, for all $r \in R$, $rN = 0$ or $rN = N$. If N is a second submodule of M, then $p = Ann_R(N)$ is a prime ideal of R. In this case, N is called a p-second submodule of M.
The dual notion of Zariski topology on all second submodules of M was studied in [1], [2], [5], [6], [8] and [9].

Let $Spec^s(M)$ be the set of all second submodules of M and $V^s(N) = \{P \in Spec^s(M) : P \subseteq N\}$ for a submodule N of M. Then $V^s(0)$ is the empty set and $V^s(M)$ is $Spec^s(M)$. It is easy to see that $\bigcap\limits_{i\in\Lambda} V^s(N_i)$ is equivalent to $V^s\left(\bigcap\limits_{i\in\Lambda} N_i\right)$ for any family of submodules N_i of M. Besides $V^s(N) \cup V^s(L) \subseteq V^s(N + L)$, where N and L are submodules of M.

M is said to be a comultiplication R-module if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$.

Let I and J be any ideals of R. Then we have

$$\begin{aligned} V^s((0 :_M I) \cup V^s((0 :_M J)) &= V^s((0 :_M I) + (0 :_M J)) \\ &= V^s((0 :_M I \cap J)) = V^s((0 :_M IJ)). \end{aligned}$$

$\Gamma^s(M) = \{V^s((0 :_M I))\}$, where I is an ideal of R, satisfies the axioms for closed sets in a topological space, which induces a topology on $Spec^s(M)$. If M is a comultiplication R-module, there exists a topology Γ^s on $Spec^s(M)$ having $\xi^s(M) = \{V^s(N) : N \leq M\}$ as the collection of all closed sets.

Let N be a submodule of an R-module M. The sum of all second submodules of M contained in N is called the second radical of N denoted by $Sec(N)$. In case N does not contain any second submodule, the second radical of N is defined to be (0). It is said that $N \neq 0$ is a second submodule of M if $Sec(N) = N$ ([5]).

Lemma 1. *[8] Let $N = (0 :_M I)$ be a submodule of a comultiplication R-module M, $\mathcal{X}_N^s = Spec^s(M)\backslash V^s((0 :_M I))$ and $\tilde{V}^s((0 :_M J)) = V^s((0 :_M J))\backslash V^s((0 :_M I))$, where I and J are ideals of R. Then*

$$\Gamma_N^s = \left\{\tilde{V}^s((0 :_M J))\right\}$$

satisfies the axioms for closed sets of a topological space on $\mathcal{X}_N^s$.

This topology is called as the complement dual Zariski topology of N in M.
Note that

$$\begin{aligned} V^s((0 :_M J)) &= V^s\left(\left(0 :_M \sum_{r_i \in J} Rr_i\right)\right) \\ &= V^s\left(\bigcap_{r_i \in J} (0 :_M Rr_i)\right) = \bigcap_{r_i \in J} V^s((0 :_M Rr_i)). \end{aligned}$$

Proposition 2. *[8] Let $N = (0 :_M I)$ be a submodule of a comultiplication R-module M, where I is an ideal of R. For any ideal J of R, the set $(\mathcal{X}_N^s)^{(0:_M J)} = \mathcal{X}_N^s\backslash\tilde{V}^s((0 :_M J))$ forms a base for the complement dual Zariski topology of N in M on $\mathcal{X}_N^s$.*

Main Results

We fix the module M as a comultiplication module in this study.

Lemma 3. *Let I,J and K be proper ideals of R. Then we have the following.*
i) Any open set of $\mathcal{X} = Spec^s(M)$ is of the form $\mathcal{X}_{(0:_M I)}$.
ii) $\mathcal{X}_{(0:_M I)} = \mathcal{X}_{(0:_M J)}$ if and only if $Sec((0 :_M I)) = Sec((0 :_M J))$.

Proof. i) It is clear.
ii) Let $\mathcal{X}_{(0:_M I)} = \mathcal{X}_{(0:_M J)}$. Then we have

$$Spec^s(M)\backslash V^s((0 :_M I)) = Spec^s(M)\backslash V^s((0 :_M J))$$

implying $V^s((0 :_M I)) = V^s((0 :_M J))$. It follows $Sec((0 :_M I)) = Sec((0 :_M J))$.
Let $Sec((0 :_M I)) = Sec((0 :_M J))$. Then we have $V^s((0 :_M I)) = V^s((0 :_M J))$, implying $Spec^s(M)\backslash V^s((0 :_M I)) = Spec^s(M)\backslash V^s((0 :_M J))$. Thus we get that $\mathcal{X}_{(0:_M I)} = \mathcal{X}_{(0:_M J)}$. □

Lemma 4. *Let I,J and K be proper ideals of R. Then we have $\mathcal{X}_{(0:_M I)} \cap \mathcal{X}_{(0:_M J)} = \mathcal{X}_{(0:_M K)}$ if and only if $Sec((0 :_M IJ)) = Sec((0 :_M K))$.*

Proof. Let $\mathcal{X}_{(0:_M I)} \cap \mathcal{X}_{(0:_M J)} = \mathcal{X}_{(0:_M K)}$. Then we have

$$(Spec^s(M)\backslash V^s((0 :_M I))) \cap (Spec^s(M)\backslash V^s((0 :_M J))) = Spec^s(M)\backslash V^s((0 :_M K))$$

implying $V^s((0 :_M IJ)) = V^s((0 :_M K))$. It follows $Sec((0 :_M IJ)) = Sec((0 :_M K))$.
Let $Sec((0 :_M IJ)) = Sec((0 :_M K))$. Then we have

$$\begin{aligned} Spec^s(M)\backslash V^s((0 &: {}_M IJ)) = Spec^s(M)\backslash V^s((0 :_M K)) \\ &= (Spec^s(M)\backslash V^s((0 :_M I))) \cap (Spec^s(M)\backslash V^s((0 :_M J))) \end{aligned}$$

Thus we get that $\mathcal{X}_{(0:_M I)} \cap \mathcal{X}_{(0:_M J)} = \mathcal{X}_{(0:_M K)}$. □

The following corollary is a consequence of Lemma 4.

Corollary 5. *Let I, J be proper ideals of R. Then $\mathcal{X}_{(0:_M I)} \cap \mathcal{X}_{(0:_M J)} = \emptyset$ if and only if $Sec((0 :_M IJ)) = Sec(M)$.*

Acknowledgements

This paper has been presented at **"The 2nd Mediterranean International Conference of Pure&Applied Mathematics and Related Areas"**, Paris-France, August 28-31, 2019.

References

[1] H. Ansari-Toroghy, F. Farshadifar, On the dual notion of prime submodules, Algebra Colloquium 19 (2012), 1109-1116.

[2] H. Ansari-Toroghy and F. Farshadifar, The Zariski Topology on the Second Spectrum of a Module, Algebra Colloquium 21 (2014), 671-688.

[3] M. F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra, London: Addison-Wesley; 1969.

[4] N. Bourbaki, Elements of Mathematics General Topology Part 1 and Part 2, Massachusetts: Addison-Wesley; 1966.

[5] S. Çeken and M. Alkan, Dual Zariski Topology for Modules, AIP Conf. Proc. 1389 (2011), 357-360 .

[6] S. Çeken and M. Alkan, Second Spectrum Of Modules And Spectral Spaces, Bulletin of the Malaysian Mathematical Society Series, DOI: 10.1007/s40840-017-0473-0, 2017.

[7] C. P. Lu, Modules with Noetherian Spectrum, Communication in Algebra 38 (2010), 807-828.

[8] O. Öneş, Dual Zariski Topology on Comultiplication Modules, Sakarya University Journal of Science 23 (2019), 390-395.

[9] O. Öneş and M. Alkan, The Structure of Some Topologies on A Module 1863 (2017), 300010(1)-300010(4).

[10] O. Öneş and M. Alkan, Multiplication modules with prime spectrum, Turkish Journal of Mathematics 43 (2019), 2000-2009.

[11] O. Öneş and M. Alkan, Zariski Subspace Topologies On Ideals, Hacettepe Journal of Mathematics and Statistics, Doi: 10.15672/HJMS.2018.597.

[1,2]Department of Mathematics, Faculty of Science, University of Akdeniz, Antalya TR-07058, Turkey.

E-mail : ortacns@akdeniz.edu.tr, alkan@akdeniz.edu.tr

A Generalization of Prime Ideals in a Commutative Ring

Ortaç Öneş[1]

Abstract

The purpose of this study is to study $*$-prime ideals associated with any ideal in a commutative ring and examine the connections between prime ideals and $*$-prime ideals.

2010 Mathematics Subject Classifications : 16N60
Keywords: Prime ideal

Introduction

In this study, all rings are commutative with identity and R will denote a ring.

The subject of prime ideal is one of the most important concepts in ring theory and so many useful results have been proved in this subject ([2], [5], [7], [8], [13]).

A proper ideal P of R is called prime if for $a, b \in R$ such that $ab \in P$, $a \in P$ or $b \in P$ holds. We define a generalization of this concept as follows:

Let I and P be ideals of a ring R. An ideal P of R is said to be I-prime ideal if for every element a, b in I such that $ab \in P$, either $a \in P$ or $b \in P$ holds.
Clearly, one can observe that every prime ideal is also $*$-prime ideal but the converse of this is not true. We give some examples for this situation and investigate some properties of this concept. We also examine the relationships between prime ideals and $*$-prime ideals.

Main results

Definition 1. *Let I be an ideal of R. An ideal P of R is said to be I-prime ideal if for every element a, b in I such that $ab \in P$, either $a \in P$ or $b \in P$ holds.*

It is clear that Definition 1 is equivalent to the known definition of prime ideal in a commutative ring R when $I = R$.

We say $*$-prime instead of I-prime when we don't mention a certain ideal I of R.

One can observe that every prime ideal of a commutative ring is also $*$-prime but the converse is not true. For both this situation and $*$-prime ideals, we give some examples as follows:

Example 2. *Let $\mathbb{Z}$ be an integer ring.*

i) $6\mathbb{Z}$ is not prime ideal of $\mathbb{Z}$ but $6\mathbb{Z}$ is $2\mathbb{Z}$-prime ideal of $\mathbb{Z}$. It is clear that $6\mathbb{Z}$ is not prime ideal of $\mathbb{Z}$. Let a and b be in $2\mathbb{Z}$ such that $ab \in 6\mathbb{Z}$. Since a and b are in $2\mathbb{Z}$, 2 divides both a and b. Besides 3 must divide ab because 3 divides 6. Assume that 3 divides a. Thus 6 divides a because a is divided by both 2 and 3, which implies that $a \in 6\mathbb{Z}$. Consequently $6\mathbb{Z}$ is $2\mathbb{Z}$-prime ideal of $\mathbb{Z}$.

ii) $6\mathbb{Z}$ and $10\mathbb{Z}$ are not prime ideal of $\mathbb{Z}$ but $6\mathbb{Z}$ is a $10\mathbb{Z}$-prime ideal of $\mathbb{Z}$ and $10\mathbb{Z}$ is a $6\mathbb{Z}$-prime ideal of $\mathbb{Z}$.

Lemma 3. *Let I, J and P be ideals of R. Then the followings hold.*

i) Let P be an I-prime ideal of R such that $J \subseteq I$. Then P is a J-prime ideal of R.

ii) P is an I-prime ideal of R such that $P \subsetneq I$ if and only if $(I+P)/P$ has any nonzero divisors.

Proof. *i*) It is clear from the fact that $J \subseteq I$ and P is an I-prime ideal of R.

ii) Let P be an I-prime ideal of R. Take the elements $P \neq a+P, b+P$ in $(I+P)/P$ such that $P = (a+P)(b+P)$. We can assume $ab \in P$, where $a, b \in I$. It follows that $a \in P$ or $b \in P$ with the hypothesis. This means that $P = a+P$ or $P = b+P$, which is a contradiction. Thus $(I+P)/P$ has any nonzero divisors.

Assume that $(I+P)/P$ has any nonzero divisors. Take the elements a, b in I such that $ab \in P$. Then $P \neq a+P, b+P \in (I+P)/P$ such that $P = ab+P = (a+P)(b+P)$. Thus we have $a+P = P$ or $b+P = P$ with the hypothesis. It follows that $a \in P$ or $b \in P$, which completes the proof. □

Proposition 4. *Let R and S be any rings, $\phi : R \to S$ a ring epimorphism and $Ker\phi \subseteq P$. Then P is an I-prime ideal of R if and only if $\phi(P)$ is a $\phi(I)$-prime ideal of S.*

Corollary 5. *Let I be an ideal of R. Then P is an I-prime ideal of R if and only if P/J is a I/J-prime ideal of R/J, where J is an ideal of R and $J \subseteq I, P$.*

Acknowledgements

This paper has been presented at **"The 2nd Mediterranean International Conference of Pure&Applied Mathematics and Related Areas"**, Paris-France, August 28-31, 2019.

References

[1] I. Akray, I-prime ideals, Journal of Algebra and Related Topics 4 (2016), 41-47.

[2] M. Alkan and Y. Tıraş, On prime submodules, The Rocky Mountain Journal of Mathematics 37 (2007), 709-722.

[3] D. D. Anderson and E. Smith, Weakly prime ideals, Houston Journal of Mathematics 29 (2003), 831-840.

[4] D. D. Anderson and M. Bataineh, Generalizations of prime ideals, Communications in Algebra 36 (2008), 686-696.

[5] M. Ebrahimpor and R. Nekooei, On generalizations of prime ideals, Communications in Algebra 40 (2012), 1268-1279.

[6] M. Ferrero and R. Wisbauer, Unitary strongly prime rings and related radicals, Journal of Pure and Applied Algebra 181 (2003), 209-226.

[7] I. Kaplansky, Commutative Rings, The university of Chicago: Chicago and London; 1974.

[8] C. P. Lu, Unions of prime submodules, Houston Journal of Mathematics 23 (1997), 203-213.

[9] R. L. McCasland and M. E. Moore, Prime submodules, Communications in Algebra 20 (1992), 1803-1817.

[10] D. G. Northcott, Lesson on rings, modules, and multiplicities, London: Cambridge University Press; 1968.

[11] Y. Tıraş, A. Harmancı and P. F. Smith, A characterization of prime submodule, Journal of Algebra 212 (1999), 743-752.

[12] Y. Tıraş and M. Alkan, Prime modules and submodules, Communications in Algebra 31 (2003), 5253-5261.

[13] O. Öneş and M. Alkan, On the left O-prime ideals over a noncommutative ring, Advanced Studies in Contemporary Mathematics 27 (2017), 107-113.

[1]Department of Mathematics, Faculty of Science, University of Akdeniz, Antalya TR-07058, Turkey.

E-mail : ortacns@akdeniz.edu.tr